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Optimal Schedules of Periodic Preventive Maintenance Model with Different PM Effect

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Abstract. In this paper, we consider a periodic preventive maintenance policy in which each preventive maintenance reduces the hazard rate of amount proportional to the failure intensity, which increases since the system started to operate. And the effect of preventive maintenance at each preventive maintenance epoch is different. The expected cost rate per unit time for the proposed model is obtained. We discuss the optimal number N of the periodic preventive maintenance and the optimal period x, which minimize the expected cost rate per unit time and obtain the optimal preventive maintenance schedule for given cost structures of the model. A numerical example is given for the purpose of illustrating our results when the failure time distribution is Weibull distribution.

Key Words: *periodic preventive maintenance, minimal repair, improvement factor, hazard rate, expected cost rate*

1. INTRODUCTION

As most of industrial systems become more complex and multiple-function oriented, it is extremely important to avoid the catastrophic failure during actual operation as well as to slow down the degradation process of the system. One way of achieving these goals is to take the preventive maintenance (hereafter, PM) while the system is still in operation. Although more frequent PM's certainly would keep the manufacturing system less likely to fail during its operation, such PM policy inevitably requires a higher cost of maintaining the system. Since Barlow and Hunter(1960) propose two types of PM policies, many authors have addressed the problem of designing the optimal schedule for the PM by determining the length of time interval between PM's to minimize the average cost rate of the system. Different types of PM policies studied in many literature are summarized in Pham and Wang(1996) and Wang(2002).

In most of the PM policies discussed earlier, the effect of PM is categorized into two types. The first one is that PM reduces only the hazard rate of system. That is, PM makes the manufacturing system less likely to fail during its operation. Nakagawa(1980) considers an imperfect PM policy for which the system has a reduced age at each PM intervention. If the size of age reduction at each PM is equal to the PM period, then such a policy becomes a perfect PM policy. Doyen and Gaudoin(2004) propose two classes of imperfect PM models based on reduction of failure intensity or virtual age, which are arithmetic reduction of intensity(ARI) model and arithmetic reduction of age(ARA) model. In model, which is a specific case of ARI model, PM reduces the failure intensity of amount proportional to the failure intensity, which increases since the last PM. And it is assumed that the wear-out speed is the same as before PM.

The second type of the effect of PM is to slow down the degradation process of the system by taking the preventive measure while the system is still in operation. Canfield(1986) considers a periodic PM policy for which the PM slows the degradation process of the system, while the hazard rate keeps monotone increase. Park, Jung and Yum(2000) derive the optimal PM schedules by associating the Canfield's PM model with various cost structures of operating the system.

Most of PM models try to achieve both of two types of the effect of PM. Lim and Park(2007) propose a periodic PM policy in which each PM reduces the hazard rate of amount proportional to the failure intensity, which increases since the last PM and slows down the wear-out speed to that of new one. And the proportion of reduction in hazard rate decreases with the number of PMs.

In this paper, we consider a periodic PM policy in which each PM reduces the hazard rate of amount proportional to the failure intensity while the wear-out speed is the same as that of system without any PM. The expected cost rate per unit time for the proposed model is obtained. We discuss the optimal number N of the periodic PM and the optimal period x, which minimize the expected cost rate per unit time and obtain the optimal PM schedule for given cost structures of the model.

Section 2 describes the periodic PM model under consideration and its assumptions. In Section 3, we derive the expression for the expected cost rate for the proposed PM policy. Section 4 presents the solutions for the optimal period and the optimal number of PM's which minimize the expected cost rate and thereby proposes the optimal PM schedule for the periodic PM policy with improvement factor. In Section 5, the optimal schedules are computed numerically the underlying failure times follow a Weibull distribution.

Notation

$h(t)$	hazard rate without PM
$h_{pm}(t)$	hazard rate with PM
x	period of PM
N^*	number of PM's conducted before replacement
p_k	improvement factor in hazard rate at the i-th PM
C_{mr}	cost of minimal repair at failure
C_{pm}	cost of PM
C_{re}	cost of replacement
$C(x,N)$	expected cost rate per unit time

2. MODEL AND ASSUMPTIONS

We consider a periodic PM model with an improvement factor which reduces the hazard rate of the system after PM. The followings are assumed :

(1) The system begins to operate at time $t=0$.
(2) The PM is done at periodic time kx (k = 1, 2, ...) where $x > 0$, and is replaced by new one at the N-th PM.
(3) For $kx < t \le (k+1)x$, $h_{pm}(t)$, which is the hazard rate during the k-th PM period, is the reduced hazard rate due to PM at kx, i.e. $h(t) - p_k h(kx)$ for all, where $0 \le p_k \le 1$. Here, $p_k h(kx)$ can be considered as the effect of PM at kx. When $p_k = 1$, the system after PM is restored to as good as new one and when $p_k = 0$, the system right after PM has the same hazard rate as that just prior to PM. For $0 \le p_k \le 1$, the hazard rate of the system is somewhat reduced after PM. After each PM, the ware-out speed is the same as that of the system without any PM. Then, between two PM's, the failure rate is virtually parallel to the initial hazard rate regardless of the magnitude of p_k.
(4) For any $k = 1,2,\ldots$, p_k satisfies that $p_k h(kx) \ge p_{k-1} h((k-1)x)$.
(5) The system undergoes only minimal repairs at failures between PM's.
(6) The repair and PM times are negligible.
(7) $h(t)$ is strictly increasing and convex in t.

Assumption (4) is to guarantee that the PM at the time epoch kx ($k = 1,2,\ldots$) make the system more reliable than before the PM. That is, the hazard rate right after PM is lower than hazard rate just prior to the PM.

3. EXPECTED COST RATE PER UNIT TIME

In this paper, we propose a periodic PM model with different improvement factor at each PM. Under this model, the hazard rate $h_{pm}(t)$ is given by

$$h_{pm}(t) = \begin{cases} h(t) & 0 < t \le x \\ h(t) - p_k h(kx) & kx < t \le (k+1)x \end{cases} \tag{3.1}$$

for $k = 1,2,\ldots$. For $t = 0$, $h_{pm}(0) = h(0)$ and x is the time interval between PM interventions. In the equation (3.1), $h(t)$ represents the initial hazard rate at time t and $p_k h(kx)$ is the amount of reduction in hazard rate due to the k-th PM.

Since it is well-known from Lemma 1.1 in Fontenot and Proschan(1984) that the number of minimal repairs during the period k of PM is nonhomogeneous Poisson process(NHPP) with intensity function $\int_{kx}^{(k+1)x} h_{pm}(t)dt$, the expected cost rate per unit time can be obtained in the following manner:

Expected Cost Rate Per Unit Time
= [(expected cost of minimal repairs in [0, Nx))
+ (expected cost of PM in [0, Nx))
+ (expected cost of replacement)]/Nx.

Each expected cost given in the expected cost rate per unit time is obtained as follows:

(i) Expected cost of minimal repairs in [0, Nx) $= C_{mr} \sum_{k=0}^{N-1} \int_{kx}^{(k+1)x} h_{pm}(t)dt$, where

$h_{pm}(t)$ is given in the equation (3.1)

(ii) Expected cost of PM in [0, Nx) = $(N-1)C_{pm}$.

(iii) Expected cost of replacement = C_{re}.

Using (i), (ii) and (iii), the expected cost rate per unit time for running the periodic PM with improvement factor during *[0, Nx]* is obtained as follows:

$$C(x,N) = \frac{1}{Nx}\left[C_{mr}\int_0^{Nx} h_{pm}(t)dt + (N-1)C_{pm} + C_{re}\right]$$

$$= \frac{1}{Nx}\left[C_{mr}\sum_{k=1}^{N}\int_0^{x}[h(t+(k-1)x) - p_{k-1}h((k-1)x)]dt + (N-1)C_{pm} + C_{re}\right]$$

$$= \frac{1}{Nx}\left[C_{mr}\sum_{k=1}^{N}\int_0^{x}\gamma_k(t)dt + (N-1)C_{pm} + C_{re}\right], \quad (3.2)$$

where $\gamma_k(t) = h(t+(k-1)x) - p_{k-1}h((k-1)x)$.

4. OPTIMAL SCHEDULES FOR PERIODIC PM

To design the optimal schedules for the periodic PM, we need to find an optimal PM period x^* and an optimal number N^* of PM needed before replacing the system by a new one. The decision criterion to adopt is to minimize the expected cost rate during the life cycle of the system.

4.1. PM period x is known

We first consider the case when the PM period x is known. In order to determine the optimal N^* which minimizes $C(x,N)$ of (3.2), we solve the following two inequalities for N.

$$C(x,N+1) \geq C(x,N) \text{ and } C(x,N) < C(x,N-1).$$

It can easily be shown from (3.2) that $C(x,N+1) \geq C(x,N)$ implies

$$\sum_{k=0}^{N} \int_0^x [\gamma_{N+1}(t) - \gamma_k(t)]dt \geq \frac{C_{re} - C_{pm}}{C_{mr}}. \tag{4.1}$$

Similarly, the inequality $C(x,N) < C(x,N-1)$ implies

$$\sum_{k=0}^{N} \int_0^x [\gamma_N(t) - \gamma_k(t)]dt < \frac{C_{re} - C_{pm}}{C_{mr}}. \tag{4.2}$$

Let $L(x,N) = \sum_{k=0}^{N} \int_0^x [\gamma_{N+1}(t) - \gamma_k(t)]dt$. Then $L(x,0) = 0$ and by combining (4.1) and (4.2), we have the inequalities

$$L(x,N) \geq \frac{C_{re} - C_{pm}}{C_{mr}} \quad \text{and} \quad L(x,N-1) < \frac{C_{re} - C_{pm}}{C_{mr}}.$$

In order to prove the existence and uniqueness of the optimal N, we need to prove that $L(x,N)$ is increasing in N and goes to infinity. Instead of showing it directly, we use the consequence discussed by Nakagawa(1986) to find the optimal number of PM's, N^*, which minimizes the expected cost rate per unit time.

Suppose that the period x is known. Nakagawa(1986) shows that if $\gamma_k(t)$ is increasing in k and $\gamma_N(t)$ goes to infinity as N goes to infinity then there exists a finite and unique N^* which minimizes $C(x,N)$ for a given x.

Theorem 4.1. Let $\gamma_k(t) = h(t + (k-1)x) - p_{k-1} h((k-1)x)$. Then

(i) For any positive integer k, $\gamma_k(t) \leq \gamma_{k+1}(t)$ for all $0 < t < x$.

(ii) $\lim_{N \to \infty} \gamma_N(t) = \infty$ for all $t > 0$

Proof : For all $0 < t < x$,

$\gamma_{k+1}(t) - \gamma_k(t) = h(t + kx) - p_k h(kx) - [h(t + (k-1)x) - p_{k-1} h((k-1)x)]$

$= [h(t + kx) - h(t + (k-1)x)] - [p_k h(kx) - p_{k-1} h(k-1)x)] \geq 0$.

The last inequality holds since $h(t)$ is convex and strictly increasing in $t > 0$.

And $\gamma_N(t) = h(t + (N-1)x) - p_{N-1} h((N-1)x) > (1 - p_{N-1}) h((N-1)x)$ becomes infinity as $N \to \infty$ since $h(t)$ goes to infinity as t goes to infinity.

4.2. Number of PM N is known

In this case, we assume that the number of PM's conducted before replacement, N, is known. In order to find the optimal PM period x^* for a given N which minimizes $C(x,N)$, given in (3.2), we differentiate $C(x,N)$ with respect to x and set it equal to 0. Then, we have

$$\sum_{k=0}^{N}\left[x\int_{0}^{x}\frac{d\gamma_k(t)}{dx}dt + x\gamma_k(x) - \int_{0}^{x}\gamma_k(t)dt\right] = \frac{(N-1)C_{pm}+C_{re}}{C_{mr}}, \tag{4.3}$$

where $\gamma_k(t) = h(t+(k-1)x) - p_{k-1}h((k-1)x)$.

Let $g(x)$ and C denote the left-hand side and the right-hand side of (4.3), respectively. Then

$$\frac{d}{dx}C(x,N) \overset{sign}{=} g(x) - C,$$

where $g(0)=0$ and $C>0$. To prove the main result, we first need to prove that $g(x)$ is an increasing function of $x \geq 0$.

Nakagawa(1986) also shows that if $\gamma_k(t)$ is differentiable and strictly increasing to infinity as t goes to infinity then there exists a finite and unique x^* which minimizes $C(x,N)$ for a given N.

Theorem 4.2. Let $\gamma_k(t) = h(t+(k-1)x) - p_{k-1}h((k-1)x)$. Then $\gamma_k(t)$ is differentiable and strictly increasing to infinity as t goes to infinity

Proof: Since $h(t)$ is differentiable, it is clear that $\gamma_k(t)$ is differentiable. And since $h(t)$ is strictly increasing to infinity, $\gamma_k(t)$ is strictly increasing to infinity as t goes to infinity.

5. NUMERICAL EXAMPLE

Suppose that the failure time distribution F is Weibull distribution with a scale parameter λ and a shape parameter β. The failure rate is $h(t) = \beta\lambda^{\beta-1}t^{\beta-1}$ for $\beta>0$ and $t>0$. We assume that λ=1.

As an improvement factor, we take the following age dependent function of k.

$$p_k = e^{-2k} \quad \text{for } k = 1,2,\text{K}, N$$

The cost for minimal repair (C_{mr}) and cost for PM (C_{pm}) are assumed to be 1 and 1.5, respectively.

5.1. Optimal PM schedule when PM period x is known.

Straightforward computation yields

$L(x,N)$

$$= x^{\beta}\left\{N[(N+1)^{\beta} - N^{\beta} - p_N\beta N^{\beta-1}] - \sum_{k=1}^{N}[k^{\beta} - (k-1)^{\beta} - p_{k-1}\beta(k-1)^{\beta-1}]\right\}. \tag{5.1}$$

The value of N^* is obtained by solving the following two inequalities simultaneously.

$$L(x,N) \geq \frac{C_{re} - C_{pm}}{C_{mr}} \text{ and } L(x,N-1) < \frac{C_{re} - C_{pm}}{C_{mr}}. \tag{5.2}$$

Table 5.1 and 5.2 present the values of N^* for various combinations of x and C_{re} when the failure time distribution is Weibull distribution with $\beta = 2.0$ and $\beta = 2.5$, respectively. As for C_{re} and x, we take $C_{re} = 3.0$ to 15.0(2) so that the ratio $(C_{re} - C_{pm})/C_{mr}$ varies 2 to 10(2) and take $x = 0.1$ to 0.9(0.2). It is apparent from both Table 5.1 and 5.2 that the value of N^* increases as the cost for replacement gets higher and the PM period gets shorter. And it is also noted that the values of N^* for the case of $\beta = 2.5$ are smaller than the values of N^* for the case of $\beta = 2.0$. It is quiet natural since the wear-out speed for the case of $\beta = 2.5$ si faster than that of $\beta = 2.0$.

Table 5.1.Values of N^*(listed in parenthesis) and corresponding expected cost rate $C(x, N^*)$ for given x when the failure time distribution is Weibull distribution with $\beta = 2.0$.

Cre	x				
	0.1	0.3	0.5	0.7	0.9
3	12	4	3	2	2
	17.4469	7.4231	5.4323	4.5196	4.1782
5	19	6	4	3	2
	18.7402	8.7263	6.7051	5.8293	5.2893
7	23	8	5	3	3
	19.6897	9.6781	7.6639	6.7817	6.3005
9	27	9	5	4	3
	20.4764	10.4657	8.4639	7.5586	7.0413
11	31	10	6	4	3
	21.1633	11.1558	9.1365	8.2729	7.7820
13	34	11	7	5	4
	21.7812	11.7750	9.7599	8.8780	8.3804
15	37	12	7	5	4
	22.3477	12.3409	10.3313	9.4494	8.9359

Table 5.2. Values of N^* (listed in parenthesis) and corresponding expected cost rate $C(x, N^*)$ for given x when the failure time distribution is Weibull distribution with $\beta = 2.5$.

Cre	x				
	0.1	0.3	0.5	0.7	0.9
3	10	3	2	2	1
	17.4984	7.4949	5.4402	4.7717	4.1871
5	14	5	3	2	2
	19.1554	9.1538	7.1153	6.2003	5.8816
7	17	6	3	2	2
	20.4509	10.4566	8.4486	7.6289	6.9927
9	19	6	4	3	2
	21.5655	11.5677	9.5342	8.6661	8.1038
11	21	7	4	3	2
	22.5662	12.5551	10.5342	9.6185	9.2150
13	23	8	5	3	3
	23.4874	13.4993	11.5170	10.5709	10.2293
15	24	8	5	3	3
	24.3424	14.3326	12.3170	11.5233	10.9701

5.2. Optimal PM schedule when number of PM N is known.

When $p_k = e^{-2k}$ for $k = 1,2,\Lambda, N$, by solving the equation (4.3) for x, we obtain

$$x^* = \left\{ \frac{(N-1)C_{pm} + C_{re}}{(\beta-1)C_{mr} \sum_{k=1}^{N} \left[k^\beta - (k-1)^\beta - p_{k-1}\beta(k-1)^{\beta-1} \right]} \right\}^{\frac{1}{\beta}}. \quad (5.3)$$

The values of x^* and its corresponding expected cost rate $C(x^*, N)$ are listed for $N = 1$ to 30(5) and $C_{re} = 3.0$ to 15(2) in Table 5.3 and 5.4 when when the failure time distribution is Weibull distribution with $\beta = 2.0$ and $\beta = 2.5$, respectively. Both Table 5.3 and 5.4 show that as N increases, the value of x^* gets shorter for fixed C_{re}, while the value of x^* increases when the replacement of the system costs higher. It suggests that when the system is more expensive to replace and the number of PM's required before replacement is fixed, the PM should be done less frequently to reduce the cost of conducting the periodic PM. And it is also noted that the optimal periods for the case of $\beta = 2.5$ are shorter than the values for the case of $\beta = 2.0$. It is due to the different wear-out speed.

Table 5.3. Values of x^* (listed in upper cell) and corresponding expected cost rate $C(x, N^*)$ (listed in lower cell) for given N when the failure time distribution is Weibull distribution with $\beta = 2.0$.

N	Cre						
	3	5	7	9	11	13	15
1	1.7321	2.2361	2.6458	3.0000	3.3166	3.6056	3.8730
	3.4641	4.4721	5.2915	6.0000	6.6332	7.2111	7.7460
5	0.6044	0.6682	0.7264	0.7803	0.8306	0.8782	0.9232
	5.9565	6.5851	7.1588	7.6898	8.1864	8.6545	9.0986
10	0.4069	0.4309	0.4536	0.4752	0.4959	0.5157	0.5348
	8.1093	8.5867	9.0390	9.4696	9.8816	10.2770	10.6577
15	0.3269	0.3402	0.3531	0.3654	0.3774	0.3890	0.4003
	9.7901	10.1898	10.5745	10.9456	11.3046	11.6525	11.9903
20	0.2808	0.2895	0.2980	0.3063	0.3144	0.3222	0.3299
	11.2199	11.5706	11.9110	12.2419	12.5641	12.8783	13.1849
25	0.2499	0.2562	0.2624	0.2684	0.2743	0.2801	0.2857
	12.4864	12.8025	13.1111	13.4125	13.7073	13.9959	14.2787
30	0.2273	0.2322	0.2369	0.2416	0.2461	0.2506	0.2550
	13.6354	13.9256	14.2098	14.4885	14.7619	15.0303	15.2940

Table 5.4. Values of x^* (listed in upper cell) and corresponding expected cost rate $C(x, N^*)$ (listed in lower cell) for given N when the failure time distribution is Weibull distribution with $\beta = 2.5$.

N	Cre						
	3	5	7	9	11	13	15
1	1.3195	1.6186	1.8518	2.0477	2.2188	2.3721	2.5119
	3.7893	5.1483	6.3001	7.3254	8.2627	9.1338	9.9527
5	0.4110	0.4454	0.4762	0.5042	0.5301	0.5542	0.5769
	7.2988	8.2326	9.1006	9.9165	10.6899	11.4276	12.1349
10	0.2611	0.2733	0.2848	0.2956	0.3059	0.3156	0.3249
	10.5316	11.2800	11.9966	12.6857	13.3507	13.9944	14.6188
15	0.2021	0.2087	0.2150	0.2210	0.2268	0.2324	0.2377
	13.1920	13.8410	14.4703	15.0819	15.6774	16.2582	16.8254
20	0.1690	0.1732	0.1773	0.1812	0.1850	0.1887	0.1923
	15.5318	16.1162	16.6868	17.2447	17.7907	18.3259	18.8508
25	0.1473	0.1502	0.1531	0.1559	0.1587	0.1613	0.1639
	17.6561	18.1940	18.7214	19.2391	19.7477	20.2477	20.7395
30	0.1317	0.1339	0.1361	0.1382	0.1403	0.1423	0.1443
	19.6218	20.1239	20.6178	21.1039	21.5827	22.0545	22.5196

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