International Journal of Reliability and Applications
Vol. 9, No. 1, pp. 95-112, 2008

Reliability Equivalence of a Non-identical Components Parallel System

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Abstract. This paper gives the reliability equivalence factors of a parallel system with n independent and non-identical components. It is assumed here that, the failure rates of the system's components are constants. We used three different methods to improve the system given. Two reliability characteristics (the mean time to failure and the reliability function) are used to perform the system improvement. For this purpose, the reliability functions and the mean times to failures of the original and improved systems are obtained. The results given in this paper generalize the results given in the literatures by setting $n = 1, 2$. An illustrative numerical example is presented to compare the different reliability factors obtained.

Key Words : *Exponential distribution, coherent system, mean time to failure.*

1. INTRODUCTION

In case of no repair, equivalent of different designs of the same system with respect to a reliability characteristic such as mean time to failure or survival function is needed. The concept of reliability equivalence has been introduced by Rade (1989). Rade (1990, 1991, 1993a,b) and Sarhan (2000, 2002) have applied this concept on various systems.
Rade (1993) has considered three different methods to improve the quality of a system. He suggested that the reliability function of the system can be improved by: (*i*) improving the quality of one or several components by decreasing their failure rates, (*ii*) adding a hot

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component to the system, (*iii*) adding a cold redundant component to the system. Sarhan (2000) has considered more general methods to improve the quality of a system. He suggested the following four methods,

1) Improving the quality of some components by reducing their failure rates by a factor ρ, $0 < \rho < 1$.
2) Hot duplications method.
3) Cold duplications method.
4) Cold duplication with imperfect switch method.

Rade (1993a,b) and Sarhan (2000) used the survival function as a performance measure of the system reliability to compare different system designs. Rade (1993a,b) has obtained the reliability equivalence factors for a single component and for two independent and identical components series and parallel systems. Sarhan (2002) has obtained the reliability equivalence factors of n independent and non-identical series system. Sarhan (2002) used the survival function and mean time to failure as characteristics to compare different system designs. He has derived two different types of reliability equivalence factors of a basic series/parallel system. Sarhan (2005) has obtained the reliability equivalence factors of a parallel system with n independent and identically components. He assumed that the failure rates of the components to be constant. In this paper, we derive the reliability equivalence factors of a parallel system with n independent and non-identically components. We assume that the lives of the system components are exponentially distributed with different parameters. The survival function and mean time to failure are used as performance measures to compare the design of original system and that for the improved designs. The results presented here generalize the results given in Sarhan (2005).

We need the following definition.

Definition 1.1. [Sarhan 2002] A reliability equivalence factor of a system defined as that factor by which a characteristic of components of a system design has to be multiplied in order to reach equality of a characteristic of this design and a different design.

The paper is organized as follows. Section 2 gives the description of the original system studied here. The reliability functions and mean time to failures of the original and the improved systems are presented in Section 3. Also, a theorem that establishes a comparison among the mean time to failures of systems that improved according to the methods used is given in Section 3. In Section 4, we obtain the reliability equivalence factors of the system. The α −fractiles of the original and improved systems are obtained in Section 5. An illustrative numerical example is given in Section 6.

2. THE ORIGINAL SYSTEM

The system considered here consists of n independent but not-identical components connected in parallel. Scheme 1 shows the configuration of the system. It is assumed that the lifetime of component i, $i = 1,2,\Lambda\ ,n$, is exponentially distributed with parameter λ_i. That is, the failure rate of the component i is λ_i and its reliability function is

$$R_i(t) = \exp\{-\lambda_i\, t\},\ \lambda_i > 0; t \geq 0. \tag{2.1}$$

The system can be improved according to one of the following three different methods:

1. Reduction method: in this method, we reduce the failure rates of set A components, $A \subseteq \{1,2,\Lambda\ ,n\}$, by the same factor, say ρ, $0 < \rho < 1$.
2. Hot duplication method: it is assumed, in this method that each component belongs to the set B components, $B \subseteq \{1,2,\Lambda\ ,n\}$, is duplicated by hot redundant standby component.
3. Cold duplication method: it is assumed in this method that each component belongs to the set B components, $B \subseteq \{1,2,\Lambda\ ,n\}$, is duplicated by cold redundant standby component.

Then we will make equivalence of the improved system obtained by reduction method to: (1) the system improved by hot duplication method; (2) the system improved by cold duplication method. The first gives the hot reliability equivalence and the second provides the cold reliability equivalence. We will use both the reliability function and mean time to failure to make the equivalence. For this purpose, we give the reliability function and the mean time to failure of the original system.

The system reliability function is

$$\begin{aligned} R(t) &= 1 - \prod_{i=1}^{n} R_i(t) \\ &= 1 - \prod_{i=1}^{n} \left(1 - \exp\{-\lambda_i\, t\}\right) \end{aligned} \tag{2.1}$$

The following result is needed. One can verify that

$$\prod_{i=1}^{n}\left(1-\exp\{-\lambda_i t\}\right)=\sum_{l=0}^{n}(-1)^l \Upsilon_l(t) \tag{2.2}$$

where

$$\Upsilon_l(t)=\sum_{1\le j_1<j_2<\Lambda<j_l\le n}\exp\left\{-\sum_{i=1}^{l}\lambda_{j_i}t\right\},\quad \Upsilon_0(t)=1,$$

Using (2.1) and (2.2), we can write the reliability function $R(t)$ as in the following form

$$R(t)=\sum_{l=1}^{n}(-1)^{l+1}\Upsilon_l(t) \tag{2.3}$$

From (2.3) and according to the well know relation between the reliability function and the mean time to failure, one can get the system mean time to failure as

$$MTTF=\sum_{l=1}^{n}(-1)^{l+1}\sum_{1\le j_1<j_2<\Lambda<j_l\le n}\frac{1}{\sum_{i=1}^{l}\lambda_{j_i}}. \tag{2.4}$$

3. The improved systems

In this section, we present the improved systems which can be derived according to the three methods mentioned above.

Reduction method.

It is assumed in this method that the system can be improved by reducing the failure rates of the set A components, $A\subseteq\{1,2,\Lambda\ ,n\}$, by the factor ρ, $0<\rho<1$. Let $R_{A,\rho}(t)$ be the reliability function of the system improved according to the reduction method. One can derive $R_{A,\rho}(t)$ as in the following form

$$R_{A,\rho}(t)=1-\prod_{i\in A}\left(1-\exp\{-\rho\lambda_i t\}\right)\prod_{i\in \bar{A}}\left(1-\exp\{-\lambda_i t\}\right), \tag{3.1}$$

Here $\bar{A}=\{1,2,\Lambda\ ,n\}\setminus A$, the complementary set of A.

Applying (2.2) on $\prod_{i\in A}\left(1-\exp\{-\rho\lambda_i t\}\right)$ and $\prod_{i\in \bar{A}}\left(1-\exp\{-\lambda_i t\}\right)$, we get

$$\prod_{i\in A}\left(1-\exp\{-\rho\lambda_i t\}\right)=\sum_{l=0}^{|A|}(-1)^l \Upsilon_l^{(A)}(\rho t), \tag{3.2}$$

and

$$\prod_{i \in \bar{A}} \left(1 - \exp\{-\lambda_i t\}\right) = \sum_{l=0}^{|\bar{A}|} (-1)^l \, \Upsilon_l^{(\bar{A})}(t), \tag{3.3}$$

where $\Upsilon_0^{(C)}(t) = 1$ and

$$\Upsilon_l^{(C)}(t) = \sum_{j_1 < j_2 < \Lambda < j_l :\, j_1, j_2, \Lambda, j_l \in C} \exp\left\{-\sum_{i=1}^{l} \lambda_{j_i} t\right\} \; C \subseteq \{1, 2, \Lambda, n\}, \tag{3.4}$$

Therefore

$$\prod_{i \in A} \left(1 - \exp\{-\rho \lambda_i t\}\right) \prod_{i \in \bar{A}} \left(1 - \exp\{-\lambda_i t\}\right) = 1 + \sum_{l=1}^{|A|} (-1)^l \, \Upsilon_l^{(A)}(\rho t) + \sum_{l=1}^{|\bar{A}|} (-1)^l \, \Upsilon_l^{(\bar{A})}(t)$$

$$+ \sum_{l=}^{|A|} \sum_{k=1}^{|\bar{A}|} (-1)^{l+k} \, \Upsilon_l^{(A)}(\rho t) \, \Upsilon_k^{(\bar{A})}(t)$$

Thus, $R_{A,\rho}(t)$ can be written as in the following form

$$R_{A,\rho}(t) = \sum_{l=1}^{|A|} (-1)^{l+1} \, \Upsilon_l^{(A)}(\rho t) + \sum_{l=1}^{|\bar{A}|} (-1)^{l+1} \, \Upsilon_l^{(\bar{A})}(t) + \sum_{l=}^{|A|} \sum_{k=1}^{|\bar{A}|} (-1)^{l+k+1} \, \Upsilon_l^{(A)}(\rho t) \, \Upsilon_k^{(\bar{A})}(t). \tag{3.5}$$

Let $MTTF_{A,\rho}$ be the mean time to failure of the system improved by improving the set A components according to the reduction method. From (3.5), we may write

$$MTTF_{A,\rho} = \sum_{l=1}^{|A|} (-1)^{l+1} \int_0^{\infty} \Upsilon_l^{(A)}(\rho t)\, dt + \sum_{l=1}^{|\bar{A}|} (-1)^{l+1} \int_0^{\infty} \Upsilon_l^{(\bar{A})}(t)\, dt$$

$$+ \sum_{l=}^{|A|} \sum_{k=1}^{|\bar{A}|} (-1)^{l+k+1} \int_0^{\infty} \Upsilon_l^{(A)}(\rho t) \, \Upsilon_k^{(\bar{A})}(t)\, dt$$

But

$$\int_0^{\infty} \Upsilon_l^{(A)}(\rho t)\, dt = \sum_{j_1 < j_2 < \Lambda < j_l :\, j_1, j_2, \Lambda, j_l \in A} \frac{1}{\rho \sum_{i=1}^{l} \lambda_{j_i}},$$

$$\int_0^\infty \mathrm{Y}_k^{(\bar{A})}(t)\,dt = \sum_{s_1<s_2<\Lambda<s_k:\, s_1,s_2,\Lambda,s_k\in\bar{A}} \frac{1}{\sum_{i=1}^{k}\lambda_{s_i}},$$

$$\int_0^\infty \mathrm{Y}_l^{(A)}(\rho t)\,\mathrm{Y}_k^{(\bar{A})}(t)\,dt = \frac{1}{\rho}\sum_{j_1<j_2<\Lambda<j_l:\, j_1,j_2,\Lambda,j_l\in A}\ \sum_{s_1<s_2<\Lambda<s_k:\, s_1,s_2,\Lambda,s_k\in\bar{A}} \frac{1}{\sum_{i=1}^{l}\lambda_{j_i}\sum_{i=1}^{k}\lambda_{s_i}},$$

Therefore, $MTTF_{A,\rho}$ becomes

$$MTTF_{A,\rho} = \frac{1}{\rho}\sum_{l=1}^{|A|}(-1)^{l+1}\sum_{j_1<j_2<\Lambda<j_l:\, j_1,j_2,\Lambda,j_l\in A}\frac{1}{\sum_{i=1}^{l}\lambda_{j_i}}$$

$$+\sum_{l=1}^{|\bar{A}|}(-1)^{l+1}\sum_{s_1<s_2<\Lambda<s_k:\, s_1,s_2,\Lambda,s_k\in\bar{A}}\frac{1}{\sum_{i=1}^{k}\lambda_{s_i}}$$

$$+\frac{1}{\rho}\sum_{l=}^{|A|}\sum_{k=1}^{|\bar{A}|}(-1)^{l+k+1}\sum_{j_1<j_2<\Lambda<j_l:\, j_1,j_2,\Lambda,j_l\in A}\ \sum_{s_1<s_2<\Lambda<s_k:\, s_1,s_2,\Lambda,s_k\in\bar{A}}\frac{1}{\sum_{i=1}^{l}\lambda_{j_i}\sum_{i=1}^{k}\lambda_{s_i}}. \tag{3.6}$$

Hot duplication method.

It is assumed in this method that the system can be improved by improving the set B components, $B\subseteq\{1,2,\Lambda\ ,n\}$, according to hot duplication method. The component i is said to be improved by the hot duplication method if it is duplicated with another identical component (that is, it is connected in parallel with an identical component). Let $R_B^H(t)$ be the reliability function of the system improved according to the hot duplication method by improving the set B components. One can obtain $R_B^H(t)$ as follows

$$R_B^H(t) = 1-\prod_{i\in B}\left[1-R_i^H(t)\right]\prod_{i\in\bar{B}}\left[1-R_i(t)\right], \tag{3.7}$$

where $R_i^H(t) = \left(2-e^{-\lambda_i t}\right)e^{-\lambda_i t},\ i\in B$, $R_i(t)=e^{-\lambda_i t},\ i\in\bar{B}$ and $\bar{B}=\{1,2,\Lambda\ n\}\setminus B$. The function $R_B^H(t)$ can be written as

$$R_B^H(t) = 1-\exp\{-\Lambda_B\,t\}\prod_{i\in B}\left(2-\exp\{-\lambda_i\,t\}\right)\prod_{i\in\bar{B}}\left(1-\exp\{-\lambda_i\,t\}\right), \tag{3.8}$$

where $\Lambda_B=\sum_{i\in B}\lambda_i$. Similar to (3.3), we can express $\prod_{i\in B}\left(2-\exp\{-\lambda_i\,t\}\right)$ as in the following form

$$\prod_{i\in B}\left(2-\exp\{-\lambda_i t\}\right)=\sum_{l=0}^{|B|}(-1)^l 2^{|B|-l}\Upsilon_l^{(B)}(t), \tag{3.9}$$

where $\Upsilon_l^{(B)}(t)$ are given in (3.4). From (3.3) and (3.9), we have

$$\prod_{i\in B}\left(2-\exp\{-\lambda_i t\}\right)\prod_{i\in \bar{B}}\left(1-\exp\{-\lambda_i t\}\right)=1+\sum_{l=1}^{|B|}(-1)^l 2^{|B|-l}\Upsilon_l^{(B)}(t)+\sum_{l=1}^{|\bar{B}|}(-1)^l\Upsilon_l^{(\bar{B})}(t)$$
$$+\sum_{l=1}^{|B|}\sum_{k=1}^{|\bar{B}|}(-1)^{l+k}2^{|B|-l}\Upsilon_l^{(B)}(t)\,\Upsilon_k^{(\bar{B})}(t)$$

Thus, $R_B^H(t)$ can be written as

$$R_B^H(t)=1-\exp\{-\Lambda_B t\}+\exp\{-\Lambda_B t\}\left\{\sum_{l=1}^{|B|}(-1)^{l+1}2^{|B|-l}\Upsilon_l^{(B)}(t)+\sum_{l=1}^{|\bar{B}|}(-1)^{l+1}\Upsilon_l^{(\bar{B})}(t)\right.$$
$$\left.+\sum_{l=}^{|B|}\sum_{k=1}^{|\bar{B}|}(-1)^{l+k+1}2^{|B|-l}\Upsilon_l^{(B)}(t)\,\Upsilon_k^{(\bar{B})}(t)\right\}. \tag{3.10}$$

Let $MTTF_B^H$ be the mean time to failure of the system improved by improving the set B components according to the hot duplication method. From (3.10), we have

$$MTTF_B^H=\int_0^\infty\left[1-e^{-\Lambda_B t}\right]dt+\sum_{l=1}^{|B|}(-1)^{l+1}2^{|B|-l}\int_0^\infty e^{-\Lambda_B t}\Upsilon_l^{(B)}(t)dt$$
$$+\sum_{l=1}^{|\bar{B}|}(-1)^{l+1}\int_0^\infty e^{-\Lambda_B t}\Upsilon_l^{(\bar{B})}(t)dt$$
$$+\sum_{l=}^{|B|}\sum_{k=1}^{|\bar{B}|}(-1)^{l+k+1}2^{|B|-l}\int_0^\infty e^{-\Lambda_B t}\Upsilon_l^{(B)}(t)\,\Upsilon_k^{(\bar{B})}(t)dt$$

Solving the integrals above gives

$$MTTF_B^H=\frac{1}{\Lambda_B}+\sum_{l=1}^{|B|}(-1)^{l+1}2^{|B|-l}\sum_{j_1<j_2<\Lambda<j_l:\ j_1,j_2,\Lambda,j_l\in B}\frac{1}{\Lambda_B+\sum_{i=1}^{l}\lambda_{j_i}}$$
$$+\sum_{l=1}^{|\bar{B}|}(-1)^{l+1}\sum_{s_1<s_2<\Lambda<s_k:\ s_1,s_2,\Lambda,s_k\in\bar{B}}\frac{1}{\Lambda_{\bar{B}}+\sum_{i=1}^{k}\lambda_{s_i}}$$

$$+\sum_{l=1}^{|B|}\sum_{k=1}^{|\bar{B}|}\sum_{j_1<j_2<\Lambda<j_l:\, j_1,j_2,\Lambda,j_l\in B}\;\sum_{s_1<s_2<\Lambda<s_k:\, s_1,s_2,\Lambda,s_k\in\bar{B}}\frac{(-1)^{l+k+1}2^{|\bar{B}|-l}}{\Lambda_{\bar{B}}+\sum_{i=1}^{l}\lambda_{j_i}+\sum_{i=1}^{k}\lambda_{s_i}}\,. \tag{3.11}$$

Cold duplication method.

It is assumed in this method that the system can be improved by improving the set B components, $B\subseteq\{1,2,\Lambda,n\}$, according to cold duplication method. The component i is said to be improved by the cold duplication method if it is duplicated with another identical component via a perfect switch.. Let $R_B^C(t)$ be the reliability function of the system improved according to the cold duplication method by improving the set B components. The function $R_B^C(t)$ can be obtained as follows

$$R_B^C(t)=1-\prod_{i\in B}R_i^C(t)\prod_{i\in\bar{B}}R_i(t), \tag{3.12}$$

where $R_i^C(t)=(1+\lambda_i t)e^{-\lambda_i t}$, $i\in B$, see Billinton and Allan (1983). The function $R_B^C(t)$ can be written as

$$R_B^C(t)=1-\exp\{-\Lambda_B t\}\prod_{i\in B}(1+\lambda_i t)\prod_{i\in\bar{B}}(1-\exp\{-\lambda_i t\})\ , \tag{3.13}$$

where $\Lambda_B=\sum_{i\in B}\lambda_i$. The express $\prod_{i\in B}(1+\lambda_i t)$ can be written as in the following form

$$\prod_{i\in B}(1+\lambda_i t)=\sum_{l=0}^{|B|}a_l t^l,\quad a_l=\sum_{i_1<i_{21}<\Lambda<i_l\in B}\lambda_{i_1}\lambda_{i_2}\Lambda\,\lambda_{i_l},\ a_0=1, \tag{3.14}$$

Replacing $\bar{A}$ with $\bar{B}$ in (3.3) we get $\prod_{i\in\bar{B}}(1-\exp\{-\lambda_i t\})$. From (3.3) and (3.14), we have

$$\prod_{i\in B}(1+\lambda_j t)\prod_{i\in\bar{B}}(1-\exp\{-\lambda_i t\})=1+\sum_{l=1}^{|B|}a_l t^l+\sum_{l=1}^{|\bar{B}|}(-1)^l\Upsilon_l^{(\bar{B})}(t)$$

$$+\sum_{l=1}^{|B|}\sum_{k=1}^{|\bar{B}|}(-1)^k a_l t^l\,\Upsilon_k^{(\bar{B})}(t)$$

Thus, $R_B^C(t)$ can be written as

$$R_B^C(t) = 1 - \exp\{-\Lambda_B t\} - \exp\{-\Lambda_B t\}\left\{\sum_{l=1}^{|B|} a_l t^l + \sum_{l=1}^{|\bar{B}|} (-1)^l \Upsilon_l^{(\bar{B})}(t) + \sum_{l=1}^{|B|}\sum_{k=1}^{|\bar{B}|} (-1)^k a_l t^l \Upsilon_k^{(\bar{B})}(t)\right\}. \tag{3.15}$$

Let $MTTF_B^C$ be the mean time to failure of the system improved by improving the set B components according to the cold duplication method. From (3.15), we have

$$MTTF_B^C = \int_0^\infty \left[1 - e^{-\Lambda_B t}\right]dt - \sum_{l=1}^{|B|} a_l \int_0^\infty t^l e^{-\Lambda_B t} \Upsilon_l^{(B)}(t)dt + \sum_{l=1}^{|\bar{B}|}(-1)^{l+1}\int_0^\infty e^{-\Lambda_B t}\Upsilon_l^{(\bar{B})}(t)dt + \sum_{l=1}^{|B|}\sum_{k=1}^{|\bar{B}|}(-1)^{k+1} a_l \int_0^\infty t^l e^{-\Lambda_B t}\Upsilon_k^{(\bar{B})}(t)dt \tag{3.16}$$

But

$$\int_0^\infty t^l e^{-\Lambda_B t}\Upsilon_k^{(C)}(t)\,dt = \sum_{j_1<j_2<\Lambda<j_k\in C} \frac{\Gamma(l+1)}{\left[\Lambda_B + \sum_{i=1}^k \lambda_{j_i}\right]^{l+1}}, \quad C = B, \bar{B},$$

Substituting from integral into (3.16), gives

$$MTTF_B^C = \frac{1}{\Lambda_B} - \sum_{l=1}^{|B|} a_l \frac{\Gamma(l+1)}{[\Lambda_B]^{l+1}} - \sum_{l=1}^{|B|}\sum_{j_1<j_2<\Lambda<j_k\in\bar{B}} \frac{(-1)^k}{\Lambda_B + \sum_{i=1}^k \lambda_{j_i}} - \sum_{l=1}^{|B|}\sum_{k=1}^{|\bar{B}|}\sum_{j_1<j_2<\Lambda<j_k\in\bar{B}} \frac{(-1)^k a_l \Gamma(l+1)}{\left[\Lambda_B + \sum_{i=1}^k \lambda_{j_i}\right]^{l+1}}. \tag{3.17}$$

4. RELIABILITY EQUIVALENCE FACTORS

In this section, we derive two reliability equivalence factors: (1) the survival reliability equivalence factor (SREF); (2) the mean reliability equivalence factor (MREF). The following are the definitions of these two factors.

Definition 4.1 (SREF) [Sarhan (2002)] The hot (cold) SREF, say $\rho_{A,B}^D, D = H(C)$, is defined as that factor by which the failure rates of the set A components should be reduced in order to improve the system reliability to be as that reliability of the system improved by assuming hot (cold) duplication of the set B components.

Definition 4.2 (MREF) [Sarhan (2002)] The hot (cold) MREF, say $\zeta_{A,B}^{D}, D = H(C)$, is defined as that factor by which the failure rates of the set A components should be reduced in order to improve the system MTTF to be as that MTTF of the system improved by assuming hot (cold) duplication of the set B components.

Based on the definition (4.1), the hot (cold) SREF $\zeta_{A,B}^{D}, D = H(C)$ can be derived by solving the following equation, with respect to $\rho = \zeta_{A,B}^{D}$:

$$MTTF_B^D = MTTF_{A,\rho}\,,\ D = H(C), \alpha \in (0,1) \tag{4.1}$$

Using (4.1), when D=H, (3.5) and (3.10), we get the following system of two non-linear equations

$$\alpha = \sum_{l=1}^{|A|}(-1)^{l+1}\Upsilon_l^{(A)}(\rho t) + \sum_{l=1}^{|\bar{A}|}(-1)^{l+1}\Upsilon_l^{(\bar{A})}(t) + \sum_{l=}^{|A|}\sum_{k=1}^{|\bar{A}|}(-1)^{l+k+1}\Upsilon_l^{(A)}(\rho t)\Upsilon_k^{(\bar{A})}(t), \tag{4.2}$$

$$\alpha = 1 - \exp\{-\Lambda_B t\} + \exp\{-\Lambda_B t\}\left\{\sum_{l=1}^{|B|}(-1)^{l+1}2^{|B|-l}\Upsilon_l^{(B)}(t) + \sum_{l=1}^{|\bar{B}|}(-1)^{l+1}\Upsilon_l^{(\bar{B})}(t)\right.$$
$$\left. + \sum_{l=}^{|B|}\sum_{k=1}^{|\bar{B}|}(-1)^{l+k+1}2^{|B|-l}\Upsilon_l^{(B)}(t)\Upsilon_k^{(\bar{B})}(t)\right\}. \tag{4.3}$$

To get the hot SREF $\rho_{A,B}^{H}(\alpha)$, we have to solve the system of the non-linear equations (4.2) and (4.3). As it seems, this system has no analytical solution, therefore we have to use numerical technique to get $\rho_{A,B}^{H}(\alpha)$. For this purpose, we used the MathCad package.

Similarly, using (4.1), when D=C, (3.5) and (3.15), we get the following system of two non-linear equations

$$\alpha = \sum_{l=1}^{|A|}(-1)^{l+1}\Upsilon_l^{(A)}(\rho t) + \sum_{l=1}^{|\bar{A}|}(-1)^{l+1}\Upsilon_l^{(\bar{A})}(t) + \sum_{l=}^{|A|}\sum_{k=1}^{|\bar{A}|}(-1)^{l+k+1}\Upsilon_l^{(A)}(\rho t)\Upsilon_k^{(\bar{A})}(t), \tag{4.4}$$

$$\alpha = 1 - \exp\{-\Lambda_B t\} - \exp\{-\Lambda_B t\}\left\{\sum_{l=1}^{|B|}a_l t^l + \sum_{l=1}^{|\bar{B}|}(-1)^l\Upsilon_l^{(\bar{B})}(t)\right.$$
$$\left. + \sum_{l=1}^{|B|}\sum_{k=1}^{|\bar{B}|}(-1)^k a_l t^l\,\Upsilon_k^{(\bar{B})}(t)\right\}. \tag{4. 5}$$

Thus, to obtain the cold SREF $\rho^{C}_{A,B}(\alpha)$, we have to solve the system of the non-linear equations (4.4) and (4.5). As it seems, this system has no analytical solution, therefore we have to use numerical technique to get $\rho^{C}_{A,B}(\alpha)$. For this purpose, we used the MathCad package.

Following the definition (4.2), and using equations (3.6), (3.11) and (3.17), the hot (cold) MREF $\rho = \zeta^{D}_{A,B}, D = H(C)$ can be derived by solving the following equation:

$$MTTF_B^D = \frac{1}{\rho}\sum_{l=1}^{|A|}\sum_{j_1<j_2<\Lambda<j_l\in A}\left[\frac{(-1)^{l+1}}{\sum_{i=1}^{l}\lambda_{j_i}} + \sum_{k=1}^{|\bar{A}|}\sum_{s_1<s_2<\Lambda<s_k\in\bar{A}}\frac{(-1)^{l+k+1}}{\sum_{i=1}^{l}\lambda_{j_i}+\sum_{i=1}^{k}\lambda_{s_i}}\right]$$

$$+\sum_{l=1}^{|\bar{A}|}(-1)^{l+1}\sum_{s_1<s_2<\Lambda<s_k\in\bar{A}}\frac{1}{\sum_{i=1}^{k}\lambda_{s_i}}. \tag{4.6}$$

where $MTTF_B^H$ and $MTTF_B^C$ is given by (3.11) and (3.17), respectively. Solving equation (4.6), we get the $\zeta^{D}_{A,B}, D = H(C)$ as

$$\zeta^{D}_{A,B} = \frac{\sum_{l=1}^{|A|}\sum_{j_1<j_2<\Lambda<j_l\in A}\left[\frac{(-1)^{l+1}}{\sum_{i=1}^{l}\lambda_{j_i}} + \sum_{k=1}^{|\bar{A}|}\sum_{s_1<s_2<\Lambda<s_k\in\bar{A}}\frac{(-1)^{l+k+1}}{\sum_{i=1}^{l}\lambda_{j_i}+\sum_{i=1}^{k}\lambda_{s_i}}\right]}{MTTF_B^D + \sum_{l=1}^{|\bar{A}|}(-1)^{l}\sum_{s_1<s_2<\Lambda<s_k\in\bar{A}}\frac{1}{\sum_{i=1}^{k}\lambda_{s_i}}}. \tag{4.7}$$

5. THE α -FRACTILES

We discuss the α -fractiles of the original and improved systems. The following definition gives the α -fractile.

Definition 5.1 For a given $\alpha \in (0,1)$, the α -fractile of a system with the reliability function R(t) is the solution of the following equation with respect to (w.r.t.) $L = L(\alpha)$

$$R\left(\frac{L(\alpha)}{\Lambda}\right) = \alpha, \tag{5.1}$$

where $\Lambda = \sum_{i=1}^{n}\lambda_i$.

Using relations (2.3) and (5.1), we can get the α -fractile of the original system by solving the following non-linear equation w.r.t. $L = L(\alpha)$

$$\alpha = \sum_{l=1}^{n} \sum_{1 \le j_1 < j_2 < \Lambda < j_l \le n} (-1)^{l+1} \exp\left\{ -\sum_{i=1}^{l} \lambda_{j_i} \frac{L(\alpha)}{\Lambda} \right\}. \tag{5.2}$$

Similarly, we can get the α -fractiles of the systems improved according to the hot (cold) duplication method, we have to solve similar equation to the equation (5.1) when the reliability function of the improved system replaced the function R. That is, the α -fractile of the system improved by improving the set B components according to the HDM, $L = L_B^H(\alpha)$, is the solution of the following equation

$$\alpha = 1 - \exp\left\{-\tfrac{\Lambda_B}{\Lambda} L\right\} + \exp\left\{-\tfrac{\Lambda_B}{\Lambda} L\right\} \left\{ \sum_{l=1}^{|B|} (-1)^{l+1} 2^{|B|-l} \Upsilon_l^{(B)}\left(\tfrac{L}{\Lambda}\right) + \sum_{l=1}^{|\bar{B}|} (-1)^{l+1} \Upsilon_l^{(\bar{B})}\left(\tfrac{L}{\Lambda}\right) + \sum_{l=}^{|B|} \sum_{k=1}^{|\bar{B}|} (-1)^{l+k+1} 2^{|B|-l} \Upsilon_l^{(B)}\left(\tfrac{L}{\Lambda}\right) \Upsilon_k^{(\bar{B})}\left(\tfrac{L}{\Lambda}\right) \right\}. \tag{5.3}$$

and the α -fractile of the system improved by improving the set B components according to the CDM, $L = L_B^C(\alpha)$, is the solution of the following equation

$$\alpha = 1 - \exp\left\{-\tfrac{\Lambda_B}{\Lambda} L\right\} - \exp\left\{-\tfrac{\Lambda_B}{\Lambda} L\right\} \left\{ \sum_{l=1}^{|B|} a_l \left(\tfrac{L}{\Lambda}\right)^l + \sum_{l=1}^{|\bar{B}|} (-1)^l \Upsilon_l^{(\bar{B})}\left(\tfrac{L}{\Lambda}\right) + \sum_{l=1}^{|B|} \sum_{k=1}^{|\bar{B}|} (-1)^k a_l \left(\tfrac{L}{\Lambda}\right)^l \Upsilon_k^{(\bar{B})}\left(\tfrac{L}{\Lambda}\right) \right\}. \tag{5.4}$$

The above equations do not have analytic solutions. Therefore, numerical technique methods should be used to get the fractiles.

6. AN ILLUSTRATIVE EXAMPLE

In this section, we assume a parallel system with three independent components. The lifetime of component i $(i = 1,2,3)$ is exponential with parameter λ_i, where $\lambda_1 = 0.5$, $\lambda_2 = 0.15$ and $\lambda_3 = 0.2$. The mean time to failure of this system is 9.019. We improved this system according to the methods mentioned in the section 3. Table 6.1 shows the mean time to failures of the system improved by the duplication methods with different possible sets. Figures 6.1 and 6.2 give $MTTF_{A,\rho}$ against ρ for different set A. It seems from figure 1 that reducing the failure rate of a single component with smaller failure rate gives a better system in the sense of having higher mean time to failure. From Figure 2, one can conclude that reducing the failure rates of two components, for which the sum of their failure rates is smaller than that of any other two components, produces a modified system with higher mean time to failure. Reducing the failure rate of the component with highest failure rate gives slightly improvement specially when $\rho > 0.2$.

The survival and mean reliability equivalence factors of this system are computed for different possible sets and listed respectively in Tables 6.2, 6.3, 6.3 and 6.5.

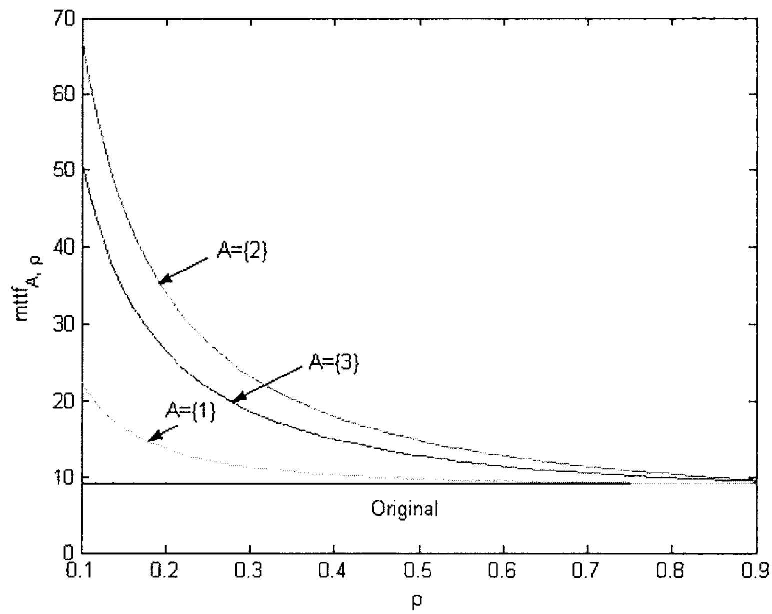

Figure 6.1. The behavior of $MTTF_{A,\rho}$ against ρ, when $|A| \geq 2$.

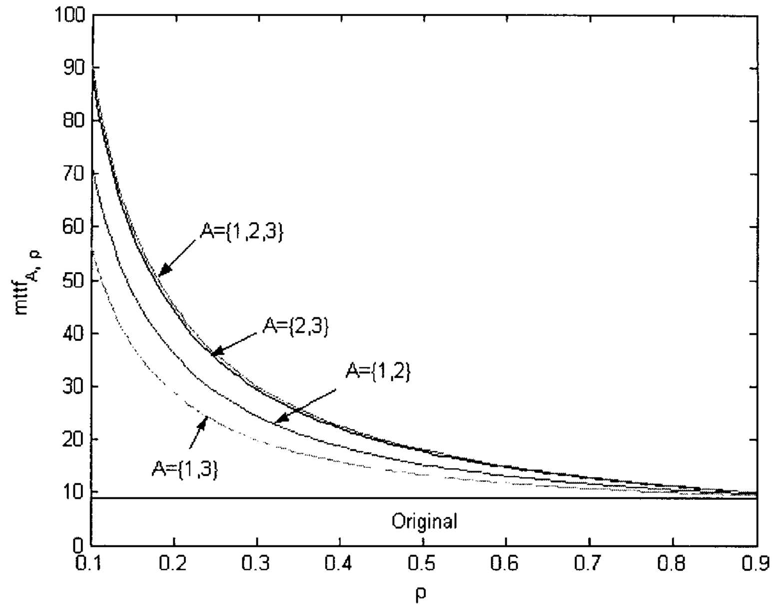

Figure 6.2. The behavior of $MTTF_{A,\rho}$ against ρ, when $|A| \geq 2$.

Table 6.1. The mean time to failures of the improved systems.

B	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$MTTF_B^H$	9.190	11.383	10.387	11.469	10.488	12.341	12.397
$MTTF_B^C$	9.507	14.314	12.255	14.500	12.486	16.248	16.349

Table 6.2. The hot SREF $\rho_{A,B}^H(\alpha)$ for different sets A and B.

α	A={1}						
	B						
	{1}	{2}	{3}	{1,2}	{1,3}	{2, 3}	{1,2,3}
0.1	0.914012	0.299936	0.399646	0.299876	0.399308	0.273988	0.273957
0.2	0.89401	0.299618	0.398476	0.29924	0.396998	0.261418	0.261254
0.3	0.871259	0.298842	0.39625	0.297713	0.392677	0.250105	0.249655
0.4	0.845981	0.297354	0.392639	0.294833	0.385873	0.238804	0.237864
0.5	0.818159	0.294789	0.387203	0.289985	0.375943	0.226789	0.225082
0.6	0.786415	0.290565	0.379206	0.28226	0.361916	0.213317	0.210475
0.7	0.747925	0.28365	0.367342	0.27014	0.342072	0.197294	0.192812
0.8	0.697246	0.271822	0.348788	0.250552	0.312814	0.176603	0.169746
0.9	0.618207	0.248361	0.31498	0.214783	0.263716	0.145327	0.134929
	A={1,2}						

	B						
	{1}	{2}	{3}	{1,2}	{1,3}	{2, 3}	{1,2,3}
0.1	0.999247	0.786228	0.891309	0.786144	0.891041	0.748245	0.748197
0.2	0.99593	0.738317	0.843279	0.737825	0.842019	0.68435	0.684099
0.3	0.989178	0.702367	0.804745	0.700951	0.801599	0.635732	0.635062
0.4	0.978518	0.672681	0.771351	0.669636	0.765293	0.594963	0.593586
0.5	0.963596	0.646605	0.740902	0.640973	0.730705	0.558583	0.55612
0.6	0.943781	0.622151	0.711552	0.612632	0.695699	0.524068	0.52
0.7	0.917581	0.597222	0.681125	0.581957	0.657555	0.488841	0.482412
0.8	0.88118	0.568542	0.645975	0.544504	0.611477	0.449002	0.438987
0.9	0.821999	0.527485	0.596237	0.488472	0.54426	0.394625	0.378551
α	*A*={1,2,3}						
	B						
	{1}	{2}	{3}	{1,2}	{1,3}	{2, 3}	{1,2,3}
0.1	0.995782	0.665987	0.795372	0.66594	0.794997	0.625483	0.625466
0.2	0.984417	0.642652	0.757539	0.64228	0.755882	0.585834	0.585713
0.3	0.968205	0.625638	0.730156	0.62436	0.726119	0.556009	0.555627
0.4	0.948596	0.611319	0.707326	0.60821	0.699632	0.530414	0.529514
0.5	0.926194	0.598348	0.686818	0.592104	0.673991	0.506878	0.505071
0.6	0.900855	0.585826	0.667213	0.574684	0.647494	0.483988	0.480689
0.7	0.871512	0.572726	0.647058	0.554279	0.61819	0.460278	0.454586
0.8	0.835247	0.557189	0.62397	0.527861	0.582647	0.433336	0.423736
0.9	0.782609	0.533753	0.591341	0.486805	0.531202	0.396496	0.379893

Table 6.3. The cold SREF $\rho^{C}_{A,B}(\alpha)$ for different sets *A* and *B*.

α	*A*={1}						
	B						
	{1}	{2}	{3}	{1,2}	{1,3}	{2, 3}	{1,2,3}
0.1	0.738427	0.264269	0.264269	0.194166	0.26401	0.177537	0.17753
0.2	0.695625	0.25305	0.25305	0.185709	0.25186	0.160884	0.160836
0.3	0.65984	0.245161	0.245161	0.179634	0.242159	0.14803	0.147871
0.4	0.627354	0.238476	0.238476	0.174082	0.232594	0.136669	0.136287
0.5	0.595501	0.23209	0.23209	0.168103	0.222078	0.125859	0.125084
0.6	0.562339	0.225293	0.225293	0.160855	0.209694	0.114964	0.113545
0.7	0.52546	0.217174	0.217174	0.151262	0.194256	0.103305	0.100878
0.8	0.480585	0.206004	0.206004	0.137442	0.173587	0.089766	0.085794
0.9	0.41582	0.186715	0.186715	0.114433	0.141756	0.071436	0.065103
α	*A*={1, 2}						
	B						
	{1}	{2}	{3}	{1,2}	{1,3}	{2, 3}	{1,2,3}
0.1	0.993568	0.596017	0.732689	0.59597	0.732258	0.556442	0.556427
0.2	0.975054	0.548871	0.671341	0.548499	0.669454	0.493656	0.493543
0.3	0.948315	0.516449	0.628283	0.515165	0.623692	0.449071	0.448715
0.4	0.916601	0.491608	0.594484	0.488471	0.585783	0.413578	0.412749

0.5	0.881664	0.471289	0.566145	0.46497	0.55175	0.383271	0.381626
0.6	0.84385	0.453527	0.540811	0.442229	0.518875	0.355683	0.352711
0.7	0.802039	0.436575	0.516271	0.417856	0.484494	0.328689	0.323612
0.8	0.75266	0.418022	0.489492	0.388332	0.444624	0.29951	0.291063
0.9	0.684147	0.391801	0.452995	0.344776	0.389179	0.261565	0.247301
α	A={1, 2, 3}						
	B						
	{1}	{2}	{3}	{1,2}	{1,3}	{2, 3}	{1,2,3}
0.1	0.995782	0.665987	0.795372	0.66594	0.794997	0.625483	0.625466
0.2	0.984417	0.642652	0.757539	0.64228	0.755882	0.585834	0.585713
0.3	0.968205	0.625638	0.730156	0.62436	0.726119	0.556009	0.555627
0.4	0.948596	0.611319	0.707326	0.60821	0.699632	0.530414	0.529514
0.5	0.926194	0.598348	0.686818	0.592104	0.673991	0.506878	0.505071
0.6	0.900855	0.585826	0.667213	0.574684	0.647494	0.483988	0.480689
0.7	0.871512	0.572726	0.647058	0.554279	0.61819	0.460278	0.454586
0.8	0.835247	0.557189	0.62397	0.527861	0.582647	0.433336	0.423736
0.9	0.782609	0.533753	0.591341	0.486805	0.531202	0.396496	0.379893

Table 6.4. The α -fractiles of the original system and systems improved by HDM.

α	B							
	Original	{1}	{3}	{1, 3}	{2}	{1, 2}	{2, 3}	{1,2,3}
0.1	14.9052	14.9123	16.1272	16.1302	17.7133	17.7177	18.4251	18.426
0.2	11.2377	11.266	12.5561	12.569	13.8026	13.8094	14.6159	14.62
0.3	9.0634	9.123	10.4051	10.4325	11.4354	11.4518	12.3032	12.3129
0.4	7.4891	7.5871	8.8188	8.8652	9.6848	9.7158	10.5819	10.6
0.5	6.2321	6.372	7.525	7.5947	8.2545	8.3049	9.1634	9.1931
0.6	5.1598	5.3434	6.3964	6.4932	7.0056	7.0808	7.9114	7.9564
0.7	4.1923	4.4193	5.3524	5.48	5.8506	5.9562	6.7379	6.8027
0.8	3.2611	3.529	4.3189	4.4814	4.7093	4.8515	5.5578	5.6488
0.9	2.2628	2.5643	3.1694	3.3713	3.4442	3.6313	4.2177	4.3449

Table 6.5. The α -fractiles of the original system and systems improved by CDM.

α	B							
	Original	{1}	{3}	{1, 3}	{2}	{1, 2}	{2, 3}	{1,2,3}
0.1	14.9052	14.9673	18.7411	18.7499	22.3819	22.3835	23.8309	23.8316
0.2	11.2377	11.4156	14.8345	14.8670	17.4863	17.4963	19.1812	19.1863
0.3	9.0634	9.3610	12.4128	12.4817	14.4867	14.5164	16.3009	16.3121
0.4	7.4891	7.8951	10.5882	10.7046	12.2510	12.3137	14.1196	14.1436
0.5	6.2321	6.7287	9.0739	9.2465	10.4155	10.5253	12.2950	12.3390
0.6	5.1598	5.7278	7.7335	7.9690	8.8078	8.9786	10.6612	10.7343
0.7	4.1923	4.8103	6.4790	6.7815	7.3199	7.5635	9.1083	9.2221
0.8	3.2611	3.9044	5.2264	5.5971	5.8528	6.1780	7.5257	7.6962
0.9	2.2628	2.8914	3.8265	4.2597	4.2395	4.6483	5.7069	5.9565

From the above tables, we can conclude that

1) Improving the set $B=\{1\}$ component according to the hot duplication method increases:

   1.1) The system mean time to failure from 9.019 to 9.190, see Table 6.1.

   1.2) The 0.1-fractile of the original system from 14.9052 to 14.9123, see table 7.

2) The same result can be reached by doing the following

   2.1) Reducing the failure rate of the same component by the factor $\rho^{H}_{\{1\},\{1\}}(0.1) = 0.914012$, see Table 6.2.

   2.2) Reducing the failure rates of the set $\{1,2\}$ components by the factor $\rho^{H}_{\{1,2\},\{1\}}(0.1) = 0.999247$, see Table 6.2.

3) Improving the set $B=\{1\}$ component according to the cold duplication method increases:

   3.1) The system mean time to failure from 9.019 to 9.507, see Table 6.1.

   3.2) The 0.1-fractile of the original system from 14.9052 to 14.9673, see table 8.

4) The same result can be reached by doing the following

   4.1) Reducing the failure rate of the same component by the factor $\rho^{C}_{\{1\},\{1\}}(0.1) = 0.738427$, see Table 6.3.

   4.2) Reducing the failure rates of the set $\{1,2\}$ components by the factor $\rho^{C}_{\{1,2\},\{1\}}(0.1) = 0.993568$, see Table 6.3..

5) In the same manner, one can read the reset of the results in tables 1 and 5.

6. CONCLUSION

In this paper we derived two reliability equivalence factors of a parallel system consisting of n independent and non-identical components. We assumed that the failure rates of the system's components are constants. We discussed three different methods to improve the system. We derived both the reliability function and the mean time to failure of each improved system. We illustrated the problem on a numerical example to explain how one can utilize the theoretical results obtained. The problem studied in this paper can be extended to many cases such as: when the components are not independent; the failure rates of the components are not constant with the independency assumption; non-constant failure rate and non-independency assumptions.

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