International Journal of Reliability and Applications
Vol. 9, No. 1, pp. 71-78, 2008

A Goodness of Fit Approach to Testing Exponential Better than Used (EBU) Life Distributions

S. E. Abu-Youssef*
Department of Statistics and O.R., King Saud University, Riyadh 11451, Saudi Arabia

Abstract. Based on the goodness of fit approach, a new test is presented for testing exponentiality versus exponential better (worse) than used (EBU (EWU)) class of life distributions. The new test is much simpler to compute, asymptotically normal, enjoys good power and performs better than previous tests in terms of power and Pitman asymptotic efficiencies for several alternatives.

Key Words : *EBU (EWE);Goodness of fit approach; Exponentiality; Efficiency; Asymptotic normality; Power; Monte Carlo method.*

1. INTRODUCTION

Notion of positive aging play an important role in reliability theory, survival analysis and other fields. therefore a multitude of classes life distributions have been introduced to describe several criteria of aging. Among the most well known families of life distributions are the classes of increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), decreasing mean residual life (DMRL), new better than used in expectation (NBUE) and harmonic new better than used in expectation (HNBUE). For some properties and interrelationships of these criteria we refer to Barlow and Proschan (1981) and Bryson and Siddiqui (1969).

The problem of testing exponentiality versus the classes (like IFR, IFRA, NBU, DMRL, NBUE, HNBUE and EBU) of life distributions has seen a good deal of literature for examples: Proschan and Pyke (1967), Ahmad (1994), Hollander and Proschan (1972 and 1975), Kanjo(1993) and Abu-Youssef (2002, 2003 and 2004). Let$X$ be a non negative continuous random variable with distribution function F, survival function $\overline{F}$. At age t, we define the random residual life by X_t with survival function $\overline{F}_t = \frac{\overline{F}(x+t)}{\overline{F}(t)}$.

*E-mail address: abuyousf@ksu.edu.sa

Definition 1.1. A life distribution F, with $F(0) = 0$, survival function $\overline{F}$ and finite mean μ is said to be EBU if

$$\overline{F}(x+t) \leq \overline{F}(t)e^{\frac{-x}{\mu}}, \qquad x, t > 0 \tag{1.1}$$

or

$$\overline{F}_t \leq e^{\frac{-x}{\mu}}, x, t > 0.$$

The dual class of life distributions that is EWU is defined by reversing the inequality sign of relation (1.1).

Note that, the above definition is motivated by comparing the life length X_t of a component of age t with another new component of life length Y which is exponential with the same mean as X, this leads to X is EBU if and only if $X_t \leq_{st} Y$ for all $t \geq 0$. El-Batal (2002) introduced the above class of life distribution. He investigated their relationships to other classes of life distributions, closure properties under reliability operations, moment inequality and heritage property under shock model. The implications among EBU, NBUE and HNBUE classes of life distribution are:

$$EBU \rightarrow NBUE \rightarrow HNBUE$$

The null distribution for EBU is the exponential. Thus we often encounter testing H_0 : A life distribution is exponential versus H_1 : It is EBU (EWE) and not exponential. This testing problem was discussed by Abu-Youssef (2003 and 2004) and Hendi et al (2005). However in contrast to goodness of fit problems, where the test statistics is based on a measure of departure from H_0 that depends on both H_0 and H_1. Most tests of life testing setting included those refereed above did not use the null distribution in devising the test statistics, which resulted in test statistics that are often difficult to work with require programming to evaluate.

Recently Ahmad et al (2001) and El-Bassiouny and El-Wasel (2003) used a new methodology for testing by incorporating both H_0 and H_1 in devising the test statistics for testing H_0 against the alternative the life distribution is IFR , NBUE , HNBUE and DMRL. They obtained very simple statistics that are not asymptotically equivalent in distribution and efficiency to classical procedure but also better in finite sample behaviors.

In the current work we use similar methodology to obtain a very simple statistics for testing H_0 against H_1. The thread that connects most work mentioned here is that a measure of departure from H_0, which is strictly positive under H_1 and is zero under H_0. Then, a sample version of this measure is used as test statistics and its properties are studied. In section 2, we propose a test statistic, based on the goodness of fit approach, for testing H_0 : F is exponential against H_1 : F is EBU (EWE) and not exponential. We then present Monte Carlo null distribution critical points for sample sizes $n = 5(1)40$. In section 3 we calculate the efficiency of the test statistic for some common alternatives and compared them to other procedures. In section 4 we give simulated values of the power estimates of the test. Finally in section 5 we consider an application in medical science, based on a set of real data from Susarla and Vanryzin (1978).

2. TESTING EBU(EWU) CLASS OF LIFE DISTRIBUTION

Let $X_1, X_2, \ldots, X_n$ represent a random sample from a population with distribution F. We wish to test the null hypothesis H_0 : $\overline{F}$ is exponential with mean μ against H_1 : $\overline{F}$ is EBU (EWU) and not exponential, that is

$$\overline{F}(x+t) \leq \overline{F}(t)e^{\frac{-x}{\mu}}, \qquad x, t > 0.$$

To test H_0 against H_1, Abu-Yousf (2004) defined the following measure of departure from H_0

$$\Delta_F = \int_0^\infty \int_0^\infty [\overline{F}(t)e^{\frac{-x}{\mu}} - \overline{F}(x+t)]dF(t)dF(x). \tag{2.1}$$

In a similar fashion , if F_0 denotes the exponential distribution, we can take in place of (2.1) the following measure of departure from H_0

$$\Delta_F = \int_0^\infty \int_0^\infty [\overline{F}(t)e^{\frac{-x}{\mu}} - \overline{F}(x+t)]dF_0(t)dF_0(x), \tag{2.2}$$

for testing the hypothesis that H_0: F is exponential versus H_0: F is EBU (EWU) and not exponential. Since this measure is scale invariant, then with out loss of generality we take $\mu = 1$ and thus $F_0(x) = 1 - e^{-x}$. In order to derive an expression for Δ_F we need the following lemma.

Lemma 2.1. Let T be a variable with distribution function F. Then

$$\Delta_F = -\frac{1}{2} + \frac{1}{2}Ee^{-T} + ETe^{-T}. \tag{2.3}$$

Proof. Note that Δ_F can be written as the following

$$\begin{aligned} \Delta_F &= \frac{1}{2}\int_0^\infty EI(T > t)e^{-t}dt - \int_0^\infty \int_0^\infty EI(T > x+t)e^{-x-t}dxdt \\ &= I1 - I2 \end{aligned} \tag{2.4}$$

where

$$I1 = \frac{1}{2}\int_0^\infty EI(T > t)e^{-t}dt$$

and

$$I2 = \int_0^\infty \int_0^\infty EI(T > x+t)e^{-x-t}dxdt.$$

Now

$$I1 = \frac{1}{2}\int_0^\infty EI(T > t)e^{-t}dt = \frac{1}{2}E\int_0^T e^{-t}dt = \frac{1}{2}E(1 - e^{-T}) \tag{2.5}$$

and

$$\begin{aligned} I2 &= \int_0^\infty \int_0^\infty EI(T > x + t)e^{-x-t}dxdt = E\int_0^T e^{-t}\int_0^{T-t} e^{-x}dxdt \\ &= E(1 - e^{-T} - Te^{-T}) \end{aligned} \tag{2.6}$$

Using (2.5)and (2.6) in (2.4), we get

$$\Delta_F = -\frac{1}{2} + E(\frac{1}{2}e^{-T} + Te^{-T}), \tag{2.7}$$

and the lemma is proved.

Note that under $H_0 : \Delta_F = 0$, while under $H_1 : \Delta_F > (<)0$. Based on a random sample $X_1, X_2, \ldots, X_n$ from distribution F, we wish to estimate Δ_F by $\hat{\Delta}_{F_n}$ and then $\hat{\Delta}_{F_n}$ is given by using (2.3) as the following:

$$\hat{\Delta}_{F_n} = \frac{1}{n}\sum_i(-\frac{1}{2} + \frac{1}{2}e^{-X_i} + X_i e^{-X_i}). \tag{2.8}$$

Theorem 2.1. As $n \to \infty, \sqrt{n}(\hat{\Delta}_{F_n} - \Delta_{F_n})$ is asymptotically normal with mean 0 and variance σ^2 is given in (2.9). Under $H_0, \Delta_F = 0$ and $\sigma_0^2 = \frac{1}{54}$.

Proof. Noting that $\hat{\Delta}_{F_n}$ is just an average, it is straightforward by using the central limit theorem the result follows. For the variance

$$\sigma^2 = E[-\frac{1}{2} + \frac{1}{2}e^{-x_1} + X_1 e^{-X_1}]^2. \tag{2.9}$$

Under $H_0, \Delta_F = 0$ and

$$\sigma_0^2 = \int_0^\infty [-\frac{1}{2} + \frac{1}{2}e^{-x_1} + X_1 e^{-X_1}]^2 e^{-x}dx = \frac{1}{54}.$$

3. MONTE CARLO NULL DISTRIBUTION CRITICAL POINTS FOR $\hat{\Delta}_{F_n}$ TEST

In practice, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst. we have simulated the upper percentile points for 95%, 98%, 99%. Table 3.1 gives these percentile points of statistic $\hat{\Delta}_{F_n}$ in (2.8) and the calculations are based on 5000 simulated samples of sizes $n = 5(1)40$. The percentile values change slowly as n increase.

Table 3.1 Critical Values of $\hat{\Delta}_{F_n}$

n	95%	98%	99%
5	.0809	.0890	.0929
6	.0764	.0841	.0891
7	.0729	.0818	.0862
8	.0693	.0796	.0836
9	.0663	.0764	.0812
10	.0638	.0740	.0783
11	.0609	.0703	.0748
12	.0591	.0690	.0744
13	.0564	.0665	.0716
14	.0539	.0639	.0695
15	.0527	.0626	.0681
16	.0512	.0604	.0668
17	.0503	.0599	.0658
18	.0486	.0584	.0632
19	.0473	.0568	.0614
20	.0465	.0554	.0603
21	.0468	.0555	.0605
22	.0452	.0538	.0591
23	.0428	.0517	.0564
24	.0424	.0508	.0573
25	.0412	.0502	.0550
26	.0408	.0498	.0555
27	.0400	.0483	.0526
28	.0401	.0484	.0529
29	.0393	.0469	.0525
30	.0386	.0463	.0512
31	.0378	.0457	.0508
32	.0371	.0449	.0501
33	.0364	.0444	.0490
34	.0366	.0446	.0490
35	.0358	.0439	.0485
36	.0352	.0430	.0477
37	.0346	.0420	.0477
38	.0340	.0412	.0468
39	.0336	.0410	.0464
40	.0336	.0410	.0450

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{F_n}/\sigma_0^2$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$.

4. ASYMPTOTIC RELATIVE EFFICIENCY (ARE)

We compare our test $\hat{\Delta}_{F_n}$ to tests $\hat{\Delta}_n$ and $\hat{\Delta}_{E_n}$ presented by Kango (1993) and Abu-Youssef and El-Sherbiny (2003) for NBUE and EBU classes of life distributions respectively . The comparisons are achieved by using Pitman asymptotic relative

efficiency (PARE), which is defined as follows:
Let T_{1_n} and T_{2_n} be two statistics for testing H_o: $F_\theta \epsilon \{F_{\theta_x}\}, \theta_n = \theta + \frac{c}{\sqrt{n}}$ with c an arbitrary constant, then PARE of T_{1_n} relative to T_{2_n} is defined by

$$e(T_{1_n}, T_{2_n}) = \frac{\mu_1'(\theta_o)}{\sigma_1(\theta_o)} / \frac{\mu_2'(\theta_o)}{\sigma_2(\theta_o)}$$

where $\mu_i'(\theta_o) = \lim_{n\to\infty} \frac{\partial}{\partial\theta} E(T_{in})_{\to\theta_o}$ and $\sigma_i^2(\theta_o) = \lim_{n\to\infty} VarE(T_{in}), i = 1, 2.$ Three of the most commonly used alternatives (cf. Hollander and Proschan (1972)) are:

(i) Linear failure rate family : $\bar{F}_{1\theta} = e^{-x-\frac{\theta x^2}{2}}$, $x > 0, \theta > 0$
(ii) Makeham family : $\bar{F}_{2\theta} = e^{-x-\theta(x+e^{-x}-1)}$, $x > 0, \theta > 0$
(iii) Weibull family : $\bar{F}_{3\theta} = e^{-x^\theta}$, $x \geq 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for linear failure rate and Makeham families and $\theta = 1$ for Weibull family. Direct calculations of PARE of $\hat{\Delta}_n$ and$\hat{\Delta}_{E_n}$ and $\hat{\Delta}_{F_n}$ are summarized in Table 4.1.

Table 4.1 PARE of $\hat{\Delta}_n$, $\hat{\Delta}_{E_n}$ and $\hat{\Delta}_{F_n}$

Distribution	$\hat{\Delta}_n$	$\hat{\Delta}_{E_n}$	$\hat{\Delta}_{F_n}$
Linear failure rate(F_1)	0.433	1.83	0.919
Makeham(F_2)	0.088	0.966	0.510
Weibull(F_3)	0.144	1.275	3.50

The efficiencies in Table 4.1 show clearly our statistic ($\hat{\Delta}_{F_n}$) performs better than $\hat{\Delta}_{E_n}$for F_1, F_2 and F_3 and it performs better than $\hat{\Delta}_{E_n}$ for F_3 only.

In Table 4.2 we give PARE's of $\hat{\Delta}_{F_n}$ with respect to $\hat{\Delta}_{E_n}$ and $\hat{\Delta}_{E_n}$ whose PARE are mentioned in Table 4.1.

Table 4.2 PARE of $\hat{\Delta}_{F_n}$ with respect to $\hat{\Delta}_n$and $\hat{\Delta}_{E_n}$

Distribution	$e_{F_i}(\hat{\Delta}_{F_n}, \hat{\Delta}_n)$	$e_{F_i}(\hat{\Delta}_{F_n}, \hat{\Delta}_{E_n})$
Linear failure rate(F_1)	2.12	0.5
Makeham(F_2)	8.5	0.53
Weibull(F_3)	29.17	2.75

It is clear from Table 4.2 that the statistic $\hat{\Delta}_{F_n}$ performs well for $\overline{F}_1$, $\overline{F}_2$ and $\overline{F}_3$ and it is more efficient than $\hat{\Delta}_{E_n}$ for $\overline{F}_3$ only.

Finally, the power of the test statistics $\hat{\Delta}_{F_n}$ is considered for 95% percentiles in Table 4.3 for three of the most commonly used alternatives [see Hollander and Proschan (1975)], they are

(i) Linear failure rate : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$, $x > 0, \theta > 0$
(ii) Makeham : $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$, $x \geq 0, \theta > 0$
iii) Weibull family : $\bar{F}_{3\theta} = e^{-x^\theta}$, $x \geq 0, \theta > 0$

These distributions are reduced to exponential distribution for appropriate values of θ .

Table 4.3 Power Estimate of $\hat{\Delta}_{E_{1n}}$

Distribution	θ	Sample Size		
		n=10	n=20	n=30
Linear failure rate(F_1)	1	0.344	0.710	0.887
	2	0.669	0.978	0.999
	3	0.844	0.999	1.000
Makeham(F_2)	1	0.223	0.449	0.626
	2	0.477	0.844	0.966
	3	0.669	0.971	0.999
Weibull(F_2)	1	0.052	0.049	0.048
	2	0.222	0.503	0.730
	3	0.344	0.823	0.971

5. NUMERICAL EXAMPLE

Consider the data in Susarla and Van Ryzin (1978). These data represent 81 patients of melanoma. Of them 46 represent whole life time (non-censored data) and the ordered values are: 13, 14, 19, 19, 20, 21, 23, 23, 25, 26, 26, 27, 27, 31, 32, 34, 34, 37, 38, 38, 40, 46, 50, 53, 54, 57, 58, 59, 60, 65, 65, 66, 70, 85, 90, 98, 102, 103, 110, 118, 124, 130, 136, 138, 141, 234.

Using equation (2.8), the value of test statistics, based on the above data is $\hat{\Delta}_{F_n} = -0.49$. This value leads to H_o is not rejected at the significance level $\alpha = 0.05$. See Table (3.1). Therefore the data has not EBU Property. This result agree with the result of Abu-Youssef and Elshewrpieny (2003).

ACKNOWLEDGEMENTS

The author would likes to thank the referee and Editor for their careful reading of the manuscript and giving valuable comments.
This study was supported by Faculty of Science-Research center project No(Stat/2007/:
King Saud University.

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