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Reliability Estimation of Generalized Geometric Distribution

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Abstract. In this paper generalized version of the geometric distribution is introduced. This distribution can be considered as a two-parameter generalization of the discrete geometric distribution. The main statistical and reliability properties of this distribution are discussed. Two methods of estimation, namely maximum likelihood method and the method of moments are used to estimate the parameters of this distribution. Simulation is utilized to calculate these estimates and to study some of their properties. Also, asymptotic confidence limits are established for the maximum likelihood estimates. Finally, the appropriateness of this new distribution for a set of real data, compared with the geometric distribution, is shown by using the likelihood ratio test and the Kolmogorove-Smirnove test.

Keywords: *hazard-rate, reliability function, method of moments, maximum likelihood estimation, confidence interval, Likelihood ration test, approximate Komlogorove-Smirnove statistic.*

1. INTRODUCTION

Discrete distributions have been shown to be important in many real life situations. For instance, life length data can be measured by number of runs, cycles or shocks, among many others, and hence it is represented by discrete variable. Most of the known univariate discrete distributions have been reviewed in Johnson, Kotz and Kemp (2005). It is known that most of the known discrete distribution are defined for non-negative variables and hence are utilized for modelling life data.

Some of the discrete distributions can be viewed as analogous to continuous distributions. An important example is the geometric distribution which is considered as the discrete version of exponential distribution. Nakagawa and Osaki (1975) introduced a discrete type-I Weibull distribution. Later, Stein and Dattero (1984) introduced another version of discrete Weibull distribution, called type II. Estimation of parameters for type-

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II Weibull distribution was studied by Ali Khan, Khalique and Abouammoh (1989) via a new method known by the method of proportion.

Many authors have considered discrete versions of reliability functions and the discrete forms of various classes of aging distribution, see Abramowitz and Stegun (1972), Abouammoh (1991), Evans, Hastings and Peacock (2000), Wilmmer and Altmann (1999), among many others.

The sequel of this paper is as follows: the definition of the new generalized geometric distribution and its statistical and reliability properties are given in section 2. Estimation of the parameters by maximum likelihood and moment methods for this model are derived in sectio 3. Further, properties of these estimates and the estimate of the relibility function, based on these estimates, are studied via simulation.

2. THE GENERALIZED GEOMETRIC DISTRIBUTION

In comparison with the generalized exponential distribution, see Gupta and Kundu (1999), one can introduce the following:

Definition 2.1 A discrete random variable X is said to have a generalized geometric distribution with parameters *q and* α, *denoted by* (GGD(q,α)), if its distribution function is give by:

$$P_x=P(X\leq x)=(1-q^x)^\alpha \qquad x\in Z \;\; 0<q<1\,,\ \alpha>0 \tag{2.1}$$

where Z={1,2,3,...}.

The probability mass function (pmf) can be given by:

$$p_x=P(X=x)=(1-q^x)^\alpha-(1-q^{x-1})^\alpha \quad ,x\in Z \tag{2.2}$$

One can see that the pmf in (2.2) satisfies the following properties.

1) $p_x \geq 0$, this is noticed since

$q^x < q^{x-1}$, $x \in z$

and

2) $\sum_x p_x = 1$.

The later property can be obtained by noting that

$$\sum_{x=1}^{\infty} p_x = \lim_{m\to\infty} \sum_{x=1}^{m} p_x = \lim_{m\to\infty}(1-q^m)^\alpha = 1$$

Hence, this new discrete distribution with pmf (2.2) can viewed as the discrete analogous of continuous generalized exponential distribution. This distribution implies the geometric distribution as a special case when α=1.

2.1 The reliability functions

Here, the survival functions, the failure rate and mean remaining life (MRL) are derived GGD(q,α) and some of their properties are pointed out.

The survival function of this distribution is given by

$$\overline{P}_x = 1-(1-q^x)^\alpha \tag{2.3}$$

Whereas the corresponding failure rate is given by,

$$r_x = \frac{p_x}{\overline{P}_{x-1}} = \frac{(1-q^x)^\alpha - (1-q^{x-1})^\alpha}{1-(1-q^{x-1})^\alpha}. \tag{2.4}$$

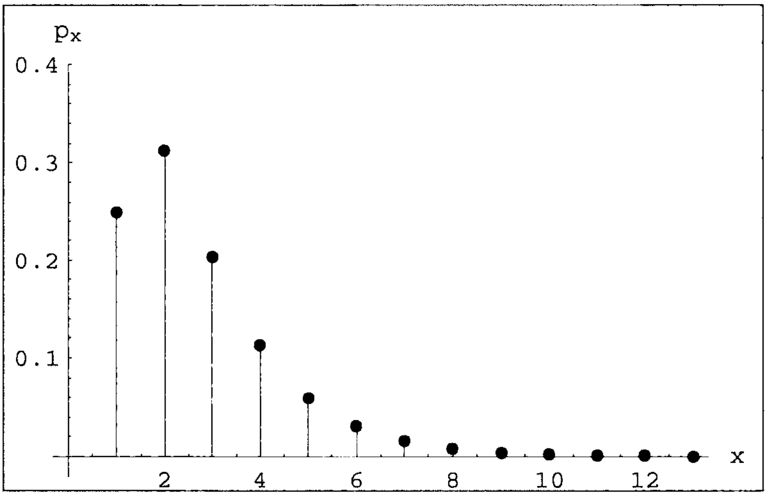

Figure 2.1. pmf of GGD(0.5, 2).

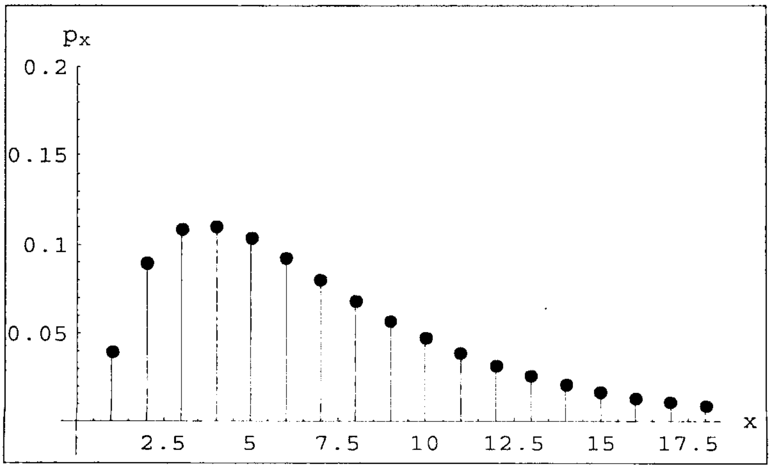

Figure 2.2. pmf of GGD(0.2, 2)

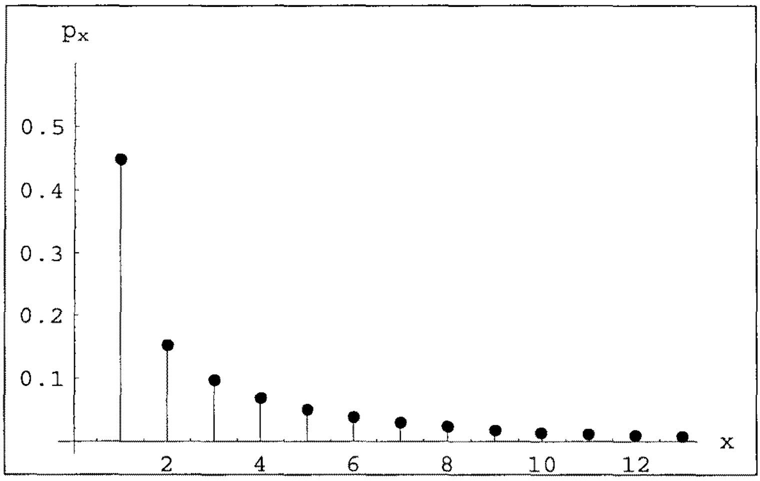

Figure 2.3. pmf of GGD(0.8, 2)

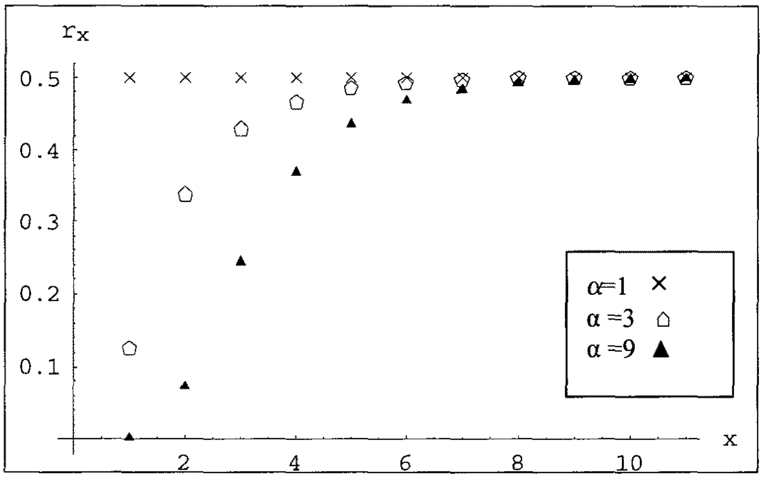

Figure 2.4. Failure rate of GGD (0.5, α)

Figures 2.1, 2.2 and 2.3 represent the graph of the pmf GGD with q=0.5, 0.2, 0.8, α=2. Whereas Figure 2.4 represents the failure rate for q=0.5, α=1, 3, 9, respectively.

One can notice the wide variety of shapes the pmf and the failure rate can produce for different values of q and α

The mean remaining life (MRL) of the GGD(q,α) which is also called the mean residual life, by some authors, is defined by

$$\mu(x) = \frac{\sum_{k=x}^{\infty} \overline{P}_k}{\overline{P}_{x-1}}, \; x \in Z$$

This form can be reduced to the following,

$$\mu(x) = \frac{\sum_{k=x}^{\infty}\{1-(1-q^k)^\alpha\}}{1-(1-q^{x-1})^\alpha} = \frac{\sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^i \frac{q^{ix}}{1-q^i}}{1-(1-q^{x-1})^\alpha} \tag{2.5}$$

One can easily see that for x=0, the MRL is reduced the mean ot E(X) of the GGD (q,α), *which is derived in the following part*.

2.2 Basic statistical properties

The main basic statistical properties, namely the mean, the variance and moment generating function

It is known that the mean has the form:

$$\mu = \sum_x x\, p_x$$
$$= \sum_{x=1}^{\infty} x\left((1-q^x)^\alpha - (1-q^{x-1})^\alpha\right)$$

Thus for integer α the mean becomes

$$\mu = \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^i)}. \tag{2.6}$$

Note that we can get the mean also by putting x=0 in the MRL that is given by (2.5). Now the variance of the GGD (q,α) random variable is calculated by using :

$$\sigma^2 = E(x^2) - \mu^2 \ , \tag{2.7}$$

But one can show that, see Arnold, Balakrishnan, and Nagaraja (1992), if the support of the distribution is a subset of nonnegative integers, then:

$$E(X^2) = 2\sum_{x=0}^{\infty} x(1-P(X \le x)) + \mu$$

Substitutions with our distribution in (2.1) we get

$$E(X^2) = 2\sum_{x=1}^{\infty} x\,[1-(1-q^x)^\alpha] + \mu$$

But for integer values of α, one has

$$E(X^2) = 2\sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{q^i}{(1-q^i)^2} + \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^i)}, \tag{2.8}$$

or

$$E(X^2) = 2\sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^i)} \tag{2.9}$$

Substituting from relations (2.7) the variance of the GGD becomes

$$\sigma^2 = 2\sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^i)} - \left(\sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^i)}\right)^2 \tag{2.10}$$

Now the moment generating function (mgf) of GGD(q,α) is given by

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=1}^{\infty} e^{tx}((1-q^x)^\alpha - (1-q^{x-1})^\alpha)$$

$$= \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^i\left(\sum_{x=1}^{\infty}(e^t q^i)^x - \frac{1}{q^i}\sum_{x=1}^{\infty}(e^t q^i)^x\right),$$

but

$$\sum_{x=1}^{\infty}(e^t q^i)^x = \frac{e^t q^i}{1-e^t q^i}.$$

Hence

$$M_x(t) == \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^i\left(\frac{e^t(q^i-1)}{(1-e^t q^i)}\right) \tag{2.11}$$

To verify this formula one may evaluate the mean using (2.11) i.e.

$$\frac{\partial}{\partial t}M_x(t) = \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^i(q^i-1)\frac{\partial}{\partial t}\left(\frac{e^t}{(1-e^t q^i)}\right)$$

$$= \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^i(q^i-1)\frac{1}{(1-e^t q^i)^2}$$

Therefore

$$\left[\frac{\partial}{\partial t}M_x(t)\right]_{t=0} = \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^i \frac{(q^i-1)}{(1-q^i)^2} = \sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{1-q^i}$$

and this is the same as (2.6).

Note that in the earlier statistical or reliability properties the corresponding forms are reduced to the forms of the geometric distribution for $\alpha=1$. Note also that for real α, the sum in all the above formulae will go to infinity rather than till α.

2.3 Distribution of extreme value

In many practical situations, one may be interested to find out the distribution of parallel system composed of n components with identical GGD(q,α) or different forms, GGD (q,α_i), i=1, 2, ..., n life lengths. Now let $X_1, X_2, \ldots, X_n$ be an iid sample of the GGD(q,α) and let $X_{(n)}=\max(X_1, X_2, \ldots, X_n)$, then $X_{(n)}$ has a GGD($q, n\alpha$) life length. This result is shown by

$$F_{(n)}(x) = p(X_{(n)} < x) = \prod_{i=1}^{n} p(X_i < x) = \prod_{i=1}^{n} (F_i(x)) = (1-q^x)^{n\alpha} \tag{2.12}$$

Where $F_{(n)}$ and F are the df of the n[th] order statistic and the parent distribution respectively. In the case of non identical distributions, that is if X_i has a GGD(q, α_i) equation (2.12) becomes

$$F_{(n)}(x) = (1-q^x)^{\sum_{i=1}^{n}\alpha_i} \tag{2.13}$$

Equations (2.13) indicates that the maximum of a sample of GGD has the same distribution with a second parameter equal to the sum of parameters α_i,, i=1, 2, ..., n. .

3. PARAMETER ESTIMATION

In order to evaluate the reliability estimate of systems composed of GGD components we consider the problem of estimation for the parameters of the GGD, namely q and α. In this section, the method of maximum likelihood and the method of moment are used to estimate the parameters. In both methods, we consider the general case when both parameters q and α are unknown.

3.1 Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of the GGD having the corresponding values $x_1, x_2, \ldots, x_n$, the likelihood function is then given by

$$l(q,\alpha)=\prod_{i=1}^{n} p_{x_i} = \prod_{i=1}^{n}\left[(1-q^{x_i})^{\alpha}-(1-q^{x_i-1})^{\alpha}\right]$$

Therefore, the $\ln l(q,\alpha)$ likelihood is

$$\ln l(q,\alpha)=\sum_{i=1}^{n}\ln\{(1-q^{x_i})^{\alpha}-(1-q^{x_i-1})^{\alpha}\}. \tag{3.1}$$

Hence

$$\frac{\partial}{\partial q}\ln l(q,\alpha)=\sum_{i=1}^{n}\frac{-\alpha\, x_i\, q^{(x_i-1)}(1-q^{x_i})^{\alpha-1}+\alpha\,(x_i-1)\,q^{(x_i-2)}(1-q^{x_i-1})^{\alpha-1}}{(1-q^{x_i})^{\alpha}-(1-q^{x_i-1})^{\alpha}} \tag{3.2}$$

And

$$\frac{\partial}{\partial \alpha}\ln l(q,\alpha)=\sum_{i=1}^{n}\frac{(1-q^{x_i})^{\alpha}\ln(1-q^{x_i})-(1-q^{x_i-1})^{\alpha}\ln(1-q^{x_i-1})}{(1-q^{x_i})^{\alpha}-(1-q^{x_i-1})^{\alpha}} \tag{3.3}$$

Equating the quantities in (3.2) and (3.3) to zero, we get the normal equations, which have no explicit solution so they need to be solved numerically.

Method of moment estimation

Here, we consider the method of moment estimation for the parameters q and α of GGD (q,α), where both parameters are unknown. Note that (2.6) and (2.10) give the mean and variance of this distribution. Since the mean and variance are expressed in terms of infinite series for real α, we restrict the estimation here to the case of integer-valued α. Hence, if we have random sample of size n of GGD (q,α) with sample mean $\bar{X}$ and ample variance s^2, the method of moment estimators of q and α are obtained by solving the following equations for both q and α,

$$\bar{X}=\Sigma_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^{i})} \tag{3.4}$$

$$s^2=2\sum_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^{i})}-\left(\Sigma_{i=1}^{\alpha}\binom{\alpha}{i}(-1)^{i+1}\frac{1}{(1-q^{i})}\right)^{2} \tag{3.5}$$

Here again there are no explicit solutions to the above equations and hence Mathematic is used to obtain numerical solution for tem that give simulated estimation for q and α.

Calculation and discussions

We have already noted that the estimators obtained have no explicit forms, and hence they have to be calculated numerically. A sample of GGD (q,α) can be generated from a standard uniform sample u_i, i=1,2,...,n by using the following relation

$$x_i = [\frac{\ln(1-u^{1/\alpha})}{\ln q}]+1, \ i=1,2,...,n. \qquad (3.6)$$

where [z] is the integer part of z.

Mathematica software is used to generate samples of GGD using (3.6) and solve the numerical equations to find the estimates.

For the maximum likelihood estimates (MLEs) we consider sample sizes of 15, 20 (10)50 and 100, the population values of α are 3, 3.5, 4, 4.5, 5, 5.5, 7 and 9 while the q value is 0.8. MLEs of the parameters, when both of the parameters are unknown, are evaluated based on 1000 replicates.

Table 3.1 represents the MLEs of α, the ratio ($\hat{\alpha}/\alpha$), bias and mean square error (mse) is also presented. It is observed from the results that the mse decreases as the sample size increases. The mse, on the other hand, increases as the value of α increases. It is noted here also that the cases are tend to be more overestimate for large values of n.

In Table 3.2, the MLEs of q are presented. Note that the mean square error (mse) of q decreases as n gets large. Also, the absolute value of bias gets smaller for larger sample sizes. As the values of α change in the same sample size mse remains almost fixed. This means that the estimation of q does not depend on the value of α. Results for the estimate of q , $\hat{q}$, tend to be more underested.

Table 3.3 represents the MLE of reliability function, based on the MLEs' of the parameters, compared with the true value of the reliability function. These two functions are evaluated at x=5 and denoted by $\hat{S}_{MLE}$ and S respectively. It is observed that in general, the absolute value of bias decreases as n gets large. Also the estimated Survival function (sf) increases as the value of α gets large. Also, one can note that $\hat{S}_{MLE}$ in Table 3.3 shows underestimation.

Tables 3.4, 3.5 and 3.6 represent MMEs of α, q, and sf at x=5 respectively. The summation in (3.4) and (3.5) are restricted for integer values of α, i.e. 3, 5, 7, and 9 where the value of q in these tables is taken to eaual 0.8. On the other hand Tables 3.7, 3.8 and 3.9 represent MMEs of α, q, and sf at for q equal 0.4. In both cases one can note that, in general the mse of the estimates decreasing as n increasing. In term of mse, the estimation of α is better for small values of α, while the estimate of q is better for large values of α. The estimates of α tend to be more over estimation while the estimates of q tend to be more under estimation. As expected, the value of the estimated survival function increases as α increases and the absolute value of the bias decreases as the sample size gets larger. Comparing Tables 3.4, 3.5 and 3.6 where q=0.8 with Tables 3.7, 3.8 and 3.9 where q=0.4, it is observe, almost for all the cases, that the estimates of α and q are better(based on the mse) when for q=0.8, evenao the ratio for the survivals ($\hat{S}_{MME}/S$) is higher in the case of q=0.8 which may indicate that the estimates of the parameters and survival function are is better (base on the mse) for larger values of q.

To conclude and by comparing the last simulated results in six tables it is observed that the MME has given preferable results than MLE for estimating the parameters of GGD(q,α).

4. ASYMPTOTIC CONFIDENCE INTERVALS

Here we will derive the asymptotic confidence intervals (CIs) for the parameters of the new discrete distribution considered in this paper, depending on the asymptotic distribution of the MLEs of the parameters.

To derive the asymptotic CIs for q and α we have to find the 2nd derivative of the log likelihood function wrt to the parameters, where the 1st derivatives are given by (3.2) and (3.3).

Now

$$\frac{\partial^2}{\partial q^2}\ln l(\theta)=\sum_{i=1}^{n}\frac{\alpha\, q^{x_i-4}\kappa_i(\theta)\,\eta_i(\theta)-\alpha\, q^{x_i}\,\tau_i(\theta)}{(\kappa_i(\theta))^2} \tag{4.1}$$

$$\frac{\partial^2}{\partial \alpha^2}\ln l(\theta)=\sum_{i=1}^{n}\frac{\kappa_i(\theta)\omega_{i2}(\theta)-(\omega_{i1}(\theta))^2}{(\kappa_i(\theta))^2} \tag{4.2}$$

$$\frac{\partial^2}{\partial \alpha\partial q}\ln l(\theta)=\sum_{i=1}^{n}\frac{q^{x-2}\,\alpha\,\varepsilon_i(\theta)\,\omega_{i1}(\theta)+\kappa_i(\theta)\,v_i(\theta)}{(\kappa_i(\theta))^2} \tag{4.3}$$

Where one finds that:

$$\kappa_i(\theta)=(1-q^{x_i})^{\alpha}-(1-q^{x_i-1})^{\alpha} \tag{4.4}$$

$$\begin{aligned}\eta_i(\theta)&=q(x_i-1)(x_i-2)(1-q^{x_i-1})^{\alpha-1}-q^2x_i\,(x_i-1)(1-q^{x_i})^{\alpha-1}\\&\quad-q^{x_i}(\alpha-1)(x_i-1)^2(1-q^{x_i})^{\alpha-2}+q^{x_i+2}x_i^2\,(\alpha-1)(1-q^{x_i})^{\alpha-2}\end{aligned} \tag{4.5}$$

$$\tau_i(\theta)=((x_i-1)(1-q^{x_i-1})^{\alpha-1}-qx_i(1-q^{x_i})^{\alpha-1})^2 \tag{4.6}$$

$$\omega_{i1}(\theta)=(1-q^{x_i})^{\alpha}\ln(1-q^{x_i})-(1-q^{x_i-1})^{\alpha}\ln(1-q^{x_i-1}) \tag{4.7}$$

$$\omega_{i2}(\theta)=(1-q^{x_i})^{\alpha}[\ln(1-q^{x_i})]^2-(1-q^{x_i-1})^{\alpha}[\ln(1-q^{x_i-1})]^2 \tag{4.8}$$

$$\varepsilon_i(\theta)=q(1-q^{x_i})^{\alpha-1}x-(x-1)(1-q^{x-1})^{\alpha-1} \tag{4.9}$$

$$\begin{aligned}\upsilon_i(\theta)&=(x-1)(1-q^{x_i-1})^{\alpha-1}-qx_i(1-q^{x_i})^{\alpha-1}+\alpha(x_i-1)(1-q^{x_i-1})\ln(1-q^{x_i-1})\\&\quad-\alpha(x_i-1)(1-q^{x_i-1})^{\alpha-1}\ln(1-q^{x_i-1})+q\alpha x_i(1-q^{x_i})^{\alpha-1}\ln(1-q^{x_i})\end{aligned} \tag{4.10}$$

and $\theta=(q,\alpha)$.

If $\hat{\theta}=(\hat{q},\hat{\alpha})$ is the MLE of θ, the observed information matrix would be

$$I(\theta)=\begin{pmatrix}\dfrac{-\partial^2 \ln l(\theta)}{\partial \alpha^2} & \dfrac{-\partial^2 \ln l(\theta)}{\partial \alpha\, \partial q} \\ \dfrac{-\partial^2 \ln l(\theta)}{\partial \alpha\, \partial q} & \dfrac{-\partial^2 \ln l(\theta)}{\partial q^2}\end{pmatrix}_{\alpha=\hat{\alpha},q=\hat{q}} \qquad (4.11)$$

Hence the variance covariance matrix would be $I^{-1}(\theta)$. The approximate (1-δ) 100% CIs for α and q are: $\hat{\alpha}\pm\xi_{\delta/2}\sqrt{V(\hat{\alpha})}$ and $\hat{q}\pm\xi_{\delta/2}\sqrt{V(\hat{q})}$ respectively, where $V(\hat{\alpha})$ and $V(\hat{q})$ are the variances of $\hat{\alpha}$ and $\hat{q}$ which are given by the 1st and the 2nd diagonal element of $I^{-1}(\theta)$, and $\xi_{\delta/2}$ is the upper (δ/2) percentile of standard normal distribution.

5. DATA ANALYSIS

Here we use a real data to represents the survival times (in days) of 40 patients suffering from blood cancer. The data are collected in one of the Ministry of Health Hospitals in Saudi Arabia; see Abouammoh et al. (1994).

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852

We make the likelihood ratio test to determine whether this data comes from GGD or GD distributions, i.e. our test is

H_0: α=1 (GD)

H_1: $\alpha \neq 1$ (GGD)

The estimates and the LRT statistic for the blood cancer data of the test are displayed in the following table

	$\hat{q}$	$\hat{\alpha}$	*ln l(θ)*	X^2	*p-value*
Under H_0	0.9991	-----	-321.428	11.1628	0.00051
Under H_1	0.9989	1.48652	-315.847		

Looking at this result we recommend that this data comes from GGD. The approximate Kolmogorov- Smirnove (K-S) distance for GGD and GD are

$K\text{-}S^*_{GGD} = 0.244$

$K\text{-}S^*_{GD} = 0.303$

Which confirm the result of the LRT above.

To find the **95% CIs** for the parameters in our example we obtain the observed information matrix as follows

$$I(\theta) = \begin{pmatrix} 18.108 & 1890 \\ 1890 & 4.68 \times 10^{7} \end{pmatrix}$$

Hence the variance covariance matrix would be

$$I^{-1}(\theta) = \begin{pmatrix} 0.096 & -0.00004 \\ -0.0004 & 3.696 \times 10^{-8} \end{pmatrix}$$

The 95% confidence intervals for the MLEs of $\hat{\alpha}$ and $\hat{q}$ are (0.97813, 1.9950) and (0.9986, 0.9992), respectively.

6. DISCUSSIONS

In this paper, a new probability model, namely the generalized geometric distribution, is introduced. The geometric distribution is obtained as a special case if the second parameter $\alpha = 1$. It is shown that the GGD can fit some life data better than the geometric distribution. MLE's and MME's are simulated for the both parameters of GGD(q,α).

In fact one may consider other methods of estimation for the parameters of this model. The relations of this distribution with other existing discrete distributions such as different forms of discrete Weibull are worthy of investigation. It is also of interest to researcher to study the compounding of the GGD with other discrete or continuous distributions. Estimation of the parameter under various censoring schemes such as random and progressive censoring are to be investigated. The authors hope to consider some of these problems in forthcoming work.

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REFERENCES

Abouammoh, A. M.(1991). On the mean remaining life of discrete random variables. J.of Inform. And Optim. Science, 12, 145-158.

Abouammoh, A. M., Abdulghani, S. A. and. Qamber, I. S. (1994) On partial orderings and testing of new better than renewal used classes. Reliability Eng. Syst. Safety, 43, 37-41.

Abramowitz, M. and Stegun, I. A. (1972). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York, Dover.

Ali Khan, M.S , Khalique, A. and Abouammoh A. M. (1989). On estimating parameters in a discrete Weibull distribution. IEEE Trans. Reliability, 38, 348-350.

Arnold,B.C, Balakrishnan,N. and Nagaraja, H. N. (1992). A first course in order statistics. John Wiley &Sons.

Evans, M.; Hastings, N. and Peacock, B. (2000). Statistical Distributions, 3rd ed. New York: Wiley.

Johnson; N. L., Kotz S. and Kemp, A. W. (2005). Univariate Discrete Distributions. 4th edition Hoboken, N.J.: Wiley-Interscience, New York.

Gupta, R. D. and Kundu, D. (1999). Generalized exponential distribution. Australian & New Zealand J. Statist., 41, 173-188.

Nakagawa, T. and Osaki, S. (1975). This discrete Weibull distribution. IEEE Trans. Reliability, 24, 300-301.

Stein ,W. E. and Dattero, R. (1984). A new discrete Weibull distribution, IEEE Trans. Reliability, R-33, 196-197.

Wilmmer, G. and Altmann, G. (1999). Thesaurus of Univariate Discrete Probability Distributions. Essen, Germany: STAMM.

Table 3.1. MLEs for α for GGD

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{\alpha}$	3.46831	3.41934	3.59841	3.19157	3.07607	3.44097
$\hat{\alpha}/\alpha$	1.1561	1.13978	1.19947	1.06386	1.02536	1.14699
bias	0.46831	0.41934	0.59841	0.19157	0.07607	0.44097
mse	1.03105	0.81541	1.07944	0.32523	0.39224	0.29471
(α=3.5)						
$\hat{\alpha}$	3.54149	3.57388	3.55089	3.58301	3.37949	3.8984
$\hat{\alpha}/\alpha$	1.01185	1.02111	1.01454	1.02372	0.96557	1.11383
bias	0.04149	0.07388	0.050887	0.08301	-0.12051	0.3984
mse	0.88467	0.79472	0.60803	0.42955	0.37585	0.3368
(α=4)						
$\hat{\alpha}$	3.77178	3.76215	3.92891	4.02383	3.85815	4.25671
$\hat{\alpha}/\alpha$	0.94295	0.94054	0.98223	1.00598	0.96454	1.06418
bias	-0.2282	-0.23785	-0.07109	0.02383	-0.14185	0.25671
mse	1.04845	0.98378	0.62269	0.52584	0.50324	0.39045
(α=4.5)						
$\hat{\alpha}$	4.14513	4.14615	4.32806	4.54114	4.34621	4.68037
$\hat{\alpha}/\alpha$	0.92114	0.92137	0.96179	1.00914	0.96582	1.04008
bias	-0.35487	-0.35385	-0.17194	0.04114	-0.15379	0.18037
mse	1.25159	1.11118	0.76365	0.75709	0.68786	0.37595
(α=5)						
$\hat{\alpha}$	4.47345	4.52825	4.64977	4.94459	4.76539	5.23125
$\hat{\alpha}/\alpha$	0.89469	0.90565	0.92995	0.98892	0.95308	1.04625
bias	-.52655	-0.47175	-0.35023	0.05541	-0.23461	0.23125
mse	1.73166	1.57773	0.89733	0.86493	0.69886	0.60925
(α=5.5)						
$\hat{\alpha}$	4.93299	4.93895	5.10335	5.27133	5.13894	5.54848
$\hat{\alpha}/\alpha$	0.89691	0.89799	0.92788	0.95842	0.93435	1.00882
bias	-0.56701	-0.56105	-0.39665	-0.22867	-0.36106	0.04848
mse	2.30634	1.97147	1.23558	1.13158	1.01008	0.62762

(α=7)						
$\hat{\alpha}$	5.92540	6.24862	6.52122	6.86723	6.56350	7.00532
$\hat{\alpha}/\alpha$	0.84649	0.89266	0.93160	0.98103	0.93764	1.00076
bias	-1.07460	-0.75138	-0.47878	0.13277	-0.43650	0.00532
mse	3.86808	2.87388	1.96821	2.01563	1.72771	0.99661
(α=9)						
$\hat{\alpha}$	8.02517	8.20184	8.55943	8.74782	8.49851	9.04407
$\hat{\alpha}/\alpha$	0.65838	0.69429	0.72458	0.76303	0.72928	1.00490
bias	-0.97483	-0.79816	-0.44057	0.25218	-0.50149	0.04407
mse	5.90422	5.61540	3.57874	3.74796	3.28100	1.80931

Table 3.2. MLE of q for GGD

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{q}$	0.76931	0.77788	0.77297	0.79601	0.80465	0.79164
$\hat{q}/q$	0.96164	0.97235	0.96622	0.99501	1.00581	0.98955
ias	-0.03069	-0.02212	-0.02703	-0.00399	0.00465	-0.00836
mse	0.00409	0.00219	0.00229	0.0008	0.00088	0.00017
(α=3.5)						
$\hat{q}$	0.77433	0.78244	0.78011	0.79398	0.80098	0.79341
$\hat{q}/q$	0.96791	0.97805	0.97514	0.99247	1.00123	0.99176
bias	-0.02567	-0.01756	-0.01989	-0.006	0.00098	-0.0066
mse	0.00366	0.002	0.00198	0.00093	0.00088	0.00018
(α=4)						
$\hat{q}$	0.77267	0.78381	0.78524	0.79372	0.79686	0.79594
$\hat{q}/q$	0.96584	0.97976	0.98155	0.99215	0.99608	0.99493
bias	-0.02733	-0.01619	-0.01476	-0.00628	-0.00314	-0.00406
mse	0.00455	0.0019	0.00161	0.0008	0.00076	0.0002
(α=4.5)						
$\hat{q}$	0.77708	0.78112	0.78887	0.79196	0.79446	0.79624
$\hat{q}/q$	0.97136	0.9764	0.98609	0.98995	0.99308	0.9953
bias	-0.02292	-0.01888	-0.01113	-0.00804	-0.00554	-0.00376
mse	0.00311	0.0026	0.00118	0.00083	0.00077	0.00022
(α=5)						

$\hat{q}$	0.77331	0.78381	0.78547	0.79171	0.79394	0.79552
$\hat{q}/q$	0.96664	0.97976	0.98183	0.98964	0.99243	0.9944
bias	-0.02669	-0.01619	-0.01453	-0.00829	-0.00606	-0.00448
mse	0.00424	0.00192	0.00205	0.00082	0.00069	0.00024
(α=5.5)						
$\hat{q}$	0.7753	0.7792	0.7904	0.79323	0.79497	0.79749
$\hat{q}/q$	0.9691	0.97402	0.988	0.99153	0.99371	0.99686
bias	-0.02473	-0.02078	-0.00961	-0.00677	-0.00503	-0.00251
mse	0.0038	0.00415	0.00109	0.00077	0.00068	0.00023
(α=7)						
$\hat{q}$	0.77450	0.77830	0.78999	0.79335	0.79501	0.79792
$\hat{q}/q$	0.96813	0.97288	0.98749	0.99169	0.99376	0.99740
bias	-0.02550	-0.02170	-0.01001	-0.00665	-0.00499	-0.00208
mse	0.00400	0.00325	0.00107	0.00074	0.00061	0.00021
(α=9)						
$\hat{q}$	0.77592	0.78189	0.78759	0.79392	0.79516	0.79790
$\hat{q}/q$	0.96991	0.97736	0.98449	0.99240	0.99395	0.99738
bias	-0.02408	-0.01811	-0.01241	-0.00608	-0.00484	-0.00210
mse	0.00333	0.00228	0.00149	0.00067	0.00059	0.00020

Table 3.3. MLE of the reliability (survival) for GGD.

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{S}_{MLE}$	0.66344	0.68216	0.68711	0.70739	0.71794	0.72235
S	0.6961	0.6961	0.6961	0.69611	0.6961	0.6961
$\hat{S}_{MLE}/S$	0.95308	0.97997	0.98708	1.01621	1.03137	1.03771
bias	-0.03266	-0.01394	-0.00899	0.01129	0.02184	0.02625
(α=3.5)						
$\hat{S}_{MLE}$	0.68506	0.71075	0.70202	0.7429	0.74125	0.7704
S	0.75082	0.75082	0.75082	0.75082	0.75082	0.75082

$\hat{S}_{MLE}/S$	0.91242	0.94663	0.935	0.98948	0.98726	1.0261
bias	-0.06576	-0.04007	-0.04879	-0.0079	-0.00957	0.0196
(α=4)						
$\hat{S}_{MLE}$	0.70331	0.73275	0.75171	0.78183	0.77583	0.80567
S	0.79568	0.79568	0.79568	0.79568	0.79568	0.79568
$\hat{S}_{MLE}/S$	0.8839	0.92091	0.94474	0.98259	0.97504	1.01256
bias	-0.09238	-0.06293	-0.04397	-0.01385	-0.01986	0.00999
(α=4.5)						
$\hat{S}_{MLE}$	0.7487	0.75941	0.7936	0.81643	0.80868	0.83558
S	0.83247	0.83247	0.83247	0.83247	0.83247	0.83247
$\hat{S}_{MLE}/S$	0.89937	0.91224	0.9533	0.98073	0.97142	1.00374
bias	-0.08378	-0.07306	-0.03887	-0.01604	-0.02379	0.00311
(α=5)						
$\hat{S}_{MLE}$	0.76499	0.79571	0.80827	0.84154	0.83571	0.86559
S	0.86263	0.86263	0.86263	0.86263	0.86263	0.86263
$\hat{S}_{MLE}/S$	0.88681	0.92242	0.93698	0.97555	0.96879	1.00342
bias	-0.09764	-0.06692	-0.05436	-0.02109	-0.02692	0.00295
(α=5.5)						
$\hat{S}_{MLE}$	0.8023	0.81225	0.84777	0.86289	0.85957	0.88478
S	0.88737	0.88737	0.88737	0.88737	0.88737	0.88737
$\hat{S}_{MLE}/S$	0.90414	0.91535	0.95537	0.97241	0.96867	0.99708
bias	-0.08506	-0.07512	-0.0396	-0.02448	-0.0278	-0.00259
(α=7)						
$\hat{S}_{MLE}$	0.85568	0.87771	0.90910	0.92505	0.91857	0.93525
S	0.93791	0.93791	0.93791	0.93791	0.93791	0.93791
$\hat{S}_{MLE}/S$	0.91233	0.93582	0.96928	0.98629	0.97939	0.99717
bias	-0.08223	-0.06019	-0.02881	-0.01286	-0.01933	-0.00266
(α=9)						
$\hat{S}_{MLE}$	0.92937	0.94128	0.95451	0.96367	0.96127	0.97079
S	0.97193	0.97193	0.97193	0.97193	0.97193	0.97193
$\hat{S}_{MLE}/S$	0.95621	0.96846	0.98207	0.99149	0.98903	0.99883
bias	-0.04257	-0.03066	-0.01743	-0.00827	-0.01066	-0.00114

Table 3.4. MME of α for GGD (q=0.8)

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{\alpha}$	3.02393	3.06404	3.1362	3.0875	3.09378	3.14695
$\hat{\alpha}/\alpha$	1.00798	1.02135	1.04540	1.02917	1.03126	1.04898
bias	0.02393	0.06404	0.13620	0.08751	0.09378	0.14695
mse	0.43636	0.30341	0.16923	0.20031	0.18865	0.04635
(α=5)						
$\hat{\alpha}$	5.09459	5.12369	5.11619	5.08078	5.17539	5.18626
$\hat{\alpha}/\alpha$	1.01892	1.02473	1.02324	1.01616	1.03508	1.03725
bias	0.09459	0.12369	0.11619	0.08078	0.17539	0.18626
mse	0.79132	0.79394	0.38989	0.38684	0.18423	0.09797
(α=7)						
$\hat{\alpha}$	7.13021	7.25615	7.25504	7.13829	7.09905	7.11561
$\hat{\alpha}/\alpha$	1.01860	1.03659	1.03643	1.01976	1.01415	1.01860
bias	0.13021	0.25615	0.25504	0.13829	0.09905	0.11561
mse	2.73316	1.73803	1.34890	0.90111	0.49984	0.21317
(α=9)						
$\hat{\alpha}$	9.30225	9.26125	9.36248	9.23779	9.23536	9.25333
$\hat{\alpha}/\alpha$	1.03358	1.02903	1.04028	1.02642	1.02615	1.02815
bias	0.30225	0.26125	0.36248	0.23779	0.23536	0.25333
mse	5.14173	2.51509	2.55868	2.00101	1.34925	0.24848

Table 3.5. MME of q for GGD (q=0.8)

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{q}$	0.78458	0.78842	0.78674	0.79082	0.7915	0.79194
$\hat{q}/q$	0.98073	0.98553	0.98343	0.98852	0.98938	0.98993
bias	-0.0154	-0.01158	-0.0133	-0.0092	-0.0085	-0.0081
mse	0.00207	0.001614	0.00104	0.00073	0.00069	0.00025

(α=5)						
$\hat{q}$	0.78807	0.79057	0.79159	0.79434	0.79242	0.7942
$\hat{q}/q$	0.98509	0.98822	0.98949	0.99292	0.99053	0.99275
bias	-0.0119	-0.0094	-0.0084	-0.0057	-0.0076	-0.0058
mse	0.00132	0.00107	0.00062	0.00044	0.00037	0.00016
(α=7)						
$\hat{q}$	0.79075	0.79136	0.79207	0.79544	0.79502	0.79621
$\hat{q}/q$	0.98843	0.98920	0.99009	0.99430	0.99378	0.99526
bias	-0.00925	-0.00864	-0.00793	-0.00456	-0.00498	-0.00379
mse	0.00130	0.00082	0.00058	0.00036	0.00031	0.00012
(α=9)						
$\hat{q}$	0.78999	0.79252	0.79287	0.79516	0.79485	0.79583
$\hat{q}/q$	0.98749	0.99065	0.99109	0.99395	0.99356	0.99479
bias	-0.01001	-0.00748	-0.00713	-0.00484	-0.00515	-0.00417
mse	0.00121	0.00070	0.00054	0.00033	0.00029	0.00011

Table 3.6. MME of the reliability (survival) for GGD (q=0.8)

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{S}_{MME}$	0.65593	0.67151	0.67532	0.68099	0.68364	0.69106
S	0.6961	0.6961	0.6961	0.6961	0.6961	0.6961
$\hat{S}_{MME}/S$	0.94229	0.96468	0.97015	0.97829	0.9821	0.99276
bias	-0.0402	-0.0246	-0.0209	-0.0151	-0.0125	-0.005
(α=5)						
$\hat{S}_{MME}$	0.84214	0.84931	0.85111	0.85507	0.85612	0.86048
S	0.86263	0.86263	0.86263	0.86263	0.86263	0.86263
$\hat{S}_{MME}/S$	0.97624	0.98455	0.98664	0.99123	0.99245	0.9975
bias	-0.0205	-0.0133	-0.0115	-0.0076	-0.0065	-0.0022

(α=7)						
$\hat{S}_{MME}$	0.92843	0.93255	0.93351	0.93522	0.93366	0.93570
S	0.93791	0.93791	0.93791	0.93791	0.93791	0.93791
$\hat{S}_{MME} / S$	0.98990	0.99428	0.99531	0.99713	0.99547	0.99764
bias	-0.00947	-0.00536	-0.00440	-0.00269	-0.00425	-0.00221
(α=9)						
$\hat{S}_{MME}$	0.96731	0.96895	0.97039	0.97081	0.97054	0.97151
S	0.97193	0.97193	0.97193	0.97193	0.97193	0.97193
$\hat{S}_{MME} / S$	0.99524	0.99693	0.99841	0.99884	0.99856	0.99957
bias	-0.00462	-0.00298	-0.00155	-0.00112	-0.00139	-0.00042

Table 3.7. MME of α for GGD (q=0.4)

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{\alpha}$	3.35567	3.42543	3.31663	3.44737	3.42609	3.52529
$\hat{\alpha} / \alpha$	1.11856	1.14181	1.10554	1.14912	1.14203	1.17510
bias	0.35567	0.42543	0.31663	0.44737	0.42609	0.52529
mse	0.55167	0.47185	0.53954	0.43035	0.44888	0.35633
(α=5)						
$\hat{\alpha}$	5.82868	5.72013	5.58576	5.59616	5.57552	5.58659
$\hat{\alpha} / \alpha$	1.16574	1.14403	1.11715	1.11923	1.1151	1.11732
bias	0.82868	0.72013	0.58576	0.59616	0.57552	0.58659
mse	1.85823	0.95561	0.39551	0.36753	0.37899	0.34706
(α=7)						
$\hat{\alpha}$	8.5960	8.5679	8.4314	8.5897	8.4675	8.1439
$\hat{\alpha} / \alpha$	1.2280	1.2240	1.2045	1.2271	1.2096	1.2280
bias	1.5960	1.5679	1.4314	1.5897	1.4675	1.1439
mse	4.5183	4.4192	3.6180	4.6100	4.2044	2.9572

(α=9)						
$\hat{\alpha}$	10.7984	11.0738	10.8662	11.0918	11.0480	11.0570
$\hat{\alpha}/\alpha$	1.1998	1.2304	1.2074	1.2324	1.2276	1.2286
bias	1.7984	2.0738	1.8662	2.0918	2.0480	2.0570
mse	5.8604	6.1611	4.8435	5.5718	5.3431	5.4794

Table 3.8. MME of q for GGD (q=0.4)

	n=15	*n=20*	*n=30*	*n=40*	*n=50*	*n=100*
(α=3)						
$\hat{q}$	0.25935	0.27022	0.28096	0.28608	0.28723	0.29045
$\hat{q}/q$	0.64838	0.67556	0.70239	0.71519	0.71808	0.72614
bias	-0.14065	-0.12978	-0.11904	-0.11392	-0.11277	-0.10954
mse	0.03257	0.02582	0.02003	0.0165	0.0167	0.01367
(α=5)						
$\hat{q}$	0.30336	0.31293	0.31682	0.32247	0.32367	0.32845
$\hat{q}/q$	0.75839	0.78233	0.79205	0.80617	0.80917	0.82112
bias	-0.09664	-0.08707	-0.08318	-0.07753	-0.07633	-0.07155
mse	0.01529	0.0122	0.01002	0.00809	0.00768	0.00601
(α=7)						
$\hat{q}$	0.31459	0.32132	0.32524	0.32921	0.33029	0.33875
$\hat{q}/q$	0.78646	0.80331	0.81309	0.82302	0.82573	0.84687
bias	-0.08541	-0.07868	-0.07476	-0.07079	-0.06971	-0.06125
mse	0.01255	0.01002	0.00839	0.00685	0.00686	0.00474
(α=9)						
$\hat{q}$	0.32691	0.33015	0.33311	0.33550	0.33608	0.34534
$\hat{q}/q$	0.81728	0.82537	0.83278	0.83875	0.84019	0.86335
bias	-0.07309	-0.06985	-0.06689	-0.06450	-0.06392	-0.05466
mse	0.00990	0.00822	0.00675	0.00559	0.00546	0.00349

Table 3.9. MME of the reliability (survival) for GGD (q=0.4)

	$n=15$	$n=20$	$n=30$	$n=40$	$n=50$	$n=100$
(α=3)						
$\hat{S}_{MME}$	0.00393	0.00493	0.00579	0.00659	0.00668	0.00727
S	0.03041	0.03041	0.03041	0.03041	0.03041	0.03041
$\hat{S}_{MME}/S$	0.12932	0.16203	0.19057	0.21671	0.21977	0.23905
bias	-0.02647	-0.02548	-0.02461	-0.02382	-0.02372	-0.02314
(α=5)						
$\hat{S}_{MME}$	0.01488	0.01704	0.01770	0.01936	0.01965	0.02117
S	0.05016	0.05016	0.05016	0.05016	0.05016	0.05016
$\hat{S}_{MME}/S$	0.29666	0.33979	0.35285	0.38590	0.39164	0.42198
bias	-0.03528	-0.03312	-0.03246	-0.03080	-0.03052	-0.02899
(α=7)						
$\hat{S}_{MME}$	0.02618	0.02897	0.03027	0.03273	0.03280	0.03575
S	0.06952	0.06952	0.06952	0.06952	0.06952	0.06952
$\hat{S}_{MME}/S$	0.37656	0.41676	0.43545	0.47086	0.47183	0.51432
bias	-0.04334	-0.04054	-0.03924	-0.03678	-0.03672	-0.03376
(α=9)						
$\hat{S}_{MME}$	0.03959	0.04259	0.04368	0.04615	0.04636	0.05299
S	0.08847	0.08847	0.08847	0.08847	0.08847	0.08847
$\hat{S}_{MME}/S$	0.44748	0.48136	0.49368	0.52161	0.52399	0.59891
bias	-0.04888	-0.04589	-0.04480	-0.04233	-0.04211	-0.03549