East Asian Math. J. **24** (2008), No. 3, pp. 305–315

# ON FIXED POINT OF UNIFORMLY PSEUDO-CONTRACTIVE OPERATOR AND SOLUTION OF EQUATION WITH UNIFORMLY ACCRETIVE OPERATOR

Yuguang Xu, Zeqing Liu and Shin Min Kang*

Abstract. The purpose of this paper is to study the existence and uniqueness of the fixed point of uniformly pseudo-contractive operator and the solution of equation with uniformly accretive operator, and to approximate the fixed point and the solution by the Mann iterative sequence in an arbitrary Banach space or an uniformly smooth Banach space respectively. The results presented in this paper show that if $X$ is a real Banach space and $A : X \to X$ is an uniformly accretive operator and $(I - A)X$ is bounded then $A$ is a mapping onto $X$ when $A$ is continuous or $X^*$ is uniformly convex and $A$ is demicontinuous. Consequently, the corresponding results which depend on the assumptions that the fixed point of operator and solution of the equation are in existence are improved.

## 1. Introduction and preliminaries

To set the framework we recall some basic notations as follows.

Let $X$ be a real Banach space with dual $X^*$. $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. The mapping $J : X \to 2^{X^*}$ defined by

$$(1.1) \qquad Jx = \{j \in X^* : \langle x, j \rangle = \|x\|\|j\|, \|j\| = \|x\|\} \quad \forall x \in X$$

is called the *normalized duality mapping.*

(a) An operator $A : D(A) \subset X \to X$ is said to be *demicontinuous* if it is strong-weak continuous, that is, $x_n$ strongly converges to $x^*$ implies that $F(Ax_n)$ converges to $F(Ax^*)$ for each $F \in X^*$ and $\{x_n\} \subset D(A)$.

(b) An operator $T : X \to X$ is said to be *uniformly pseudo-contractive* if there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that for any $x, y \in X$ there exist $j(x - y) \in J(x - y)$ satisfied

$$(1.2) \qquad \langle Tx - Ty, j(x - y) \rangle \le \|x - y\|^2 - \psi(\|x - y\|).$$

---

Received April 17, 2008; Accepted May 16, 2008.

2000 *Mathematics Subject Classification.* 47H17, 47H06, 47H10.

*Key words and phrases.* Uniformly pseudo-contractive operator, uniformly accretive operator, demicontinuous, Mann iterative sequence.

* Corresponding author.

An operator $A : X \to X$ is said to be *uniformly accretive* if $I - A$ is uniformly pseudo-contractive, i.e.,

$$\langle Ax - Ay, j(x - y)\rangle \geq \psi(\|x - y\|) \tag{1.3}$$

for any $x, y \in X$.

If the function $\psi(s) = ks$ (or $\psi(s) = s\phi(s)$) in (b), the operators $T$ and $A$ are said strongly pseudo-contractive and strongly accretive (or $\phi$-strongly pseudo-contractive and $\phi$-strongly accretive) respectively, where the constant $k \in (0, 1)$ and $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ is a strictly increasing function.

*Remark* 1.1. It is easy to see that the mapping theory for accretive operators is intimately connected with the fixed point theory of pseudo-contractive operators. We like to point out: every strongly accretive (or strongly pseudo-contractive) operator is $\phi$-strongly accretive (or $\phi$-strongly pseudo-contractive), and a $\phi$-strongly accretive (or $\phi$-strongly pseudo-contractive) operator is unnecessary strongly accretive (or strongly pseudo-contractive). Furthermore, every $\phi$-strongly accretive (or $\phi$-strongly pseudo-contractive) operator must be uniformly accretive (or uniformly pseudo-contractive), and an uniformly accretive (or uniformly pseudo-contractive) operator maybe is not $\phi$-strongly accretive (or $\phi$-pseudo-contractive).

The concept of accretive operators was introduced independently by Browder [2] and Kato [13]. Typical examples where such evolution equations occur can be found in the heat and wave. The interest and importance of these operators stems mainly from the fact that many physically significant problems can be modeled in terms of an initial value problem of the form

$$\frac{dx(t)}{dt} + Ax(t) = 0, \quad x(0) = x_0, \tag{1.4}$$

where $A$ is either an accretive or strongly accretive operator in an appropriate Banach space $X$.

An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem (1.4) is *solvable* if $A$ is locally Lipschitzian and accretive on $X$. Martin generalized the result of Browder to the continuous strongly accretive operator. That is, he proved that

**Theorem MT** ([16])**.** *Let $X$ be a real Banach space, if $A : X \to X$ is strongly accretive and continuous, then $A$ is surjective, so that the equation*

$$Ax = f \tag{1.5}$$

*has a solution for any given $f \in X$.*

Later Deimling proved that

**Theorem DM** ([4])**.** *Let $X$ be a real Banach space and $F : X \to X$ strongly accretive. Then $F$ is a mapping onto $X$ if $F$ is continuous or $X^*$ is uniformly convex and $F$ is demicontinuous.*

Very recently, we proved that

**Theorem LKXC** ([14], [15]). *Let $X$ be a real Banach space and $F : X \to X$ $\phi$-strongly accretive. Then $F$ is a mapping onto $X$ if $F$ is continuous or $X^*$ is uniformly convex and $F$ is demicontinuous.*

The concept of uniformly accretive operators seems to have been first introduced in [1] under the name "uniformly sub-accretive maps". In 2001, Moore and Nnoli defined an uniformly accretive operator by multi-$A : D(A) \to 2^X$ in [17]. There is a difference, i.e., $A$ is single-valued in our definition. The reason is that many continuous multi-valued accretive operators are single-valued actually (see, Theorem 2.1 of [7]). That is, in 2003, He proved that each $\phi$-strongly accretive and lower semi-continuous multi-valued mapping is single-valued. Using Theorem 2.1 in this paper, we can prove that a lower semi-continuous and multi-valued uniformly accretive mapping also is single-valued.

*Remark* 1.2. To approximate the fixed points of operator and the solutions of operator equation by Ishikawa [12] and Mann iterative methods have been studied extensively by many researchers. The Ishikawa and Mann iteration both are important to find the fixed points of operator and the solutions of equation. In 2004, Rhoades [21] and Soltuz point out the equivalence between the convergence of Ishikawa and Mann iterations for some Lipschitzian mappings. Afterward, in 2006, Huang [10] proved that the equivalence holds also for the uniformly pseudo-contractive mapping (so-called generalized strongly $\Phi$-pseudo-contractive mapping in [10]) without Lipschitzian continuity. Therefore, we use Mann iteration to find the fixed points of uniformly pseudo-contractive operator and the solutions of equation with uniformly accretive operator.

(c) Let $K$ be a nonempty convex subset of $X$, and $T : K \to K$ be a operator. For any given $x_0 \in K$ the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \quad (n \geq 0) \tag{1.6}$$

is called *Mann iteration sequence*, where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying some conditions.

So far, many of the results depend on the suppositions that the fixed points of operator or the solutions of operator equations are in existence, i.e., suppose that $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$ or $S(A) = \{x \in X : Ax = f\} \neq \emptyset$. In the point of view of application, the research on existence should come first. Therefore, two problems are considered in this paper. First, whether or not a equation with continuous uniformly accretive operator has a solution, and, is it unique? Next, how to find it if one is in existence? We will to research the problems and to obtain affirmative answers.

The following Lemmas play crucial role in the proofs of our main results.

**Lemma 1.1** ([3]). *If $X$ be a real Banach space, then there exists $j(x+y) \in J(x+y)$ such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \quad \forall x, y \in X. \tag{1.7}$$

**Lemma 1.2** ([24]). *If $X$ be a real $q$-uniformly smooth Banach space, then following inequality holds:*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|x\|^q \tag{1.8}$$

*for all $x, y \in X$ and some real constant $d_q > 1$.*

**Lemma 1.3** ([17]). *Let $\psi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\psi(0) = 0$ such that*

$$\lambda_{n+1}^2 \leq \lambda_n^2 - \alpha_n \psi(\lambda_{n+1}) + o(\alpha_n) \tag{1.9}$$

*where $\lambda_n \in [0,+\infty)$ $(n \geq 0)$, $\alpha_n \in (0,1]$ $(n \geq 0)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{+\infty} \alpha_n = +\infty$. Then $\lambda_n \to 0$ as $n \to \infty$.*

**Lemma 1.4** ([23]). *If a nonnegative real sequence $\{\lambda_n\}_{n=1}^{\infty}$ is satisfied the following recursive inequality*

$$\lambda_{n+1} \leq \lambda_n + c_n, \quad \forall n \geq 0 \tag{1.10}$$

*and $\sum_{n=0}^{+\infty} c_n < +\infty$, then $\lim \lambda_n$ exists.*

## 2. Existence, uniqueness and approximate method for the fixed point of uniformly pseudo-contractive operator

We now turn our attention to the main results.

**Theorem 2.1.** *Let $T : X \to X$ be an uniformly pseudo-contractive operator with bounded range. Then $T$ has an unique fixed point in $X$ if $T$ is continuous or $X$ is uniformly smooth and $T$ is demicontinuous.*

*Proof.* Define $T_n : X \to X$ by

$$T_n x = x - c_n T x \quad \forall x \in X \text{ and } n \geq 1,$$

where $c_n \in (0,1)$ and $c_n \to 1$ as $n \to \infty$. Note that $T$ is uniformly pseudo-contractive, thus

$$\begin{aligned}
&\langle T_n x - T_n y,\ j(x-y)\rangle \\
&= \langle (1-c_n)x + c_n(I-T)x - (1-c_n)y - c_n(I-T)y,\ j(x-y)\rangle \\
&= (1-c_n)\|x-y\|^2 + c_n\langle (I-T)x - (I-T)y,\ j(x-y)\rangle \\
&\geq (1-c_n)\|x-y\|^2 + c_n\psi(\|x-y\|)
\end{aligned} \tag{2.1}$$

for all $x, y \in X$, $n \geq 1$ and a $j(x-y) \in J(x+y)$.

Clearly, $T_n$ is a continuous (demicontinuous) strongly accretive operator if $T$ is a continuous (demicontinuous) uniformly pseudo-contractive operator. It follows from Theorem DM that there exists an $x_n \in X$ such that $T_n x_n = 0$ for any $n \geq 1$.

The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. In fact,

$$\|x_n\| = c_n\|Tx_n\| \le \|Tx_n\| \le \sup_n\{\|Tx_n\|\} =: M \quad \forall n \ge 1.$$

And so,

$$\lim_{n\to\infty} \|x_n - Tx_n\| \le M \lim_{n\to\infty}(1 - c_n) = 0. \tag{2.2}$$

Since $T$ is uniformly pseudo-contractive, that is, $(I-T)$ is uniformly accretive, we have

$$\begin{aligned} \psi(\|x_m - x_k\|) &\le \langle (I-T)x_m - (I-T)x_k,\ j(x_m - x_k)\rangle \\ &\le (\|x_m - Tx_m\| + \|x_k - Tx_k\|)(\|x_m\| + \|x_k\|) \\ &\le 2M(\|x_m - Tx_m\| + \|x_k - Tx_k\|) \end{aligned} \tag{2.3}$$

for any $x_m, x_k \in \{x_n\}$. (2.2) and (2.3) ensure that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Consequently, $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $q \in X$.

If $T$ is continuous, we have

$$Tq = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \frac{1}{c_n} x_n = q;$$

if $T$ is demicontinuous, we have

$$F(Tq) = \lim_{n\to\infty} F(Tx_n) = \lim_{n\to\infty} F\left(\frac{1}{c_n}x_n\right) = \lim_{n\to\infty} F(x_n) = F(q)$$

for all $F \in X^*$, it follows that $Tq = q$.

Suppose that there exists a $q^* \in X$ such that $Tq^* = q^*$. Then

$$\psi(\|q - q^*\|) \le \|q - q^*\|^2 - \langle Tq - Tq^*,\ j(q - q^*)\rangle = 0,$$

which means that $q = q^*$, i.e., $q$ is an unique fixed point of $T$. □

The existence and uniqueness of fixed point of uniformly pseudo-contractive operator have been obtained in Theorem 2.1. We now turn our attention to the following approximative problems.

**Theorem 2.2.** *Let $X$ be an arbitrary real Banach space and $T : X \to X$ an uniformly continuous and uniformly pseudo-contractive operator with bounded range. Suppose the Mann iteration sequence $\{x_n\}$ defined by* (1.6) *with parameter*

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=0}^{+\infty} \alpha_n = +\infty.$$

*Then $T$ has an unique fixed point in $X$ and for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to the fixed point of $T$.*

*Proof.* From Theorem 2.1, $T$ has an unique fixed point $q \in X$.

Putting $M = \sup\{\|Tx - q\| : x \in X\} + \|x_0 - q\|$. For any $n \ge 0$, using

induction, we obtain $\|x_n - q\| \leq M$ for all $n \geq 0$.
Since

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \alpha_n \|(x_n - Tx_n)\| \leq 2M \lim_{n\to\infty} \alpha_n = 0,$$

therefore,

$$e_n := \|Tx_n - Tx_{n+1}\| \to 0 \quad (\text{as } n \to \infty)$$

by the uniformly continuity of $T$.

Using (1.6), (1.7) and (1.2) we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|(1-\alpha_n)(x_n - q) + \alpha_n(Tx_n - q)\|^2 \\
&\leq \|(1-\alpha_n)(x_n - q)\|^2 + 2\alpha_n \langle Tx_n - q,\ j(x_{n+1} - q)\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle Tx_n - Tx_{n+1},\ j(x_{n+1} - q)\rangle \\
&\quad + 2\alpha_n \langle Tx_{n+1} - q,\ j(x_{n+1} - q)\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \|x_{n+1} - q\|^2 \\
&\quad - 2\alpha_n \psi(\|x_{n+1} - q\|) + 2M\alpha_n e_n
\end{aligned} \tag{2.4}$$

for all $n \geq 0$, i.e.,

$$\begin{aligned}
&(1-2\alpha_n)\|x_{n+1} - q\|^2 \\
&\leq (1-2\alpha_n)\|x_n - q\|^2 - 2\alpha_n \psi(\|x_{n+1} - q\|) + M^2\alpha_n^2 + 2M\alpha_n e_n
\end{aligned} \tag{2.5}$$

for all $n \geq 0$. To pick a positive integer $N$ such that $\alpha_n \leq 1/3$ for all $n \geq N$. Thus,

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - 2\alpha_n \psi(\|x_{n+1} - q\|) + 3M\alpha_n(M\alpha_n + 2e_n) \tag{2.6}$$

for all $n \geq N$.

Putting $\|x_n - q\| = \lambda_n$ and $3M\alpha_n(M\alpha_n + 2e_n) = o(\alpha_n)$, (2.6) is changed into

$$\lambda_{n+1}^2 \leq \lambda_n^2 + o(\alpha_n) - 2\alpha_n \psi(\lambda_{n+1}) \quad \forall n \geq N.$$

It follows from the Lemma 1.3 that $\lim_{n\to\infty} \lambda_n = 0$. Hence, $\lim_{n\to\infty} x_n = q$. □

**Theorem 2.3.** *Let $X$ be an uniformly smooth Banach space and $T : X \to X$ a uniformly pseudo-contractive operator with bounded range. If $T$ has a fixed point and the Mann iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by (1.6) satisfying*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{+\infty} \alpha_n = +\infty \quad \text{and} \quad \sum_{n=0}^{+\infty} \alpha_n^2 < +\infty,$$

*then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to unique fixed point of $T$.*

*Proof.* Putting $M = \sup\{\|Tx - q\| : x \in X\} + \|x_0 - q\|$. For any $n \geq 0$, using induction, we obtain $\|x_n - q\| \leq M$ for all $n \geq 0$. Since $X$ is uniformly smooth, so, $X$ is 2-uniformly smooth. using (1.8), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|(1-\alpha_n)(x_n - q) + \alpha_n(Tx_n - q)\|^2 \\
&\leq \|(1-\alpha_n)(x_n - q)\|^2 + 2\alpha_n\langle (Tx_n - q),\ j((1-\alpha_n)(x_n - q))\rangle \\
&\quad + d_2\|\alpha_n(Tx_n - q)\|^2 \\
&\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(1-\alpha_n)\|x_n - q\|^2 \\
&\quad - 2\alpha_n(1-\alpha_n)\psi(\|x_n - q\|) + d_2M^2\alpha_n^2 \\
&\leq \|x_n - q\|^2 - 2\alpha_n(1-\alpha_n)\psi(\|x_n - q\|) + d_2M^2\alpha_n^2
\end{aligned} \tag{2.7}$$

for all $n \geq 0$. Furthermore, we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + d_2M^2\alpha_n^2. \tag{2.8}$$

Putting $\|x_n - q\| = \lambda_n$ and $d_2M^2\alpha_n^2 = b_n$, (2.8) implies that

$$\lambda_{n+1}^2 \leq \lambda_n^2 + b_n \quad \forall n \geq 0.$$

It follows from the Lemma 1.4 that $\lim_{n\to\infty} \lambda_n = \lambda \geq 0$. If $\lambda > 0$ then there exists a positive integer $N_1$ such that $\lambda_n \geq \lambda/2 > 0$ for all $n \geq N_1$. Thus, $\psi(\lambda_n) \geq \psi(\lambda/2) > 0$ for all $n \geq N_1$. To find another positive integer $N \geq N_1$ such that $2\alpha_n \leq 1$ for all $n \geq N$ and to simplify the recursive inequality (2.7), we obtain

$$\begin{aligned}
\lambda_{n+1}^2 &\leq \lambda_n^2 - 2\alpha_n(1-\alpha_n)\psi(\lambda/2) + b_n \\
&\leq \lambda_n^2 - \psi(\lambda/2)\alpha_n + \psi(\lambda/2)\alpha_n(2\alpha_n - 1) + b_n \\
&\leq \lambda_n^2 - \psi(\lambda/2)\alpha_n + b_n
\end{aligned} \tag{2.9}$$

for all $n \geq N$.

In induction, from the recursive inequality (2.9) we obtain a contradiction as follows

$$+\infty = \psi\Big(\frac{\lambda}{2}\Big)\sum_{j=N}^{+\infty}\alpha_j \leq \lambda_N^2 + \sum_{j=N}^{+\infty} b_j < +\infty. \tag{2.10}$$

Hence, $\lim_{n\to\infty}\lambda_n = 0$, i.e., $\lim_{n\to\infty} x_n = q$. □

**Corollary 2.1.** *Let $X$ be an uniformly smooth Banach space and $T : X \to X$ a semicontinuous uniformly pseudo-contractive operator with bounded range. If the Mann iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by (1.6) satisfying*

$$\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=0}^{+\infty}\alpha_n = +\infty \quad \text{and} \quad \sum_{n=0}^{+\infty}\alpha_n^2 < +\infty,$$

*then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to unique fixed point of $T$.*

*Proof.* It follows from Theorem 2.1 and Theorem 2.3 that $T$ has an unique fixed point $q \in X$ and the Mann iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by (1.6) converges strongly to unique fixed point of $T$. □

## 3. The solution of equation with uniformly accretive operator

We now study the existence and uniqueness of the solution of equation with uniformly accretive operator, and to approximate the solution by the Mann iterative sequence in an arbitrary Banach space or an uniformly smooth Banach space respectively.

**Corollary 3.1.** *Suppose that $A : X \to X$ is an uniformly accretive operator and $(I - A)X$ is bounded. For any given $f \in X$ the equation $Ax = f$ has an unique solution in $X$ if $A$ is continuous or $X$ is uniformly smooth and $A$ is demicontinuous.*

*Proof.* We define $T : X \to X$ by $Tx = f + x - Ax$ for all $x \in X$. It is easy to see that $T$ is uniformly pseudo-contractive and continuous (demicontinuous) if $A$ is uniformly accretive and continuous (demicontinuous), and the range of $T$ is bounded if $I - A$ is.

Clearly, $q$ is a fixed point of $T$ if and only if that $q$ is a solution of the equation $Ax = f$. It follows from the Theorem 2.1 that $Ax = f$ has an unique solution in $X$ for any given $f \in X$. □

*Remark* 3.1. The Corollary 3.1 show that the Theorem MT, Theorem DM and Theorem LKXC also are true if $A$ is uniformly accretive and $(I - A)X$ is bounded.

**Corollary 3.2.** *Let $X$ be an arbitrary Banach space. Suppose that $A : X \to X$ is an uniformly continuous and uniformly accretive operator and $(I - A)X$ is bounded. If the Mann iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + X_n - Ax_n) \quad (n \geq 0)$$

*satisfying*

$$\lim_{n\to\infty} \alpha_n = 0 \quad and \quad \sum_{n=0}^{+\infty} \alpha_n = +\infty,$$

*then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to unique solution of equation $Ax = f$ for any given $f \in X$.*

*Proof.* It follows from Theorem 3.1 that $Ax = f$ has an unique solution $x^* \in X$.

We define $T : X \to X$ by $Tx = f + x - Ax$ for all $x \in X$. It is easy to see that $T$ is uniformly pseudo-contractive and uniformly continuous and the range of $T$ is bounded since $I - A$ is.

Clearly, $x^*$ is a fixed point of $T$ if and only if that $x^*$ is a solution of the equation $Ax = f$. It follows from the Lemma 3.3 that $Ax = f$ has an unique solution in $X$ and for arbitrary $x_0 \in X$, the Mann iteration sequence $\{x_n\}$ converges strongly to the solution of equation $Ax = f$. □

**Corollary 3.3.** *Let $X$ be uniformly smooth Banach space. Suppose that $A : X \to X$ is an uniformly accretive operator and $(I - A)X$ is bounded. If the equation $Ax = f$ has a solution in $X$ for any given $f \in X$ and the Mann iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + x_n - Ax_n) \quad (n \geq 0)$$

*satisfying*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{+\infty} \alpha_n = +\infty \quad \text{and} \quad \sum_{n=0}^{+\infty} \alpha_n^2 < +\infty,$$

*then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to unique solution of equation $Ax = f$.*

*Proof.* We define $T : X \to X$ by $Tx = f + x - Ax$ for all $x \in X$. It is easy to see that $T$ is uniformly pseudo-contractive and demicontinuous and the range of $T$ is bounded since $I - A$ is.

Clearly, $x^*$ is a fixed point of $T$ if and only if that $x^*$ is a solution of the equation $Ax = f$. It follows from the Corollary 3.1 that $Ax = f$ has an unique solution in $X$ and for arbitrary $x_0 \in X$, the Mann iteration sequence $\{x_n\}$ converges strongly to the solution of equation $Ax = f$. □

**Corollary 3.4.** *Let $X$ be uniformly smooth Banach space. Suppose that $A : X \to X$ is a semicontinuous uniformly accretive operator and $(I - A)X$ is bounded. If the Mann iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + x_n - Ax_n) \quad (n \geq 0)$$

*satisfying*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{+\infty} \alpha_n = +\infty \quad \text{and} \quad \sum_{n=0}^{+\infty} \alpha_n^2 < +\infty,$$

*then for arbitrary $x_0 \in X$, $\{x_n\}$ converges strongly to unique solution of equation $Ax = f$ for any given $f \in X$.*

*Proof.* It follows from the Corollary 3.1 and Corollary 3.3 that $Ax = f$ has an unique solution $x^* \in X$ for any given $f \in X$ and the Mann iteration sequence $\{x_n\}$ converges strongly to the solution of equation $Ax = f$. □

*Remark* 3.2. Our Theorem 2.1-2.3 and Corollary 2.1, 3.1-3.3 generalize or improve a number of results (for example, Theorem 3.2, 3.4 and 4.2 of [3], Theorem 3.1 and Theorem 3.2 of [5], Theorem 3.1 and 3.2 of [6], Theorem 2.1 and Theorem 2.2 of [8], Theorem 3.1 and Theorem 3.2 of [9], Theorem 1 and Theorem 2 of [11], Theorem 3.1 of [15], Theorem 2.2 and 2.6 of [17], Theorem 2 and Corollary 1 of [18], Theorem 1 and 2 of[19], Theorem 1 of [20], Theorem of [22], Theorem 3.1 and 3.3 of [25], Theorem 3 and Theorem 4 of [26], Theorem 1 and Theorem 2 of [27] and etc.) in the following ways:

(1) The suppositions of $Fix(T) = \{x \in X : Fx = x\} \neq \emptyset$ or $G(A) = \{x \in X : Ax = f\} \neq \emptyset$ are dropped.

(2) The results for the strongly accretive (pseudo-contractive) and $\phi$-strongly accretive (pseudo-contractive) operators are generalized to uniformly accretive (pseudo-contractive) operators.

(3) Since the Mann iterative method has an unique parameter $\alpha_n$, so that it's iterative process is simple than Ishikawa iterative. A prototype for $\{\alpha_n\}$ is $\alpha_n = \frac{1}{n+1}$ $(n \geq 0)$.

**Acknowledgement.** This work is supported by the foundation of Natural Science of P. R. China (No.10561011).

## References

[1] Ya. Alber and A. Notik, *Geometric properties of Banach spaces and approximate methods for solving nonlinear equations,* Soviet Math. Dokl. **29** (1984), 611–615.

[2] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces,* Bull. Amer. Math. Soc. **73** (1967), 875–882.

[3] S. S. Chang, Y. J. Cho, B. S. Lee and S. M. Kang, *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudocontractive mappings in Banach spaces,* J. Math. Anal. Appl. **224** (1998), 149–165.

[4] K. Deimling, *Nonlinear Functional Analysis,* Berlin, Springer-Verlag, 1985.

[5] X. P. Ding, *Iterative process with errors to locally strictly pseudo-contractive maps in Banach spaces,* Comput. Math. Appl. **32(10)** (1996), 91–97.

[6] X. P. Ding, *Iterative process with errors to nonlinear $\Phi$-strongly accretive operator equations in arbitrary Banach spaces,* Comput. Math. Appl. **33(8)** (1997), 75–82.

[7] X. He, *On $\phi$-strong accretive mappings and some set-valued variational problems,* J. Math. Anal. Appl. **277** (2003), 504–511.

[8] N. J. Huang, Y. J. Cho, B. S. Lee and J. S. Jung, *Convergence of iterative processes with errors for set-valued pseudo-contractive and accretive type mappings, in Banach spaces,* Comput. Math. Appl. **40** (2000), 1127–1139.

[9] N. J. Huang, C. J. Gao and X. P. Huang, *New iteration Procedures with errors for Multivalued $\phi$-strongly pseudo-contractions and $\phi$-strongly accretive mappings,* Comput. Math. Appl. **43** (2002), 1381–1390.

[10] Z. Huang, *The equivalence theorems of the convergence between Ishikawa and Mann iterations with errors for generalized strongly successively $\Phi$-pseudo-contractive mappings without Lipschitzian assumptions,* J. Math. Anal. Appl. **329** (2007), 935–947.

[11] N. Hirano and Z. Huang, *Convergence theorems for multivalued $\Phi$-Hemicontractive operators and $\Phi$-strongly accretive operators,* Comput. Math. Appl. **46** (2003), 1461–1471.

[12] S. Ishikawa, *Fixed point and iteration of a nonexpansive mapping in a Banach spaces,* Proc. Amer. Math. Soc. **73** (1976), 65–71.

[13] T. Kato, *Nonlinear semigroups and evdution equations,* J. Math. Soc. Japan. **19** (1967), 508–520.

[14] Z. Q. Liu and S. M. Kang, *Convergence theorems for $\phi$-strongly accretive and $\phi$-hemicontractive operators,* J. Math. Anal. Appl. **253** (2001), 35–49.

[15] Z. Q. Liu, Y. G. Xu and Y. J. Cho, *Iterative solution of nonlinear equations with $\phi$-strongly accretive operators,* Arch. Math. **77** (2001), 508–516.

[16] R. H. Martin, Jr. *A global existence theorem for autonomous differential equations in Banach spaces,* Proc. Amer. Math. Soc. **26** (1970), 304–314.

[17] C. Moore and B. V. C. Nnoli, *Iterative solution of nonlinear equations involving set-valued uniformly accretive operators,* Comput. Math. Appl. **42** (2001), 131–140.

[18] M. O. Osilike, *Ishikawa and Mann iteration methods with errors for nonlinear equations of the accretive type,* J. Math. Anal. Appl. **213** (1997), 91–105.
[19] M. O. Osilike, *Iterative solution of nonlinear equations of the $\phi$-strongly accretive type,* J. Math. Anal. Appl. **200** (1996), 259–271.
[20] M. O. Osilike, *Iterative solutions of nonlinear $\phi$-strongly accretive operator equations in arbitrary Banach spaces,* Nonlinear Analysis **36** (1999), 1–9.
[21] B. E. Rhoades and S. M. Soltuz, *The equivalence between the convergence of Ishikawa and Mann iterations for an asymptotically non-expansive in the intermediate sense and strongly successively pseudo-contractive maps,* J. Math. Anal. Appl. **289** (2004), 266–278.
[22] S. Sharma and B. Deshpande, *Approximation of fixed points and convergence of generalized Ishikawa iteration,* Indian J. pure Appl. Math. **33** (2002), 185–191.
[23] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process,* J. Math. Anal. Appl. **178** (1993), 301–308.
[24] Z. B. Xu and G. F. Roach, *Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces,* J. Math. Anal. Appl. **157** (1991), 189–210.
[25] Y. G. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations,* J. Math. Anal. Appl. **224** (1998), 91–101.
[26] L. C. Zeng, *Error bounds for approximation solutions to nonlinear equations of strongly accretive operators in uniformly smooth Banach spaces,* J. Math. Anal. Appl. **209** (1997), 67–80.
[27] H. Y. Zhou, *Some convergence theorems for the Ishikawa iterative sequences of certain nonlinear operators in uniformly smooth Banach spaces,* Acta. Math. Sinica. **40** (1997), 751–758.

Yuguang Xu
Department of Mathematics, Kunming Teacher's College, Kunshi Road No: 2, Kunming Yunnan 650031, People's Republic of China
*E-mail address*: mathxu5329@163.com

Zeqing Liu
Department of Mathematics, Liaoning Normal University, P. O. Box 200, Dalian, Liaoning 116029, People's Republic of China
*E-mail address*: zeqingliu@dl.cn

Shin Min Kang
Department of Mathematics and the Research Institute of Natural Science, Gyeongsang National University, Jinju 660-701, Korea
*E-mail address*: smkang@gnu.ac.kr