# ON THE NEWTON–KANTOROVICH AND MIRANDA THEOREMS

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ABSTRACT. We recently showed in [5] a semilocal convergence theorem that guarantees convergence of Newton's method to a locally unique solution of a nonlinear equation under hypotheses weaker than those of the Newton-Kantorovich theorem [7]. Here, we first weaken Miranda's theorem [1], [9], [10], which is a generalization of the intermediate value theorem. Then, we show that operators satisfying the weakened Newton-Kantorovich conditions satisfy those of the weakened Miranda's theorem.

# 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution  $x^*$  of equation

F(x) = 0,

where, F is defined on an open convex subset S of  $\mathbb{R}^n$  (n is a positive integer) with values in  $\mathbb{R}^n$ .

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations [4], [6], [7], [8].

Newton's methods:

(1.2) 
$$x_{m+1} = x_m - F'(x_m)^{-1} F(x_m) \quad (m \ge 0), \quad (x_0 \in \mathcal{S})$$

has been used to generate a sequence  $\{x_m\}$  approximating  $x^*$ . A survey of local and semilocal convergence theorems on Newton's method (1.2) can be found in [4], [6], [8], and the references there.

We recently showed in [5] that the famous Newton-Kantorovich condition (see (2.17)) which is the sufficient hypothesis for the convergence of Newton's method can be weakened without any additional computational cost than it is already appearing in the Newton-Kantorovich theorem [8]. Here, we first

Received March 29, 2008; Accepted May 16, 2008.

<sup>2000</sup> Mathematics Subject Classification. 65H10, 65B05, 47H17, 49M15.

Key words and phrases. Newton–Kantorovich theorem, Miranda theorem, Lipschitz/center–Lipschitz condition, Miranda partition/domain/conditions, Newton–Kantorovich hypothesis.

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weaken the generalization of Miranda's theorem, which is an extension of the intermediate value theorem (Theorem 4.3 in [9]). Then, we show that operators satisfying the weakened Newton–Kantorovich conditions satisfy those of the weakened Miranda's theorem. That is our approach extends the applicability of Miranda's theorem.

## 2. Convergence analysis for Newton's method (1.2)

For brevity and to avoid repetitions, we refer the reader to [9] for the terminology introduced here.

Let  $\mathbb{R}^n$  be equipped with a norm denoted by  $\| \cdot \|$ , and  $\mathbb{R}^{n \times n}$  with a norm  $\| \cdot \|$ , such that  $\| M x \| \le \| M \| \| x \|$  for all  $M \in \mathbb{R}^{n \times n}$ , and  $x \in \mathbb{R}^n$ . Choose constants  $c_0, c_1 > 0$ , such that, for all  $x \in \mathbb{R}^n$ :

(2.1) 
$$c_0 || x ||_{\infty} \le || x || \le c_1 || x ||_{\infty}$$

since all norms on finite dimentional spaces are equivalent. Set:

(2.2) 
$$c = \frac{c_0}{c_1} \le 1.$$

**Definition 2.1.** Let  $S \in \mathbb{R}^n$  be an open convex set, and let  $G : S \longrightarrow \mathbb{R}^n$ , be a differentiable operator on S. Let  $x_0 \in S$ , and assume:

(2.3) 
$$G'(x_0) = \mathcal{I}$$
 (the identity matrix);

there exists  $\eta \geq 0$ , such that

$$(2.4) || G(x_0) || \le \eta;$$

there exist  $\ell_0 \geq 0$ , such that

(2.5) 
$$|| G'(x) - G'(x_0) || \le \ell_0 || x - x_0 ||$$
 for all  $x \in S$ 

Define

$$(2.6) h_0 = \ell_0 \ \eta.$$

We say that G satisfies the weak center-Kantorovich condition at  $x_0$  if:

$$(2.7) h_0 \le \frac{1}{2}.$$

We also say that G satisfies the strong center-Kantorovich condition at  $x_0$  if:

$$h_0 \le \frac{c^2}{2}$$

Moreover, define: (2.9)

$$r_1 = \frac{c - \sqrt{c^2 - 2h_0}}{\ell_0}, \quad r_2 = \frac{c + \sqrt{c^2 - 2h_0}}{\ell_0}, \quad \text{and} \quad R = [r_1, r_2] \quad \text{for} \quad \ell_0 \neq 0.$$

Furthermore, if  $\ell_0 = 0$ , define:

(2.10) 
$$r_1 = \frac{\eta}{c}$$
, and  $R = [0, \infty)$ .

As in [8], we need to introduce certain concepts. Let r > 0,  $x_0 \in \mathbb{R}^n$ , and define:

$$U(r) = \{z \in \mathbb{R}^{n} : || z || \le r\}, \qquad U(x_{0}, r) = \{x = x_{0} + z \in \mathbb{R}^{n} : z \in U(r)\},\$$

$$U_{k}^{+}(r) = \{z \in \mathbb{R}^{n} : || z || = r, \ z_{k} = || z ||_{\infty}\},\$$

$$U_{k}^{-}(r) = \{z \in \mathbb{R}^{n} : || z || = r, \ z_{k} = - || z ||_{\infty}\},\$$

$$U_{k}^{+}(x_{0}, r) = \{x = x_{0} + z \in \mathbb{R}^{n} : z \in U_{k}^{+}(r)\},\$$

$$U_{k}^{-}(x_{0}, r) = \{x = x_{0} + z \in \mathbb{R}^{n} : z \in U_{k}^{-}(r)\},\$$
for all  $k = 1, 2, \cdots, n$ . From now on we set:  
(2.11) 
$$G(x) = F'(x_{0})^{-1} F(x) \qquad (x \in S).$$

We show the main result which states that if G satisfies (2.8) for any norm,

then G satisfies the Miranda conditions on an appropriate scalar multiple of the unit ball in that norm.

**Theorem 2.2.** Let  $G : S \longrightarrow \mathbb{R}^n$  be a differentiable operator, defined on an open convex subset of  $\mathbb{R}^n$ . Assume that G satisfies the strong center-Kantorovich condition given by (2.8). Then, for any  $r \in \mathbb{R}$ , with  $U(x_0, r) \subset S$ , the following hold:

(2.12) (a) 
$$U = U(r) = U(x_0, r)$$
 s a Miranda domain [9],

and

(2.13) 
$$U_1 = U_1(r) = \left\{ U_1^+(x_0, r), \ U_1^-(x_0, r), \ \cdots, \ U_n^+(x_0, r), \ U_n^-(x_0, r) \right\}$$

is a Miranda partition [9] of the boundary  $\partial U$ . It is a canonical Miranda partition [9] for r > 0, and a trivial Miranda domain for r = 0;

(2.14) (b) 
$$G_k(x) \ge 0 \quad \text{for all } x \in U_k^+(x_0, r), \quad k = 1, 2, \cdots, n$$
$$G_k(x) \le 0 \quad \text{for all } x \in U_k^-(x_0, r), \quad k = 1, 2, \cdots, n;$$

(c) G satisfies the Miranda conditions;

(d) if  $G(x_0) = 0$ , and  $\ell_0 > 0$ , then G satisfies the Miranda conditions for any  $r \in \left[0, \frac{2c}{\ell_0}\right]$ , such that  $U(x_0, r) \subseteq S$ .

*Proof.* The proof follows as in Theorem 4.3 in [9], but by using (2.5), (2.8) instead of Lipschitz condition:

$$(2.15) \|G'(x) - G'(y)\| \le \ell \|x - y\| for all x, y \in \mathcal{S},$$

and strong Kantorovich condition at  $x_0$ :

$$(2.16) h = \ell \ \eta \le \frac{c^2}{2},$$

respectively.

Remark 2.3. Note that:

(2.17) 
$$h \le \frac{1}{2}$$

is the famous Kantorovich hypothesis (see the Kantorovich theorem for the semilocal convergence of Newton's method [2]–[9]).

Remark 2.4. If  $\ell = \ell_0$ , then our Theorem 2.2 becomes Theorem 4.3 in [9]. Moreover, if  $\parallel . \parallel$  is the maximum norm, then it becomes Theorem 3 [1]. However in general:

$$(2.18) \qquad \qquad \ell_0 \le \ell.$$

Then, we have:

$$(2.19) h \le \frac{c^2}{2} \Longrightarrow h_0 \le \frac{c^2}{2}.$$

Similarly, the Kantorovich condition (2.17) is such that:

$$(2.20) h \le \frac{1}{2} \Longrightarrow h_0 \le \frac{1}{2},$$

but not vice verca unless if  $\ell = \ell_0$ . If strict inequality hold in (2.18), and condition (2.16) or (2.17) are not satisfied, then the conclusions of Theorem 4.3 in [9] or Theorem 1 in [7] respectively do not necessarily hold. However, if (2.7) holds, the conclusions of our Theorem 2.2 hold. Furthemore as the following example demonstrates  $\frac{\ell}{\ell_0}$  can be arbitrarily large in general.

**Example 2.5.** Let  $x_0 = 0$ , and define function F on  $\mathbb{R}$  by:

(2.21) 
$$F(x) = a_0 x + a_1 + a_2 \sin e^{a_3 x}$$

where  $a_i$ , i = 1, 2, 3 are given parameters. Using (2.5), (2.15) and (2.21), it can easily be seen that for  $a_3$  large and  $a_2$  sufficiently small,  $\frac{\ell}{\ell_0}$  can be arbitrarily large. That is (2.7) ((2.23) (2.24), see Remark 2.6) can be satisfied but not (2.17) (or (2.16)).

Remark 2.6. According to the Kantorovich theorem [8], condition (2.17) guarantees the convergence of Newton's method (1.2) to  $x^*$ . In particular, if strict inequality holds in (2.17), the convergence is quadratic (only linear in case of equality in (2.17)). However, this is not the case for condition (2.7) (only linear). To rectify this and still use a condition weaker than (2.17) (or (2.16)) define:

$$(2.22) \overline{h} = \ell_0 \ \eta$$

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We showed in [5] that if:

 $(2.23) \overline{h} \le \alpha,$ 

where

$$\alpha = \frac{\sqrt{\ell^2 + 8 \,\ell_0 \,\ell} - \ell}{\sqrt{\ell^2 + 8 \,\ell_0 \,\ell} + \ell},$$

then, finer conclusions than the ones given by the Kantorovich theorem hold [8]. Condition

$$(2.24) \overline{h} \le \frac{c^2}{2}$$

can now replace (2.16) in Theorem 4.3 in [9].

If  $\ell_0 = \ell$ , condition (2.24) reduces to (2.16). Otherwise (2.24) is an improvement over condition (2.16), obtained under the same computational cost, since in practice, the computation of constant  $\ell$  requires that of  $\ell_0$  (see also (2.20) with  $h_0$  replaced by  $\overline{h}$ ).

This study further improves the results reported in [2].

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