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STABILITY OF THE $\overline{\partial}$-ESTIMATES AND THE MERGELYAN PROPERTY FOR WEAKLY q-CONVEX MANIFOLDS

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Abstract. Let $r \geq q$. We get the stability of the estimates of the $\overline{\partial}$-Neumann problem for (p,r)-forms on a weakly q-convex complex submanifold. As a by-product of the stability of the $\overline{\partial}$-estimates, we get the Mergelyan approximation property for (p,r)-forms on a weakly q-convex complex submanifold which satisfies property (P).

1. Introduction and preliminaries

In [6], Ho introduced the nortions of weak q-convexity and q-subharmonicity and he treated L^2-estimates and existence theorems for solutions of the $\overline{\partial}$-equation on weakly q-convex domains. Thus we know that the weak q-convexity is proper for the study of the $\overline{\partial}$-equation for (p,q)-forms.

Let M be a complex manifold of dimension n. Let $\Omega \subset\subset M$ be an open submanifold with a C^3 boundary. By applying the Gram-Schmidt process in a coordinate patch U, we can construct forms $\omega^1, \dots, \omega^n$, which for all z are orthonormal basis of $\Lambda_z^{1,0}(U)$. Furthermore we can choose $\omega^n = \sqrt{2}\,\partial\rho$ on $b\Omega$, where ρ is a boundary-defining function satisfying $|d\rho| = 1$ on $b\Omega$. We shall use Hörmander's method of weighted estimates for $\overline{\partial}$ ([7], [8]). Let $\varphi \in C^1(\overline{\Omega})$ be a real-valued function and define

$$(\Phi, \Psi)_\varphi = \int_\Omega \langle \Phi, \Psi \rangle_z e^{-\varphi} dV, \quad \Phi, \Psi \in \Lambda^{p,q}(U),$$

and $\|\Phi\|_\varphi^2 = (\Phi, \Phi)_\varphi$. We then define $L^2_{(p,q)}(\Omega, \varphi)$ as the space of all (p,q)-forms Φ such that $\|\Phi\|_\varphi < \infty$. If $q \geq 1$, the operator $\overline{\partial}$ defines, in the weak sense, closed densely defined operators

$$L^2_{(p,q-1)}(\Omega, \varphi) \xrightarrow{T} L^2_{(p,q)}(\Omega, \varphi) \xrightarrow{S} L^2_{(p,q+1)}(\Omega, \varphi).$$

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By T^* we shall mean the adjoint of T. If $\varphi \in C^2(U)$, then $\varphi_{jk}(z)$, $j,k = 1,\dots,n$, is defined by

$$\partial\overline{\partial}\varphi(z) = \sum_{j,k=1}^{n} \varphi_{j,k}(z)\omega^j \wedge \overline{\omega}^k.$$

If $\Phi = \sum_{I,J} \Phi_{I,J}\omega^I \wedge \overline{\omega}^J$ is a (p,q)-form, then we define

$$H_q(\varphi)(z,\Phi) = \sum_{I,K}\sum_{j,k=1}^{n} \varphi_{j,k}(z)\Phi_{I,jK}\overline{\Phi_{I,kK}}.$$

Let $L_1,\dots,L_n$ be a basis of $T^{1,0}(U)$ that is dual to $\omega^1,\dots,\omega^n$. If $\chi \in C^2(\mathbb{R})$ be a convex increasing function, then we get

$$\begin{aligned}(1.1)\quad \frac{1}{2}\sum_{I,J}\sum_{j=1}^{n}\|\overline{L}_j\Phi_{I,J}\|^2_{\chi(\varphi)} &+ \sum_{I,J}\int_{U\cap\Omega}\chi'(\varphi)\left(\sum_{j,k=1}^{n}\varphi_{j,k}\Phi_{I,jK}\overline{\Phi_{I,kK}}\right)e^{-\chi(\varphi)}dV \\ &+ \sum_{I,K}\sum_{j,k=1}^{n}\int_{U\cap b\Omega}\rho_{jk}\overline{\Phi_{I,kK}}e^{-\chi(\varphi)}dS \\ &\le \|T^*\Phi\|^2_{\chi(\varphi)} + 2\|S\Phi\|^2_{\chi(\varphi)} + C\|\Phi\|^2_{\chi(\varphi)}.\end{aligned}$$

To get the basic estimate we do not need the full conditions that φ is strongly plurisubharmonic and that Ω is pseudoconvex, but that the following estimates

$$\sum_{I,K}\sum_{j,k=1}^{n}\varphi_{j,k}(z)\Phi_{I,jK}\overline{\Phi_{I,kK}} > C|\Phi|^2 \quad \text{on } U\cap\overline{\Omega},$$

and that

$$\sum_{I,K}\sum_{j,k}^{n}\rho_{jk}\Phi_{I,jK}\overline{\Phi_{I,kK}} \ge 0 \quad \text{for } \Phi \in D_{T^*}\cap D_S \text{ on } U\cap b\Omega.$$

Thus we introduce the following definitions.

Definition 1.1. We say that φ is *q-subharmonic* in a set U if $H_q(\varphi)(z,\Phi) \ge 0$ for all (p,q)-forms Φ on U. If it is strictly positive, we say that φ is *strongly q-subharmonic* in U.

Definition 1.2. Let ρ be a boundary defining function of Ω. We say that $b\Omega$ is *weakly q-convex* in $U\cap b\Omega$ if at every point $z\in U\cap b\Omega$ we have $H_q(\rho)(z,\Phi)\ge 0$ for all (p,q)-forms Φ on U such that $\sum_j^n (L_j\rho)\Phi_{I,jK} = 0$ on $U\cap b\Omega$.

The nortions of weak q-convexity and q-subharmonicity in Theorems 1.1 and 1.2 are invariant under unitary change of coordinates [6]. Thus we get the following result:

Lemma 1.3. *The inequality* $H_q(\varphi)(z,\Phi) \geq 0 \quad (\ \textit{resp.} \quad \geq C|\Phi|^2\ )$ *holds if and only if the inequality*

$$\sum_{j=1}^{q} H_1(\varphi)(z,t^j) \geq 0 \quad (\ \textit{resp.} \quad \geq C\ )$$

holds for every sets of vectors $t^1,\dots,t^q$ *that satisfy* $\langle t^j,t^k\rangle_z = \delta_{jk}$.

Lemma 1.4. *The q-subharmonicity and the weak q-convexity imply the* $(q+1)$*-subharmonicity and the weak* $(q+1)$*-convexity, respectively.*

Proof. This follows from the fact that

$$\sum_{j=1}^{q+1} H_1(\varphi)(z,t^j) = \frac{1}{q}\left\{\sum_{j=1}^{q+1}\sum_{i=1,i\neq j}^{q+1} H_1(\varphi)(z,t^i)\right\}.$$

□

By using a partition of unity, the estimate (1.1) leads to the following proposition. In all that follows we let r be a nonnegative integer with $r \geq q$.

Proposition 1.5. *Let* Ω *be a weakly q-convex submanifold, let* $\varphi \in C^3(\overline{\Omega})$ *be a function such that* $\Omega_{c_0} = \{z \in \overline{\Omega};\varphi(z) < c_0\} \subset\subset \Omega$ *and strongly q-subharmonic in the set* $\overline{\Omega} \cap \Omega_{c_0}$. *Then there exist a compact subset* K *in* Ω_{c_0} *and a constant* C *such that for all convex increasing functions* $\chi \in C^2(\mathbb{R})$

$$\int \chi'(\varphi)|\Phi|^2 e^{-\chi(\varphi)}dV \leq C(\|T^*\Phi\|^2_{\chi(\varphi)} + \|S\Phi\|^2_{\chi(\varphi)} + \|\Phi\|^2_{\chi(\varphi)}) \tag{1.2}$$

for all $\Phi \in D_{T^*} \cap C^\infty(\overline{\Omega})$ *with support in* CK.

From (1.2) it follows that for large t,

$$\int_{CK} |\Phi|^2 e^{-t\varphi}dV \leq \|T^*\Phi\|^2_{t\varphi} + \|S\Phi\|^2_{t\varphi} + \int_K |\Phi|^2 e^{-t\varphi}dV. \tag{1.3}$$

We shall now derive from (1.3) the L^2-estimate and the existence theorem for solutions of the $\overline{\partial}$-equation (see [7, Theorem 3.4.1 and Theorem 1.1.4]).

Theorem 1.6. *Let the hypotheses of Proposition 1.5 be fulfilled. Let* $\{\Phi_j\}$ *be a sequence in* $D_{T^*} \cap D_S$ *such that for large t,* $\|\Phi_j\|_{t\varphi}$ *is bounded and* $T^*\Phi_j \to 0, S\Phi_j \to 0$ *in* $L^2_{(p,r\mp1)}(\Omega,t\varphi)$*, respectively. Then one can select a strongly convergent subsequence and there exists* $C > 0$ *such that*

(a) $\|\Phi\|^2_{t\varphi} \leq C(\|T^*\Phi\|^2_{t\varphi} + \|S\Phi\|^2_{t\varphi}),\ \Phi \in D_{T^*} \cap D_S,\ \Phi \perp N_{T^*} \cap N_S,$
(b) R_T *is closed,*
(c) R_T *has finite codimension in* N_S.

Thus, by [7, Theorem 1.1.4], if $S\Psi = 0$ *and* $\Psi \perp N_{T^*} \cap N_S$*, then we can find* $\Phi \in D_T$ *so that* $T\Phi = \Psi$ *and* $\|\Phi\|_{t\varphi} \leq C\|\Psi\|_{t\varphi}$ *where* C *is a constant independent of* Ψ.

We denote the quotient space

$$\overline{H}^{(p,r)}(\Omega) = R_T/N_S \cong N_{T^*} \cap N_S.$$

We also define

$$H^{(p,r)}(\Omega) = \frac{\{\Phi \in L^2_{(p,r)}(\Omega, loc) : \overline{\partial}\Phi = 0\}}{L^2_{(p,r)}(\Omega, loc) \cap \{\overline{\partial}\Phi; \Phi \in L^2_{(p,r-1)}(\Omega, loc)\}}.$$

Under the hypotheses of Proposition 1.5, we get the following isomorphism theorem, which is due to Hörmander [7, Theorem 3.4.8]. The proof essentially depends on the estimate (1.2).

Theorem 1.7. *The restriction homomorphism*

$$\overline{H}^{(p,r)}(\Omega) \to \overline{H}^{(p,r)}(\Omega_{c_0})$$

is an isomorphism.

Lemma 1.8. *Let $\varphi \in C^3(\overline{\Omega})$ be a function such that $\Omega_{c_0} = \{z \in \overline{\Omega}; \varphi(z) < c_0\} \subset\subset \Omega$ and strongly q-subharmonic on $\overline{\Omega} \cap C\Omega_{c_0}$. Then there exists a function $\psi \in C^3(\Omega)$ such that*

(1) *ψ is strongly q-subharmonic on $\overline{\Omega} \cap \Omega_{c_0}$,*
(2) *$\{z \in \overline{\Omega}; \varphi(z) < c_0\} = \{z \in \Omega; \psi < c_0\}$,*
(3) *$\{z \in \overline{\Omega}; \psi(z) < c\} \subset\subset \Omega$ for every $c \in \mathbb{R}$.*

Proof. Choose $\delta < 0$ such that $\Omega_{c_0} \subset\subset \{z \in \Omega; \rho(z) < \delta\}$. Let $\chi \in C^3(\mathbb{R})$ be a convex increasing function such that $\chi(\tau) = \frac{2}{3}\delta$ for $\tau < \delta$ and $\chi(\tau) = \tau$ for $\tau > \frac{\delta}{2}$. Set

$$\psi = C\varphi + \log\left(\frac{\frac{2}{3}\delta}{\chi \circ r}\right) - (C-1)c_0 \in C^\infty(\Omega).$$

The weak q-convexity of Ω says that $\sum_{j=1}^q H_1(\rho)(z, t^j) \geq 0$ for every sets of vectors $t^1, \ldots, t^q$, where $t^j = (t_1^j, \ldots, t_n^j)$, which satisfy $\sum_{i=1}^n t_i^j(L_i\rho) = 0$ ($j = 1, \ldots, q$) and $\langle t^j, t^k \rangle_z = \delta_{jk}$ ($j, k = 1, \ldots, q$) on $b\Omega$. We write $t^j = t_T^j + t_N^j$ where t_T^j is the tangent vector and t_N^j is the normal vector at z of the surface $\rho = \rho(z)$. As $\rho \in C^3$ it follows that if $|\delta|$ is sufficiently small, there is a constant $C_1 > 0$, so that

$$\sum_{j=1}^q H_1(\rho)(z, t_T^j) \geq -C_1|\rho(z)|(\sum_{j=1}^q |t_T^j|^2) \quad \text{for } \frac{\delta}{2} < \rho(z) < 0. \tag{1.4}$$

The bilinearity of the Levi form implies

$$\begin{aligned} H_1(\rho)(z, t^j) &= H_1(\rho)(z, t_T^j) + \mathcal{O}(|t_T^j||t_N^j|) + \mathcal{O}(|t_N^j|^2) \\ &= H_1(\rho)(z, t_T^j) + \mathcal{O}(|t_N^j|) \end{aligned}$$

for $\frac{\delta}{2} < \rho(z) < 0$. Since $|t_N^j| = \mathcal{O}(|\sum_{i=1}^n t_i^j(L_i\rho)|)$, we obtain from (1.4) that

$$\sum_{j=1}^q H_1(\rho)(z,t^j) \geq -C_1|\rho(z)| - C_2 \sum_{j=1}^q |\sum_{i=1}^n t_i^j(L_i\rho)| \quad \text{for } \frac{\delta}{2} < \rho(z) < 0. \tag{1.5}$$

Also

$$\begin{aligned} H_1\left(\log\left(\frac{\frac{2}{3}\delta}{\chi\circ r}\right)\right)(z,t^j) &= \frac{1}{|\chi(\rho(z))|}\{\chi''(\rho(z))|\sum_{i=1}^n t_i^j(L_i\rho)|^2 \\ &\quad + \chi'(\rho(z))H_1(\rho)(z,t^j)\} + \left|\frac{\chi'(\rho(z))}{\chi(\rho(z))}\right|^2 |\sum_{i=1}^n t_i^j(L_i\rho)|^2 \\ &\geq \left|\frac{\chi'(\rho(z))}{\chi(\rho(z))}\right| H_1(\rho)(z,t^j) + \left|\frac{\chi'(\rho(z))}{\chi(\rho(z))}\right|^2 |\sum_{i=1}^n t_i^j(L_i\rho)|^2 \\ &= \frac{1}{|(\rho(z))|}H_1(\rho)(z,t^j) + \frac{1}{|(\rho(z))|^2}|\sum_{i=1}^n t_i^j(L_i\rho)|^2 \end{aligned}$$

where $\frac{\delta}{2} < \rho(z) < 0$. Thus we get

$$\begin{aligned} &\sum_{j=1}^q H_1\left(\log\left(\frac{\frac{2}{3}\delta}{\chi\circ\rho}\right)\right)(z,t^j) \\ &\qquad \geq \frac{1}{|\rho(z)|}\sum_{j=1}^q H_1(\rho)(z,t^j) + \frac{1}{|\rho(z)|^2}\sum_{j=1}^q |\sum_{i=1}^n t_i^j(L_i\rho)|^2 \end{aligned} \tag{1.6}$$

where $\frac{\delta}{2} < \rho(z) < 0$. From (1.5) and (1.6) it follows that

$$\sum_{j=1}^q H_1\left(\log\left(\frac{\frac{2}{3}\delta}{\chi\circ\rho}\right)\right)(z,t^j) \geq -(C_1 + C_2^2), \quad \frac{\delta}{2} < \rho(z) < 0.$$

Let $M = \inf\{H_1(\log(\frac{2}{3}\delta/\chi\circ\rho))(z,t)\ ;\ \rho(z) \leq \frac{\delta}{2},\ \varphi(z) \geq c_0,\ |t| = 1\} > -\infty$. Since φ is strongly q-subharmonic on $\{z \in \overline{\Omega}; \varphi(z) \geq c_0\}$,

$$\sum_{j=1}^q H_1(\varphi)(z,t^j) \geq C_3 \quad \text{for } z \text{ with } \varphi(z) \geq c_0\ .$$

Choose C so that

$$CC_3 \geq \max\{C_1 + C_2^2,\ q|M|\}.$$

Then $\psi = C\varphi + \log(\frac{2}{3}\delta/\chi\circ\rho) - (C-1)c_0 \in C^\infty(\Omega)$ is strongly q-subharmonic on $\{z \in \overline{\Omega}; \varphi(z) \geq c_0\}$. It is easy to verify that properties (2) and (3) hold. This completes the proof. $\square$

From Lemma 1.8 and [7, Theorem 3.4.9], it follows that the homomorphism $H^{(p,r)}(\Omega) \to \overline{H}^{(p,r)}(\Omega_{c_0})$ is an isomorphism. Thus we get the following theorem.

Theorem 1.9. *The homomorphism $H^{(p,r)}(\Omega) \to \overline{H}^{(p,r)}(\Omega)$ is an isomorphism.*

2. Stability results

In all that follows, we shall assume that $\Omega \subset\subset M$ is an open weakly q-convex submanifold with a smooth boundary defining function ρ. Also, we suppose that there exists a function $\varphi \in C^\infty(\overline{\Omega})$ which is strongly q-subharmonic in a neighborhood of $b\Omega$. In this case, by the following lemma, we can construct such a function described in Proposition 1.5.

Lemma 2.1. *There exist a function $\psi \in C^\infty(\overline{\Omega})$, a neighborhood W of $b\Omega$, and a constant $c_0 \in \mathbb{R}$ such that*

(1) *ψ is strongly q-subharmonic in W,*
(2) *$\{z \in \Omega \cup W; \psi(z) < c_0\} \subset\subset \Omega$,*
(3) *$\{z \in \overline{\Omega}; \psi(z) \geq c_0\} \subset\subset W \cap \overline{\Omega}$.*

Proof. Let U be a neighborhood of $b\Omega$ such that φ is smooth and strongly q-subharmonic in U. With both $\delta > 0$ and $C > 0$, set $\varphi_\delta = -\log(2\delta - \rho) + C\varphi$. Let $S(\delta) = \{z \in M; 0 \leq \rho(z) \leq \delta\}$. By the similar argument as in Lemma 1.8, we can prove that for small δ and for large C, φ_δ is strongly q-subharmonic in $(U \cap \overline{\Omega}) \cup S(\delta)$. Observe that there is a constant γ independent of δ, such that if $\varphi_\delta(z) \geq \gamma$, then $z \in U$. Let $\chi \in C^\infty(\mathbb{R})$ be a convex increasing function such that $\chi(\tau) = \gamma + 1$ for $\tau \leq \gamma$ and $\chi(\tau) = \tau$ for $\tau \geq \gamma + 2$. Set

$$\psi_\delta(z) = \frac{1}{C}\{(\chi \circ \varphi_\delta)(z) + \log(2\delta)\}.$$

Now we choose small $\delta > 0$ so that $-\log(2\delta) + C\varphi \geq \gamma + 2$ on $b\Omega$. By the similar method as in Lemma 1.8, we can prove that ψ_δ is strongly q-subharmonic and $\varphi \leq \psi_\delta$ in $S(\delta)$. By the continuity of second derivatives of ψ_δ, there is a neighborhood V (in the relative topology of $\overline{\Omega}$) of $b\Omega$ such that $H_q(\psi_\delta)(z, f)$ is bounded below in V by a fixed positive constant independent of δ. Thus ψ_δ is strongly q-subharmonic on $W = V \cup S(\delta)$ and W is a neighborhood of $b\Omega$. Choose $c_0 < \inf_{S(\delta)} \varphi$. Then if we choose sufficiently small $\delta > 0$ so that

$$\sup_{\overline{\Omega} - V} \psi_\delta(z) < c_0,$$

then $\{z \in \Omega \cup S(\delta); \psi_\delta(z) < c_0\} \subset\subset \Omega$ and $\{z \in \overline{\Omega}; \psi_\delta(z) \geq c_0\} \subset\subset V$. This completes the proof. □

Definition 2.2. A family $\{\Omega_\tau\}_{0 \leq \tau}, \Omega_\tau \subset\subset M$, of complex submanifolds with C^∞ boundary defining functions ρ_τ, is said to be a *continuous family of diffeomorphic complex manifolds with diffeomorphisms* $d_\tau : \Omega_\tau \to \Omega_0$, if

(1) $d_0 : \Omega_0 \to \Omega_0$ is an identity,
(2) the complex structures on Ω_τ are C^∞ close to the complex structure on Ω_0 as $\tau \to 0$,
(3) ρ_τ and all of its derivatives depend continuously on τ in C^∞-topology,
(4) the diffeomorphisms d_τ^{-1} depend continuously on τ in C^∞-topology.

Theorem 2.3. *Let $\{\overline{\Omega}_\tau\}_{0\leq\tau}$ be a continuous family of diffeomorphic compact weakly q-convex complex manifolds in M with C^∞ defining function ρ_τ. Suppose that there is $\varphi \in C^\infty(\overline{\Omega}_0)$ which is strongly q-subharmonic in a neighborhood of $b\Omega_0$. Then there is τ_0 such that $H^{(p,r)}(\Omega_\tau) \cong H^{(p,r)}(\Omega_0)$ for all $0 \leq \tau \leq \tau_0$.*

Proof. By Lemma 2.1, we may assume that :

(2.1) ϕ is strongly q-subharmonic in a neigborhood W of $b\Omega_0$,

(2.2) $\{z \in \Omega_0 \cup W; \phi(z) < c_0\} \subset\subset \Omega_0$,

(2.3) $\{z \in \overline{\Omega}_0; \phi(z) \geq c_0\} \subset\subset W \cap \overline{\Omega}_0$.

Thus by Theorems 1.7 and 1.9, it follows that

$$H^{(p,r)}(\Omega_0) \cong \overline{H}^{(p,r)}(\Omega_{c_0}),$$

where $\Omega_{c_0} = \{z \in \overline{\Omega}_0; \varphi(z) < c_0\}$. Since $\{\overline{\Omega}_\tau\}_{0\leq\tau}$ is a continuous family, there are $\delta < 0$ and $0 < \tau_0$ such that $\Omega_{c_0} \subset\subset \{z \in \overline{\Omega}_\tau; \rho_\tau(z) < \delta\}$ for all $0 \leq \tau \leq \tau_0$. Let χ be such a function as in Lemma 1.8. Set

$$\varphi_\tau = C\varphi + \log\left(\frac{\frac{2}{3}\delta}{\chi \circ \rho_\tau}\right) - (C-1)c_0.$$

By the same argument as in Lemma 1.8, we can prove that if C is sufficiently large, then $\varphi_\tau \in C^\infty(\Omega_\tau)$ is strongly q-subharmonic on $\Omega_\tau \cap C\Omega_{c_0}$. The above formula φ_τ and (2.3) shows that $\Omega_{c_0} = \{z \in \overline{\Omega}_\tau; \varphi_\tau(z) < c_0\}$. Clearly, φ_τ is an exhaustion function for Ω_τ. By Theorems 1.7 and 1.9, we get that $H^{(p,r)}(\Omega_\tau) \cong \overline{H}^{(p,r)}(\Omega_{c_0})$. Thus we completes the proof. □

We define the Sobolev space $H^m_{(p,r)}(\Omega)$ of (p,r)-forms Φ in Ω with respect to the norm $\|\Phi\|_{m,t}$ which is defined by the L^2-norm with the density $e^{-t\varphi}$. The estimate for the Neumann operator N is given in the following estimate. For each nonnegative integer $m \geq 0$, there exist constants $T_m, C_{m,t}$, and $C'_{m,t}$ such that for all $t \geq T_m$, the following estimate holds:

$$\|\Phi\|^2_{m,t} \leq C_{m,t}\|\Box\Phi\|^2_{m,t} + C'_{m,t}\|\Phi\|^2_{m,t}, \quad \Phi \in D_\Box. \tag{2.4}$$

The proof is quite similar to that of Catlin ([1], [2]) who proves the same result on pseudoconvex submanifolds. If Ψ is a (p,r)-form with $S\Psi = 0$, then $\Phi = T^*N\Psi$ is the unique solution of $T\Phi = \Psi$ which is orthogonal to the null space of T. By the estimate (2.4), we know that

$$\|\Phi\|_{m-1,t} \leq C_m(t)\|\Psi\|_{m,t}.$$

But actually we can show that

$$\|\Phi\|_{m,t} \leq C_m(t)\|\Psi\|_{m,t}. \tag{2.5}$$

The proof is the same as was proved in the proof of Kohn [9] and Catlin [1].

Let $\mathcal{O}_{(p,r)}(\Omega)$ be the set of (p,r)-forms in Ω which satisfy the equation $\overline{\partial}\Phi = 0$. Then, by the estimate (2.5), we get the following result. The proof is the same as in [1, Proposition 3.1.4].

Lemma 2.4. *Let m be a nonnegative integer. Then $\mathcal{O}_{(p,r)}(\Omega) \cap C^{\infty}_{(p,r)}(\overline{\Omega})$ is dense in $\mathcal{O}_{(p,r)}(\Omega) \cap H^m_{(p,r)}(\Omega)$.*

Remark 2.5. Let $\{\overline{\Omega}_\tau\}_{0\leq\tau}$ be a continuous family of diffeomorphic complex manifolds. Let $d_\tau : \overline{\Omega}_\tau \to \overline{\Omega}_0$ be C^∞ diffeomorphisms. It is clear that $(d_\tau^{-1})_* T^{1,0}\overline{\Omega}_0$ is an almost complex structure if d_τ^{-1} is sufficiently close to identity, and the almost complex structures $(d_\tau^{-1})_* T^{1,0}\overline{\Omega}_0$ and $T^{1,0}\overline{\Omega}_\tau$ are C^∞ close as d_τ^{-1} becomes close to identity. Thus we get the following: if $\Phi_\tau \in D_{T^*_\tau} \cap C^\infty_{(p,r)}(\overline{\Omega}_\tau)$, then $(d_\tau^{-1})^*\Phi_\tau \in D_{T^*} \cap C^\infty_{(p,r)}(\overline{\Omega}_0)$, and

$$\|T^*((d_\tau^{-1})^*\Phi_\tau)\|_{t\varphi} = \|T^*_\tau \Phi_\tau\|_{t\varphi} + o(\tau), \tag{2.6}$$

$$\|S((d_\tau^{-1})^*\Phi_\tau)\|_{t\varphi} = \|S_\tau \Phi_\tau\|_{t\varphi} + o(\tau), \tag{2.7}$$

where $o(\tau)$ does not depend on Φ_τ.

In Theorem 1.6, we got that for large t

$$\|\Phi\|^2_{t\varphi} \leq C(\|T^*\Phi\|^2_{t\varphi} + \|S\Phi\|^2_{t\varphi}), \text{ if } \Phi \in D_{T^*} \cap D_S,\ \Phi \perp H^{(p,r)}(\Omega). \tag{2.8}$$

However, in most of applications, it is essential that the constant C is stable for small perturbations of the manifold Ω. In [4], the author get the stability result, under the perturbations of the pseudoconvex manifold, for the estimate (2.8). But we can draw the same stability result in the case of the weakly q-convex manifolds.

Theorem 2.6. *Let $\{\overline{\Omega}_\tau\}_{0\leq\tau}$ and φ be as in Theorem 2.3. Then there exists a constant C_t which does not depend on τ, and there is τ_0 such that*

$$\|\Phi_\tau\|^2_{t\varphi} \leq C_t(\|T^*_\tau\Phi_\tau\|^2_{t\varphi} + \|S_\tau\Phi_\tau\|^2_{t\varphi}), \\ \Phi_\tau \in D_{T^*_\tau} \cap D_{S_\tau} \cap (H^{(p,r)}(\Omega_\tau))^\perp, 0 \leq \tau \leq \tau_0. \tag{2.9}$$

Proof. If such constants do not exist, then there is a sequence $\{\Phi_k\}$ with:

$$\Phi_k \in D_{T^*_{\tau_k}} \cap D_{S_{\tau_k}} \cap (H^{(p,r)}(\Omega_\tau))^\perp, \quad \lim_{k\to\infty} \tau_k = 0, \quad \|\Phi_k\| = 1$$

and

$$\lim_{k\to\infty} (\|T^*_{\tau_k}\Phi_k\|^2_{t\varphi} + \|S_{\tau_k}\Phi_k\|^2_{t\varphi}) = 0.$$

Set $\Psi_k = (d_{\tau_k}^{-1})^*\Phi_k$. Since $d_\tau^{-1} \to Id$, by (2.6) and (2.7), it follows that

$$\Psi_k \in D_{T^*} \cap D_S, \quad \lim_{k\to\infty} \|\Psi_k\|_{t\varphi} = 0$$

and

$$\lim_{k\to\infty} (\|T^*\Psi_k\|^2_{t\varphi} + \|S\Psi_k\|^2_{t\varphi}) = 0. \tag{2.10}$$

By theorem 1.6, there is a sequence of $\{\Psi_k\}$, which we may assume $\{\Psi_k\}$ itself converges to Ψ in $L^2_{(p,r)}(\Omega_0, t\varphi)$. Since (2.10) implies that $T^*\Psi = 0$ and $S\Psi = 0$, it follows that $\Psi \in H^{(p,r)}(\Omega_0)$.

Let $\dim H^{(p,r)}(\Omega_{\tau_k}) = \dim H^{(p,r)}(\Omega_0) = N$ and let $\{\Phi_{j_k}; j = 1, \ldots, N\}$ be an orthonormal basis of $H^{(p,r)}(\Omega_{\tau_k})$ for all $k = 1, 2, \ldots$. Set $\Psi_{j_k} = (d_\tau^{-1})^* \Phi_{j_k}$. Then

$$\|\Psi_{j_k}\|_{t\varphi} \in D_{T_{\tau_k}} \cap D_{S_{\tau_k}}, \quad \lim_{k\to\infty} \|\Psi_{j_k}\|_{t\varphi} = 1$$

and

$$\lim_{k\to\infty} (\|T\Psi_{j_k}\|^2_{t\varphi} + \|S\Psi_{j_k}\|^2_{t\varphi}) = 0.$$

Thus we may assume that $\{\Psi_{j_k}\}$ itself converges to Ψ_{j_0} in $L^2_{(p,r)}(\Omega_0, t\varphi)$. Then $\{\Psi_{j_0}\}_{j=1}^N$ form a basis of $H^{(p,r)}(\Omega_0)$. But

$$\begin{aligned}(\Psi, \Psi_{j_0})_{t\varphi} =& (\Psi, \Psi_{j_0} - \Psi_{j_k})_{t\varphi} + (\Psi - \Psi_k, \Psi_{j_k})_{t\varphi} \\ &+ (\Psi_k, \Psi_{j_k})_{t\varphi} \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

So, $\dim H^{(p,r)}(\Omega_0) \geq N + 1$, a contradiction. □

Theorem 2.7. *Let $\{\overline{\Omega}_\tau\}_{0\leq\tau}$ and φ be as in Theorem 2.3. Then there exists a constant $C_m(t)$ which does not depend on τ, and there is τ_0 such that*

$$\|\Phi_\tau\|_{m,t} \leq C_{m,t} \|\Box_\tau \Phi_\tau\|_{m,t}, \quad \Phi_\tau \in D_{\Box_\tau}, \Phi_\tau \perp H^{(p,r)}(\Omega_\tau), 0 \leq \tau \leq \tau_0. \tag{2.11}$$

Proof. In the estimate (2.4) the constants $C_{m,t}$ and $C'_{m,t}$ were come from integration by parts and differentiations of the coefficients of the vector fields. Thus there exists τ_0 such that

$$\|\Phi_\tau\|^2_{m,t} \leq C_{m,t} \|\Box_\tau \Phi_\tau\|^2_{m,t} + C'_{m,t} \|\Phi_\tau\|^2_{t\varphi}, \ \Phi_\tau \in D_{\Box_\tau}, \ 0 \leq \tau \leq \tau_0, \tag{2.12}$$

where $C_{m,t}$ and $C'_{m,t}$ are independent of τ.

From (2.8) we get that

$$\begin{aligned}\|\Phi_\tau\|^2_{t\varphi} &\leq C_t(\|T_\tau^* \Phi_\tau\|^2_{t\varphi} + \|S_\tau \Phi_\tau\|^2_{t\varphi}) = C_t(\Box\Phi_\tau, \Phi_\tau)_{t\varphi} \\ &\leq C_t(\epsilon) \|\Box\Phi_\tau\|^2_{t\varphi} + \epsilon \|\Phi_\tau\|^2_{t\varphi},\end{aligned}$$

where $\Phi_\tau \in D_{T_\tau^*} \cap D_{S_\tau} \cap (H^{(p,r)}(\Omega_\tau))^\perp, \ 0 \leq \tau \leq \tau_0$. Thus it follows that

$$\|\Phi_\tau\|_{t\varphi} \leq C'_t \|\Box\Phi_\tau\|_{t\varphi}, \ \Phi_\tau \in D_{T_\tau^*} \cap D_{S_\tau} \cap (H^{(p,r)}(\Omega_\tau))^\perp, \ 0 \leq \tau \leq \tau_0, \tag{2.13}$$

where C'_t is independent of τ. By (2.12) and (2.13), we get the result. □

3. Approximation theorems

Definition 3.1. We shall say that the boundary of Ω *satisfies property (P)* (see [3]) if for every positive number C there is a function $\lambda \in C^\infty(\overline{\Omega})$ with $0 \leq \lambda \leq 1$, such that for all $z \in b\Omega$,

$$H_q(\lambda)(z, \Phi) \geq C|\Phi|^2, \quad \Phi \in \Lambda^{p,q}(\overline{\Omega}).$$

Theorem 3.2. *Let $\Omega \subset\subset M$ be a weakly q-convex submanifold with C^3 boundary $b\Omega$. Assume that $b\Omega$ satisfies property (P). Then there are a neighborhood W of $b\Omega$ and a new C^3 boundary defining function ρ such that ρ is q-subharmonic on W.*

Proof. A computation similar to the one in Lemma 1.8 shows that there exists a constant $C_1 > 0$ such that for z in a neighborhood W of $b\Omega$ and sets of vectors $t^1, \ldots, t^q$ that satisfy $\sum_{i=1}^n t_i^j (L_i \rho) = 0$ $(j = 1, \ldots, q)$ and $\langle t^j, t^k \rangle_z = \delta_{jk}$ $(j, k = 1, \ldots, q)$ on W,

$$\sum_{j=1}^q H_1(\rho)(z, t^j) \geq -2C_1 |\rho(z)| - 2C_1 \frac{1}{|\rho(z)|} \sum_{j=1}^q |\sum_{i=1}^n t_i^j (L_i \rho)|^2. \tag{3.1}$$

Set $\rho_0 = \varphi(\rho) e^h$, where φ and h are functions selected momentarily. Then

$$\begin{aligned} H_1(\rho_0)(z, t^j) \geq e^h \Bigg\{ & \varphi(\rho) \left[H_1(\rho)(z, t^j) + |\sum_{i=1}^n t_i^j (L_i h)|^2 \right] \\ & + \varphi'(\rho) \left(-2 \frac{\varphi(\rho)}{\varphi'(\rho)} |\sum_{i=1}^n t_i^i (L_i h)|^2 - \frac{1}{2} \frac{\varphi'(\rho)}{\varphi(\rho)} |\sum_{i=1}^n t_i^i (L_i \rho)|^2 \right) \\ & + \varphi'(\rho) H_1(\rho)(z, t^j) + \varphi''(\rho) |\sum_{i=1}^n t_i^j (L_i \rho)|^2 \Bigg\}. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), it follows that

$$\begin{aligned} \sum_{j=1}^q H_1(\rho_0)(z, t^j) \geq e^h \Bigg\{ & \varphi \left[H_1(\rho)(z, t^j) - \sum_{j=1}^q |\sum_{i=1}^n t_i^j (L_i h)|^2 \right] \\ & + \left(\varphi'' - \frac{1}{2} \frac{(\varphi')^2}{\varphi} - \frac{2C_1 \varphi'}{|\rho|} \right) \sum_{j=1}^q |\sum_{i=1}^n t_i^i (L_i h)|^2 \Bigg\}. \end{aligned}$$

Define

$$\varphi(\rho) = \begin{cases} \rho^m, & \text{if } \rho > 0 \\ 0, & \text{if } \rho < 0, \end{cases}$$

where $m > 2 + 4C_1$. Then

$$\varphi'' - \frac{1}{2} \frac{(\varphi')^2}{\varphi} - \frac{2C_1 \varphi'}{|\rho|} = m |\rho|^{m-2} \left(\frac{1}{2} m - 1 - 2C_1 \right) \geq 0.$$

Set $h = k e^\lambda$ where $\lambda \in C^\infty(\overline{\Omega})$ with $0 \leq \lambda \leq 1$ and k is a constant with $0 < k < \frac{1}{e^\lambda}$. Then

$$H_1(h)(z, t^j) - |\sum_{i=1}^n t_i^j (L_i h)|^2 - 2Cm \geq k e^\lambda H_1(\lambda)(z, t^j) - 2Cm.$$

Choose C so large that $C \geq \frac{2qCm}{k}$. Since $b\Omega$ satisfies property (P), there exists $\lambda \in C^\infty(\overline{\Omega})$ such that $0 \leq \lambda \leq 1$ and $\sum_{j=1}^q H_1(\lambda)(z, t^j) \geq C$. Therefore

$$\sum_{j=1}^q H_1(h)(z, t^j) - \sum_{j=1}^q |\sum_{i=1}^n t_i^j (L_i h)|^2 - 2qCm \geq 0 \quad \text{for } z \in W.$$

Thus ρ_0 is a new C^∞ boundary defining function of $b\Omega$ which satisfies

$$\sum_{j=1}^{q} H_1(\rho_0)(z,t^j) \geq 0 \quad \text{for } z \in W.$$

This completes the proof. $\square$

It follows easily that the level sets of the function ρ_0 give a weakly q-convex neighborhood basis for $\overline{\Omega}$. This result gives the Mergelyan approximation property on a compact weakly q-convex complex submanifold.

Theorem 3.3. *Let $\Omega \subset\subset M$ be a weakly q-convex submanifold with C^∞ boundary $b\Omega$. Assume that $b\Omega$ satisfies property (P) and that m is a nonnegative integer. Then $\mathcal{O}_{(p,r)}(\overline{\Omega}) \cap C^\infty(\overline{\Omega})$ is dense in $\mathcal{O}_{(p,r)}(\Omega) \cap H^m_{(p,r)}(\Omega)$.*

Proof. By Theorem 3.2, we may assume that ρ is a C^∞ boundary function of $b\Omega$ such that for some $\delta_0 > 0, \rho$ is q-subharmonic in $S(\delta_0) = \{z \in M; -\delta_0 < \rho(z) < \delta_0\}$. Set $\Omega_\delta = \{z \in M; \rho(z) < \delta\}$. Then $\{\overline{\Omega}_\delta\}_{0\leq\delta<\delta_0}$ is a continuous family of diffeomorphic weakly q-convex compact complex manifolds such that $\overline{\Omega} = \overline{\Omega}_0 \subset\subset \Omega_\delta$ for all $0 < \delta < \delta_0$.

By Lemma 2.4, $\mathcal{O}_{(p,r)}(\Omega) \cap C^\infty_{(p,r)}(\overline{\Omega})$ is dense in $\mathcal{O}_{(p,r)}(\Omega) \cap H^m_{(p,r)}(\Omega)$. Thus we prove that $\mathcal{O}_{(p,r)}(\overline{\Omega}) \cap C^\infty(\overline{\Omega})$ is dense in $\mathcal{O}_{(p,r)}(\Omega) \cap C^\infty_{(p,r)}(\overline{\Omega})$ for the m-th order Sobolev norm. We define $\Phi_\delta = P^\delta_t(d^*_\delta \Phi)$, where $d_\delta : \overline{\Omega}_\delta \to \overline{\Omega}$ are diffeomorphisms and $P^\delta_t : L^2_{(p,r)}(\Omega_\delta, t\varphi) \to L^2_{(p,r)}(\Omega_\delta, t\varphi) \cap \mathcal{O}_{(p,r)}(\Omega_\delta)$ are Bergman projections with respect to the weight $e^{-t\varphi}$. Then Φ_δ satisfies the equation $\overline{\partial}\Phi_\delta = 0$ and

$$\Phi_\delta = d^*_\delta\Phi - T^*_\delta N^\delta_t S_\delta d^*_\delta \Phi,$$

where N^δ_t are the Neumann operators on Ω_δ with respect to the weight $e^{-t\varphi}$. By Theorem 2.5, for any nonnegative integer $m \geq 0$,

$$\begin{aligned} \|T^*_\delta N^\delta_t S_\delta d^*_\delta \Phi\|_{m,t,\Omega_\delta} &\lesssim \|N^\delta_t S_\delta d^*_\delta f\|_{m+1,t,\Omega_\delta} \\ &\lesssim \|S_\delta d^*_\delta \Phi\|_{m+1,t,\Omega_\delta} \end{aligned}$$

uniformly for small δ. Since the complex structures on $\overline{\Omega}_\delta$ converge to the complex structure on $\overline{\Omega}$ in C^∞-topology, we can get $S_\delta d^*_\delta \Phi \to S\Phi = 0$, also in C^∞-topology as $\delta \to 0$. So $\|\Phi_\delta - d^*_\delta\Phi\|_{m,t,\Omega_\delta}$ converges to zero as $\delta \to 0$. Since the diffeomorphisms d_δ are continuous function of δ, $d^*_\delta\Phi \to \Phi$ in C^∞-topology on $\overline{\Omega}$. Thus there exists δ_1 such that $\|d^*_\delta\Phi - \Phi\|_{m,t,\Omega} < \frac{\epsilon}{2}$, for each δ with $0 \leq \delta \leq \delta_1$. Therefore $\Phi_\delta \in \mathcal{O}_{(p,r)}(\Omega_\delta) \cap H^m_{(p,r)}(\Omega_\delta)$ and $\|\Phi - \Phi_\delta\|_{m,\Omega} < \epsilon$ for each $0 \leq \delta \leq \delta_1$. Hence we get the theorem. $\square$

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