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Uninorm logic: toward a fuzzy-relevance logic(2)* †

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【Abstract】 This paper first investigates several uninorm logics (introduced by Metcalfe and Montagna in [8]) as *fuzzy-relevance* logics. We first show that the uninorm logic **UL** and its extensions **IUL**, **UML**, and **IUML** are fuzzy-relevant; *fuzzy* in Cintula's sense, i.e., the logic **L** is complete with respect to linearly ordered **L**-matrices; and *relevant* in the weak sense that $\phi \to \psi$ is a theorem only if either (i) ϕ and ψ share a sentential variable or constant, or (ii) both $\sim\phi$ and ψ are theorems. We next expand these systems to those with $\triangle$.

【Key words】 uninorm logics, fuzzy-relevance logic, algebraic and matrix completeness.

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1. Introduction

Hájek [6] introduced the basic fuzzy logic BL and showed that the well-known infinite-valued systems Ł (Łukasiewicz logic), G (Gödel-Dummett logic), and Π (Product logic) are its extensions. BL is the residuated fuzzy logic capturing the tautologies of *continuous* t-norms and their residua. Esteva and Godo [4] introduced the monoidal t-norm logic MTL, which copes with the logic of *left-continuous* t-norms and their residua, as a weakening of BL (and a strengthening of Affine multiplicative additive intuitionistic linear logic AMAILL introduced by Höhle [7]).

All the above systems may be called t-norm (based) logics in the sense that their algebraic counterparts are based on t-norms. T-norm (based) logics are *not* relevant. Because the integral condition (Int) that the greatest element 1 is the unit element ($1 * x = x$ for all $x \in [0, 1]$) corresponds to (&-E) $(\phi \,\&\, \psi) \rightarrow \phi$, a common axiom of all the above (t-norm) logics, from which the weakening (W) $\phi \rightarrow (\psi \rightarrow \phi)$ can be proved using the "residuation" below and vice versa, and (W) (and so (&-E)) allow(s) *irrelevance* between ϕ and ψ in case $\phi \rightarrow \psi$ is a theorem.

Metcalfe and Montagna [8] recently introduced the uninorm logic UL capturing the tautologies of *left-continuous conjunctive* uninorms and their residua as a weakening of MTL and a strengthening of Multiplicative additive intuitionistic linear logic MAILL, and its schematic extensions IUL, UML, IUML. They investigated them as *substructural fuzzy logics*. The present author [10] investigated the system R of Relevance with "mingle" RM and its weakening wRM as *fuzzy-relevance* logics; *fuzzy* in the sense that it satisfies the fuzzy condition (of a logic) of Cintula in [2] that it is a weakly implicative logic which is complete with respect to (w.r.t.) linearly ordered RM or wRM-matrices (or RM or wRM-algebras); and *relevant* in the *weak* sense that it satisfies the *weak* relevance principle (WRP) in [3] that $\phi \rightarrow \psi$ is a theorem only if either (i) ϕ and ψ share a sentential variable or (ii) both $\sim\phi$ and ψ are theorems.

Concerning these two facts, one interesting point to state is that while algebraic counterpart of (w)RM rejects (Int), it instead accepts all of the conditions of a uninorm, more exactly, the conditions of an isotonic commutative monoid below. Then, the above uninorm logics seem to be *fuzzy-relevant* in the above senses as well. This paper investigates the above uninorm logics as *fuzzy-relevance logics* (see [10] for the motivation for fuzzy-relevance logics).[1] More exactly, we shall show that they are all

[1] Roughly speaking, our motivation for fuzzy-relevance logic is this: 1) we argue in natural language, which is "vague" rather than "exact", i.e., we in fact treat propositions in vagueness, and 2) in argument conclusion is relevant to premise(s), i.e., we do not argue something from irrelevant premise(s). Fuzzy-relevance logic is a logic to

fuzzy in Cintula's sense and relevant in the weak sense that $\phi \rightarrow \psi$ is a theorem only if either (i) ϕ and ψ share a sentential variable or constant, or (ii) both $\sim\phi$ and ψ are theorems (calling this "weak relevance principle′" (WRP′)). Note that even if Metcalfe and Montagna [8] called the uninorm logics *fuzzy* ones, they did not exactly show that in which or whose sense they are fuzzy. We first give algebraic completeness results for them, and next show that they are both fuzzy and relevant in the above senses. We furthermore expand the above propositional systems to those with $\triangle$.

For convenience, we shall adopt the notation and terminology similar to those in [2], [4], [5], and [6], and assume being familiar with them (together with results found in them).

2. Syntax

We base uninorm logics on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **F**, f, **t**. Further definable connectives are:

dfl. $\sim\phi := \phi \rightarrow$ f, and

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consider both 1) and 2).

df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We moreover define T as ~F, and ϕ_t as $\phi \wedge t$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiom schemes and rules for UL.

Definition 2.1 UL consists of the following axiom schemes and rules:[2)]

A1. $\phi \rightarrow \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ (∧-elimination, ∧-E)
A3. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ (∧-introduction, ∧-I)
A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ (∨-introduction, ∨-I)
A5. $((\phi\rightarrow\chi)\wedge(\psi\rightarrow\chi)) \rightarrow ((\phi\vee\psi)\rightarrow\chi)$ (∨-elimination, ∨-E)
A6. $(\phi\wedge(\psi\vee\chi))\rightarrow((\phi\wedge\psi)\vee(\phi\wedge\chi))$ (∧∨-distributivity, ∧∨-D)
A7. $F \rightarrow \phi$ (ex falso quadlibet, EF)
A8. $(\phi \,\&\, \psi) \rightarrow (\psi \,\&\, \phi)$ (&-commutativity, &-C)
A9. $(\phi \,\&\, t) \leftrightarrow \phi$ (push and pop, PP)
A10. $(\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$ (prefixing, PF)
A11. $(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\phi \,\&\, \psi) \rightarrow \chi)$ (residuation, RE)
A12. $(\phi \rightarrow \psi)_t \vee (\psi \rightarrow \phi)_t$ (t-prelinearity, PL_t).
$\phi \rightarrow \psi, \phi \vdash \psi$ (modus ponens, mp)
$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj)

Definition 2.2 (ULs) A logic is a schematic extension of L if and only if (iff) it results from L by adding axiom schemes. L is a UL iff L is a schematic extension of UL.

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2) The axiomatization of UL is slightly different from that of UL in [8]. But we can easily show that they are equivalent.

In particular:

- IUL is UL plus $\sim\sim\phi \rightarrow \phi$ (double negation elimination, DNE)
- UML is UL plus $(\phi \;\&\; \phi) \leftrightarrow \phi$ (idempotence, ID)
- IUML is IUL plus (ID) and $t \leftrightarrow f$ (fixed-point, FP)

A *theory* over L is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of L or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). $T \vdash \phi$, more exactly $T \vdash_L \phi$, means that ϕ is *provable* in T w.r.t. L, i.e., there is an L-proof of ϕ in T. The relevant deduction theorem (RDT) for L is as follows:

Proposition 2.3 Let L be a UL, T a theory, and ϕ, ψ formulas. $T, \phi \vdash_L \psi$ iff $T \vdash_L \phi_t \rightarrow \psi$.

Proof: It is just Enthymematic Deduction Theorem (see [9]). □

A theory T is *inconsistent* if $T \vdash F$; otherwise it is *consistent.*

For easy reference we group the uninorm logics defined in this section together as a set.

Definition 2.4 *Logics* = {UL, IUL, UML, IUML}

For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

3. Semantics

Suitable algebraic structures for ULs are obtained as varieties of residuated lattices.

Definition 3.1 A *bounded commutative monoidal residuated lattice* (bcmr-lattice) is a structure **A** = (A, $\top$, $\bot$, $\top_t$, $\bot_f$, $\wedge$, $\vee$, $*$, $\to$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded distributive lattice with top element $\top$ and bottom element $\bot$.
(II) (A, $*$, $\top_t$) satisfies for all x, y, z $\in$ A,
(a) $x * y = y * x$ (commutativity)
(b) $\top_t * x = x$ (identity)
(c) $x \leq y$ implies $x * z \leq y * z$ (isotonicity)
(d) $x * (y * z) = (x * y) * z$ (associativity)
(III) $y \leq x \to z$ iff $x*y \leq z$, for all x, y, z $\in$ A (residuation)

(A, $*$, $\top_t$) satisfying (II-b, d) is a *monoid.* Thus (A, $*$, $\top_t$) satisfying (II-a, b, c, d) is an isotonic commutative monoid. (A, $*$, $\top_t$) satisfying (II-a, b, c, d) on [0, 1] is a *uninorm* and it is a *t-norm* in case $\top_t = \top$.

To define a bcmr-lattice we may take in place of (II-c)

(IV) $x * (y \vee z) = (x * y) \vee (x * z)$.

Using $\to$ and $\bot_f$ we can define $\top_t$ as $\bot_f \to \bot_f$, and ~

as in (df1). In a bcmr-lattice, $\sim$ is a *weak* negation in the sense that for all x, $x \leq \sim\sim x$ holds in it.

Definition 3.2 (UL-algebra) A *UL-algebra* is a bcmr-lattice satisfying the condition: for all x, y,

(pl$_t$) $\top_t \leq (x \rightarrow y)_t \vee (y \rightarrow x)_t$..

An *IUL-algebra* is a UL algebra with strong negation, i.e., for all x, $x = \sim\sim x$; a *UML-algebra* is a UL algebra satisfying (II-e) $x * x = x$ (idempotence) for all x; and an *IUML-algebra* is a UML algebra having strong negation and (FP) $\top_t = \bot_f$.

An algebra **A** is *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let **A** be an algebra. An *A-evaluation* is a function v : FOR $\rightarrow$ **A** satisfying:

$$v(\#(\phi_1, \cdots, \phi_m)) = \#_A(v(\phi_1), \cdots, v(\phi_m)),$$

where $\# \in \{\&, \rightarrow, \wedge, \vee, t, f, T, F\}$, $\#_A \in \{*, \rightarrow, \wedge, \vee, \top_t, \bot_f, \top, \bot\}$, and m is the arity of # and $\#_A$.

Definition 3.4 Let L be a propositional language, **L** a logic in L, T a theory in L, ϕ a formula, and **K** a class of A-algebras.

(i) (Tautology) ϕ is a *$\top_t$-tautology* in **A**, briefly an

A-tautology (or *A-valid*), if v(ϕ) ≥ $\top_t$ for each A-evaluation v.

(ii) (Model) An A-evaluation v is an *A-model* of T if v(ϕ) ≥ $\top_t$ for each ϕ ∈ T. By *Mod(T, A)*, we denote the class of A-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by T $\models_K$ ϕ, if Mod(T, A) = Mod(T ∪ {ϕ}, A) for each A ∈ K.

Definition 3.5 (L-algebra) Let L be a logic in *L*, T a theory in *L*, ϕ a formula, and A an algebra. A is an *L-algebra* iff whenever ϕ is L-provable in T, i.e., T $\vdash_L$ ϕ, it is a semantic consequence of T w.r.t. the set of A. By *$MOD^{(l)}(L)$*, we denote the class of (linearly ordered) L-algebras. We write T $\models^{(l)}_L$ ϕ in place of T $\models_{MOD^{(l)}L}$ ϕ.

Proposition 3.6 The class of all UL-algebras is a variety of algebras.

Proof: To prove that the class of all UL-algebras is a variety, we note first that the class of (bounded) distributive lattices is a variety and each of the conditions of (II-a, b, d) has a form of equation. Note also that in each UL-algebra the equations for (II-c) and (III) can be given; for the equations for (II-c) and (III), see (IV) and Lemma 2.3.10 in [6]. Analogously the equations for (pl_t) can be given. Thus, since each condition for a UL-algebra has a form of equation or can be defined in equation, it can be ensured that the class of all UL-algebras is a variety. □

Let A be an algebra. A_M-matrix, briefly *M-matrix*, is an A-algebra with D, a subset of A. The elements of D are usually called designated elements of matrix M. Then, in an analogy to the above, we can define a UL-matrix. Furthermore, by taking $v(\phi) \in D$ in place of $v(\phi) \geq \top_t$, we can analogously define tautology, model, semantic consequence, and L-matrix on M-matrices in place of A-algebras.

Let us take $D = \{x: x = v(\phi) \geq \top_t\}$. Then it is immediate that

Corollary 3.7 A UL-algebra A is an L-algebra iff $T \vdash_L \phi$ implies $T \vDash_L \phi$ iff a UL-matrix M = (A, D) is an L-matrix.

4. Algebraic completeness

Let L be a UL, and A a (corresponding) UL-algebra. We first note that the nomenclature of the prelinearity condition is explained by the following subdirect representation theorem.

Proposition 4.1 Each UL-algebra is a subdirect product of linearly ordered UL-algebras.

Proof: Its proof is as usual. □

We next show that classes of provably equivalent formulas form an L-algebra. Let T be a fixed theory over L. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash_L \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \,\&\, \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [F]_T$, $\top = [T]_T$, $\top_t = [t]_T$, and $\bot_f = [f]_T$. By $\boldsymbol{A}_T$, we denote this algebra.

Proposition 4.2 For T a theory over L, A_T is an L-algebra.

Proof: Note that A2 to A6 ensure that $\wedge$, $\vee$, and $\rightarrow$ satisfy (I) in Definition 3.1; that A8, A9, and the theorems (AS) $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$ and (IT) $(\phi \rightarrow \psi) \rightarrow ((\phi \,\&\, \chi) \rightarrow (\psi \,\&\, \chi))$ ensure that & satisfies (II) (a) - (d); that A11 ensures that (III) holds; and that A12 ensures that (pl_t) holds. It is obvious that $[\phi]_T \leq [\psi]_T$ iff $T \vdash_L \phi \leftrightarrow (\phi \wedge \psi)$ iff $T \vdash_L \phi \rightarrow \psi$. Finally recall that A_T is an L-algebra iff $T \vdash_L \psi$ implies $T \vDash_L \psi$, and observe that for ϕ in T, since $T \vdash_L t \rightarrow \phi$, it follows that $[t]_T \leq [\phi]_T$. Thus it is an L-algebra. □

Theorem 4.3 (Strong completeness) Let L be a UL, T a theory, and ϕ a formula. $T \vdash_L \phi$ iff $T \vDash_L \phi$ iff $T \vDash^1_L \phi$.

Proof: (i) $T \vdash_L \phi$ iff $T \models_L \phi$. Left to right follows from definition. Right to left is as follows: from Proposition 4.2, we obtain $A_T \in MOD(L)$, and for A_T-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in Mod(T, A_T)$. Thus, since from $T \models_L \phi$ we obtain that $[\phi]_T = v(\phi) \geq_T t$, $T \vdash_L t \rightarrow \phi$. Then, since $T \vdash_L t$, by (mp) $T \vdash_L \phi$, as required.

(ii) $T \models_L \phi$ iff $T \models^1_L \phi$. It follows from Proposition 4.1. □

Corollary 4.4 (Weak completeness) For each formula ϕ, ϕ is a theorem iff for each (linearly ordered) L-algebra A, ϕ is an A-tautology, i.e., $\vdash_L \phi$ iff $\models^{(l)}_L \phi$.

5. Fuzzy-relevance

5.1. Fuzziness

Following Cintula [2], let a UL L be *fuzzy* in case it is complete w.r.t. linearly ordered L-matrices, i.e., $L = \models^1_L$. We show that a logic L is so in case it is a UL, i.e., an L having A12. We first show that all the ULs above are weakly implicative logics.

Proposition 5.1 A UL L is a weakly implicative logic.

Proof: We first note that a weakly implicative logic

(WIL) is a logic having (SI), (mp), transitivity ($\phi \rightarrow \psi$, $\psi \rightarrow \chi \vdash \phi \rightarrow \chi$), and congruence w.r.t. connectives. Since L has A1, (mp), and proves (SF) $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$, it suffices to check that $\leftrightarrow$ is a congruence w.r.t. $\wedge$, $\vee$, &, and $\rightarrow$: we check one direction. Let $\vdash \phi \rightarrow \psi$. W.r.t. $\wedge$, by A2 and transitivity, $(\phi \wedge \chi) \rightarrow \psi$, and thus $(\phi \wedge \chi) \rightarrow (\psi \wedge \chi)$ by A2, A3, (adj), and (mp); w.r.t. $\vee$, analogously to $\wedge$; w.r.t. &, $(\phi \& \chi) \rightarrow (\psi \& \chi)$ and $(\chi \& \phi) \rightarrow (\chi \& \psi)$ by (IT) and A9; w.r.t. $\rightarrow$, $(\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)$ and $(\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \psi)$ by (SF) and A10. □

To show that L is fuzzy, following [2] we add more definitions on a theory T to the definitions above.

Definition 5.1 Let L be a UL.

(i) T is *linear* if T is consistent and for each pair ϕ, ψ of formulas, $T \vdash \phi \rightarrow \psi$ or $T \vdash \psi \rightarrow \phi$.

(ii) T is *prime* if for each pair ϕ, ψ of formulas such that $T \vdash \phi \vee \psi$, $T \vdash \phi$ or $T \vdash \psi$.

(iii) L has the *Linear Extension Property* (LEP) if for each theory T and formula ϕ such that $T \nvdash \phi$, there is a linear theory T' such that $T \subseteq T'$ and $T' \nvdash \phi$.

(iv) L has the *Prelinearity Property* (PP) if for each theory T, we get $T \vdash \chi$ whenever $T, \phi \rightarrow \psi \vdash \chi$ and $T, \psi \rightarrow \phi \vdash \chi$.

(v) L has the *Subdirect Decomposition Property* (SDP) if each ordered L-matrix is a subdirect product of linearly ordered L-matrices.

(vi) L has the *Prime Extension Property* (PEP) if for each theory T and formula ϕ such that $T \nvdash \phi$, there is a prime theory T' such that $T \subseteq T'$ and $T' \nvdash \phi$.

(vii) L has the *Proof by Cases Property* (PCP) if for each theory T, we get T, $\phi \vee \psi \vdash \chi$ whenever T, $\phi \vdash \chi$ and T, $\psi \vdash \chi$.

We consider L as a *finitary* logic in the sense that for each theory T and formula ϕ we have that if T $\vdash \phi$ there is a *finite* theory $T' \subseteq T$ such that $T' \vdash \phi$. Then, since a UL L is a WIL, by Lemma 17 in [2], we can obtain that

Proposition 5.2 Let L be a UL and T a theory.

(i) T is linear iff the L-matrix M_T is linearly ordered;
(ii) T is linear iff T is prime;
(iii) L has PP iff L has PCP and (PL), i.e., $(\phi \to \psi) \vee (\psi \to \phi)$.

Note that Cintula showed that as a finitary WIL L is fuzzy iff L has LEP iff L has PP iff L has SDP iff L has PCP and (PL) (see Theorem 3 and Lemma 17 in [2]). Since a finitary UL L is a finitary WIL proving (PL), it is immediate that

Corollary 5.3 For a finitary UL L,
L is fuzzy iff L has PCP iff L has PEP.

Let us consider L with deduction theorem. In an analogy to Lemma 22 in [2], we can show that

Proposition 5.4 Let L be a finitary logic with RDT. Then L is a fuzzy logic iff it holds: $\vdash_L (\phi \to \psi)_t \vee (\psi \to \phi)_t$, i.e., A12.

Proof: Left to right is obvious. For right to left, we just show that L has PP. Let T, $\phi \to \psi \vdash_L \chi$ and T, $\psi \to \phi \vdash_L \chi$. By RDT, T $\vdash_L (\phi \to \psi)_t \to \chi$ and T $\vdash_L (\psi \to \phi)_t \to \chi$. Then by A5 (together with (adj) and (mp)), T $\vdash_L ((\phi \to \psi)_t \vee (\psi \to \phi)_t) \to \chi$. Thus, by A12 and (mp), T $\vdash_L \chi$, as desired. □

Then using Proposition 5.4 (and soundness as usual), we can easily show that

Theorem 5.5 (Completeness) Let T be a theory over a finitary UL L and ϕ a formula. Then T $\vdash_L \phi$ iff T $\models^1_L \phi$.

5.2. Relevance

In this subsection we show that a UL L is relevant in the weak sense above, i.e., in the sense that it satisfies WRP′.

Proposition 5.6 Let L be a UL. Then L is *relevant* in the sense that it satisfies WRP′.

Proof: For this, it suffices to note that **IUML** is the **RM** just having (FP) and that **UL**, **IUL**, and **UML** are weakenings of **RM**. Since **RM** is relevant in the sense that it satisfies WRP and so WRP′, it is immediate. □

Then, from Propositions 5.4 and 5.6, it directly follows that

Proposition 5.7 Let L be a UL. Then, L is a fuzzy-relevance logic.

Proposition 5.8 Let L be a UML. It proves: $\sim(\phi \to \phi) \to (\psi \to \psi)$.

Proof: We first note that $t \to (\phi \to \phi)$ and $f \to t$ are theorems of L. Then, since $\sim(\phi \to \phi) \to f$ by contraposition and mp, $\sim(\phi \to \phi) \to t$ by transitivity. Then, since $t \to (\phi \to \phi)$, $\sim(\phi \to \phi) \to (\psi \to \psi)$ by transitivity. □

Note that the *strong* relevance principle (SRP) in [1] is that $\phi \to \psi$ is a theorem only if ϕ and ψ share a sentential variable. Thus, from Proposition 5.5. it directly follows that

Corollary 5.9 A UML L is not strongly relevant.

Remark 5.10 Let L be a UL having either (f) $f \to t$ or (FP) or $\phi \to (\phi \to \phi)$ (mingle). L proves such formulas as $\sim(\phi \vee \sim\phi) \to (\psi \vee \sim\psi)$ and $\sim(\phi \to \phi) \to (\psi \to \psi)$, and so it is not relevant in the strong sense any more. (But, since these formulas still satisfy WRP′, L may be instead relevant in the weak sense.) Note that **RM** is the **R** with (mingle) and that while **RM** is weakly relevant, **R** is strongly relevant in the above senses. Then, since UL and IUL do not prove any of (f), (PF) and (mingle), they seem to be *strongly* relevant. But UL and IUL have A12, i.e., (PL_t) and so they are not weakenings of **R**, which is strongly relevant. We conjecture that they are strongly relevant. It is an open problem remained in this paper.

6. The connective △

We here consider a UL **L** expanded with the unary connective △. For this we expand the language of **L** with △.

Definition 6.1 Let **L** be a UL, and $\triangle\phi_t$ be $(\triangle\phi)_t$, i.e., $\triangle\phi \wedge t$.

(i) (UL$\triangle_T$s) **L**$\triangle_T$ is **L** with the following axiom schemes and rule:

A△1. $\triangle\phi_t \vee (\triangle\phi \to F)$

A△2. $\triangle\phi \to (T \to \triangle\phi)$

A△3. $\triangle(\phi \vee \psi) \to (\triangle\phi \vee \triangle\psi)$ (△∨-distribution, △∨-D)

A△4. $(\triangle\phi \wedge \triangle\psi) \to \triangle(\phi \wedge \psi)$ (△∧-D)

A△5. $\triangle\phi_t \to \phi$ ($\triangle_t$-reflexivity, △RF_t)

A△6. $\triangle\triangle\phi \leftrightarrow \triangle\phi$

A△7. $\triangle(\phi \to \psi) \to (\triangle\phi \to \triangle\psi)$ (△-monotonicity, △MN)

A△8. $(\triangle\phi \mathbin{\&} \triangle\phi)_t \to (\triangle\phi_t \mathbin{\&} \triangle\phi_t)$ (△∧&-D_t)

A△9. $\triangle\phi \to (\triangle\phi \mathbin{\&} \triangle\phi)$ (△&-INC)

A△10. $\triangle(\phi \mathbin{\&} \phi) \to \triangle\phi$ (△&-DEC)

A△11. $\sim\sim(\triangle\phi) \to \triangle\phi$ (△DNE)

A△W_t. $\triangle\phi_t \to (\psi_t \to \phi)$ (△W_t)

$\phi \vdash \triangle\phi$ (necessitation, nec)

(ii) (UL$\triangle_x$s) **L**$\triangle_x$ is **L** with A△1, A△3, A△4, A△7, A△8,

A△9, A△W_t, (nec) in (i) and:
A△2′. $\triangle\phi_t \to (\phi \to \triangle\phi)$
A△5′. $\triangle\phi \to \phi$ (△-reflexivity, △RF)
A△6′. $\triangle\phi \to \triangle\triangle\phi$ (△-transitivity, △TRS)
A△10′. $\triangle(\phi \,\&\, \phi)_t \to \triangle\phi$ (△&-DEC_t)
A△11′. $\sim\sim(\triangle\phi_t) \to \triangle\phi$ (△DNE_t)
(iii) ($UL\triangle_{Tt}s$) $L\triangle_{Tt}$ is L with A△1, A△3, A△4, A△5′, A△6′, A△7, A△9, A△10, A△11, (nec) in (i) and (ii), and:
A△2′′. $\triangle\phi \to t$
A△Wa. $\triangle\phi \to (\psi_t \to \phi)$ (△Wa).

For brevity, by a *UL△*, we ambiguously express $UL\triangle_T$, $UL\triangle_x$, and $UL\triangle_{Tt}$ all together if we do not need distinguish them; and by *L△*, $L\triangle_T$, $L\triangle_x$, and $L\triangle_{Tt}$. We can first easily show that

Proposition 6.2 (i) L△ proves:

(1) $(\triangle\phi \to F) \to (\triangle\phi \leftrightarrow F)$
(2) $(T \to \triangle\phi) \to (\triangle\phi \leftrightarrow T)$
(3) $(\triangle\phi \,\&\, \triangle\psi) \to \triangle(\phi \,\&\, \psi)$
(4) $\triangle(\phi\&\phi)_t \leftrightarrow \triangle\phi_t \leftrightarrow (\triangle\phi_t\&\triangle\phi_t) \leftrightarrow (\triangle\phi\&\triangle\phi)_t$ (△&-ID_t)
(5) $\triangle\phi^n{}_t \leftrightarrow \triangle(\phi^n{}_t)_t$, i.e., $((\triangle\phi)_t)^n \leftrightarrow ((\triangle\phi_t)^n)_t$, for each n
(6) $\triangle(\phi \to (\psi \to \chi))_t \leftrightarrow \triangle((\phi \,\&\, \psi) \to \chi)_t$
(7) $\triangle(\phi \to \psi)_t \to \triangle((\psi \to \chi) \to (\phi \to \chi))$
(8) $\triangle(\psi \to \chi)_t \to \triangle((\phi \to \psi) \to (\phi \to \chi))$
(9) $(\triangle\phi_t \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\triangle\phi_t \,\&\, \psi) \,\&\, \chi)$ (△AS_ta)
(10) $((\triangle\phi_t \,\&\, \psi) \,\&\, \chi) \leftrightarrow ((\triangle\phi_t \,\&\, \chi) \,\&\, \psi)$ (△AS_tb)
(11) $(\triangle\phi_t \to \triangle\psi_t) \to ((\triangle\psi_t \to \chi) \to (\triangle\phi_t \to \chi))$ (△SF_ta)
(12) $\triangle(\phi \to \psi)_t \to ((\psi \to \triangle\chi_t) \to (\phi \to \triangle\chi_t))$ (△SF_tb)
(13) $(\triangle\psi_t \to \triangle\chi_t)_t \to ((\phi \to \triangle\psi_t) \to (\phi \to \triangle\chi_t))$ (△PF_ta)
(14) $(\psi \to \chi) \to ((\triangle\phi_t \to \psi) \to (\triangle\phi_t \to \chi))$ (△PF_tb)
(15) $(\triangle\phi_t \to (\triangle\phi_t \to \psi)) \to (\triangle\phi_t \to \psi)$ (△C_t)
(16) $(\triangle\phi_t \to (\psi \to \chi)) \leftrightarrow ((\triangle\phi_t \,\&\, \psi) \to \chi)$ (△RE_t)
(17) $(\triangle\phi_t \to (\psi \to \chi)) \leftrightarrow (\psi \to (\triangle\phi_t \to \chi))$ (△PM_t)
(18) $\triangle(\phi \to \psi)^n{}_t \vee \triangle(\psi \to \phi)^n{}_t$, for each n (△$PL^n{}_t$)
(19) $\triangle(\phi \to \psi)_t \vee \triangle(\psi \to \phi)_t$ (△PL_t)

(20) $\triangle(\phi \rightarrow \psi) \vee \triangle(\psi \rightarrow \phi)$ ($\triangle$PL)

(ii) L$\triangle_T$ proves:

(1) $T \leftrightarrow \triangle t$

(2) $\triangle(\phi_t) \leftrightarrow \triangle\phi$

(3) $\triangle\phi_t \rightarrow (\triangle\phi \leftrightarrow T)$

(4) $\triangle\phi \leftrightarrow (\triangle\phi \& \triangle\phi) \leftrightarrow \triangle(\phi \& \phi)$ ($\triangle$&-ID)

(5) $\sim\sim(\triangle\phi) \leftrightarrow \triangle\phi$ ($\triangle$DN)

(iii) L$\triangle_x$ proves:

(1) $\triangle\phi^n_t \leftrightarrow \triangle(\phi^n_t)$, i.e., $((\triangle\phi)_t)^n \leftrightarrow \triangle(\phi_t)^n$, for each n

(2) $\triangle\phi_t \rightarrow (\triangle\phi \leftrightarrow \phi)$

(iv) L$\triangle_t$ proves:

(1) $\triangle\phi_t \leftrightarrow \triangle\phi$

(2) $\triangle\phi_t \rightarrow (\triangle\phi \leftrightarrow t)$

(3) $\triangle(\phi \& \phi) \leftrightarrow \triangle\phi \leftrightarrow (\triangle\phi \& \triangle\phi)$ ($\triangle$&-ID)

(4) $\sim\sim\triangle\phi \leftrightarrow \triangle\phi$ ($\triangle$DN)

(5) $\triangle(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow \triangle((\phi \& \psi) \rightarrow \chi)$

(6) $\triangle(\phi \rightarrow \psi) \rightarrow \triangle((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$

(7) $\triangle(\psi \rightarrow \chi) \rightarrow \triangle((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

(8) $(\triangle\phi \& (\psi \& \chi)) \leftrightarrow ((\triangle\phi \& \psi) \& \chi)$ ($\triangle$ASa)

(9) $((\triangle\phi \& \psi) \& \chi) \leftrightarrow ((\triangle\phi \& \chi) \& \psi)$ ($\triangle$ASb)

(10) $(\triangle\phi \rightarrow \triangle\psi) \rightarrow ((\triangle\psi \rightarrow \chi) \rightarrow (\triangle\phi \rightarrow \chi))$ ($\triangle$SFa)

(11) $\triangle(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \triangle\chi) \rightarrow (\phi \rightarrow \triangle\chi))$ ($\triangle$SFb)

(12) $(\triangle\psi \rightarrow \triangle\chi) \rightarrow ((\phi \rightarrow \triangle\psi) \rightarrow (\phi \rightarrow \triangle\chi))$ ($\triangle$PFa)

(13) $(\psi \rightarrow \chi) \rightarrow ((\triangle\phi \rightarrow \psi) \rightarrow (\triangle\phi \rightarrow \chi))$ ($\triangle$PFb)

(14) $(\triangle\phi \rightarrow (\triangle\phi \rightarrow \psi)) \rightarrow (\triangle\phi \rightarrow \psi)$ ($\triangle$Ca)

(15) $\triangle(\triangle\phi \rightarrow (\triangle\phi \rightarrow \psi)) \rightarrow (\triangle\phi \rightarrow \psi)$ ($\triangle$Cb)

(16) $\triangle\phi \rightarrow (\triangle\psi \rightarrow \phi)$ ($\triangle$Wb)

(17) $(\triangle\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\triangle\phi \& \psi) \rightarrow \chi)$ ($\triangle$RE)

(18) $(\triangle\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\psi \rightarrow (\triangle\phi \rightarrow \chi))$ ($\triangle$PMa)

(19) $\triangle(\triangle\phi \rightarrow (\triangle\psi \rightarrow \chi)) \rightarrow (\triangle\psi \rightarrow (\triangle\phi \rightarrow \chi))$ ($\triangle$PMb)

Proof: We prove (i-9) as an example. We prove right to left:

1. $\triangle\phi_t \rightarrow ((\psi \& \chi) \rightarrow (\triangle\phi_t \& (\psi \& \chi)))$ (A1, re)
2. $\triangle(\triangle\phi_t) \rightarrow \triangle((\psi\&\chi) \rightarrow (\triangle\phi_t\&(\psi\&\chi)))$ (1, A$\triangle$7, mp)
3. $\triangle\phi_t \rightarrow (\triangle\triangle\phi \wedge \triangle t)$ (A$\triangle$6(′)), A2, A3, adj, mp, trans.)

4. $\triangle\phi_t \rightarrow \triangle((\psi\&\chi) \rightarrow (\triangle\phi_t\&(\psi\&\chi)))$ (2, 3, A△4, trans.)
5. $\triangle\phi_t \rightarrow \triangle((\psi\&\chi) \rightarrow (\triangle\phi_t \& (\psi \& \chi)))_t$ (4, A3, adj, mp)
6. $\triangle\phi_t \rightarrow \triangle(\psi \rightarrow (\chi \rightarrow (\triangle\phi_t\&\psi\&\chi)))_t$ (5, (i-6), trans.)
7. $\triangle\phi_t\rightarrow(\psi \rightarrow (\chi \rightarrow (\triangle\phi_t\&(\psi\&\chi))))$ (6, A△5(′), A2, trans.)
8. $((\triangle\phi_t \& \psi) \& \chi) \rightarrow (\triangle\phi_t \& (\psi \& \chi))$ (7, re)

Proof of left to right is analogous.

Proof of the rest is left to the interested reader. □

Proposition 6.3 (RDT△) Let L△ be a finitary UL△ logic, T a theory over L△, and ϕ, ψ formulas. Then, $T \cup \{\phi\} \vdash_{L\triangle} \psi$ iff $T \vdash_{L\triangle} \triangle\phi_t \rightarrow \psi$.

Proof: Right to left is easy. For left to right, we just check the rule (mp) as example. By IH, we first assume that for some j, k < i, $T \vdash_{L\triangle} \triangle\phi_t \rightarrow \psi_j$ and $T \vdash_{L\triangle} \triangle\phi_t \rightarrow \psi_k$, where $\psi_k = \psi_j \rightarrow \psi_i$, and show that $T \vdash_{L\triangle} \triangle\phi_t \rightarrow \psi_i$. By the second assumption and (pm) $\phi \rightarrow (\psi \rightarrow \chi)$) $\vdash_{L\triangle} \psi \rightarrow (\phi \rightarrow \chi)$ (as a derived rule in L and so in L△), $T \vdash_{L\triangle} \psi_j \rightarrow (\triangle\phi_t \rightarrow \psi_i)$. So, by the first assumption and transitivity, $T \vdash_{L\triangle} \triangle\phi_t \rightarrow (\triangle\phi_t \rightarrow \psi_i)$. Then, using $\triangle C_t$ and (mp), we can obtain that $T \vdash_{L\triangle} \triangle\phi_t \rightarrow \psi_i$, as desired. □

Since $L\triangle_t$ proves Proposition 6.2 (iv-1), it has Delta Deduction Theorem (DT△) as well.

Corollary 6.4 (DT△) $T \cup \{\phi\} \vdash_{L\triangle t} \psi$ iff $T \vdash_{L\triangle t} \triangle\phi \rightarrow \psi$.

But since $L\triangle_{\top}$ and $L\triangle_{x}$ both do not prove (iv-1), they do not have DT△.

Definition 6.5 (i) ($UL\triangle_{\top}$-algebras) A $UL\triangle_{\top}$-algebra is a UL-algebra expanded by a unary operation △ satisfying: for all x, y, z, there is n such that

△1. $\top_t \leq (\triangle x_{\top t} \vee (\triangle x \to \bot))$

△2. $\triangle x \leq (\top \to \triangle x)$

△3. $\triangle(x \vee y) \leq (\triangle x \vee \triangle y)$

△4. $(\triangle x \wedge \triangle y) \leq \triangle(x \wedge y)$

△5. $\triangle x \top t \leq x$

△6. $\triangle x = \triangle\triangle x$

△7. $(\triangle x * \triangle(x \to y)) \leq \triangle y$

△8. $(\triangle x * \triangle x)\top t \leq (\triangle x \top t * \triangle x \top t)$

△9. $\triangle x \leq (\triangle x * \triangle x)$

△10. $\triangle(x * x) \leq \triangle x$

△11. $\sim\sim(\triangle x) \leq \triangle x$

△Wk⊤t. $\triangle x \top t \leq (y \top t \to x)$

△⊤. $\top = \triangle \top t$.

(ii) ($UL\triangle_{x}$-algebras) A $UL\triangle_{x}$-algebra is a UL-algebra expanded by a unary operation △ satisfying:

△1, △3, △4, △7, △8, △9, $\triangle Wk_{\top t}$ in (i), and for all x,

△2′. $\triangle x_{\top t} \leq (x \to \triangle x)$

△5′. $\triangle x \leq x$

△6′. $\triangle x \leq \triangle\triangle x$

△10′. $\triangle(x * x)\top t \leq \triangle x$

△11′. $\sim\sim(\triangle x \top t) \leq \triangle x$

$\triangle$t. $\top t = \triangle \top t$.

(iii) (UL$\triangle_{\top t}$-algebras) A UL$\triangle_{\top t}$-algebra is a UL-algebra expanded by a unary operation $\triangle$ satisfying: $\triangle$1, $\triangle$3, $\triangle$4, $\triangle$5′, $\triangle$6′, $\triangle$7, $\triangle$9, $\triangle$10, $\triangle$11, $\triangle$t in (i) and (ii), and for all x, y, z,

$\triangle$2′′. $\triangle x \leq \top_t$.

$\triangle$Wk. $\triangle x \leq (y_{\top t} \rightarrow x)$.

Note that w.r.t. UL$\triangle_{\top}$-algebras, using $\triangle\top$, we can prove that (Nec.) $\top_t \leq x$ only if $\top_t \leq \triangle x$, and vice versa; and that w.r.t. UL$\triangle_x$- and UL$\triangle_{\top t}$-algebras, using $\triangle$t, we can prove (Nec.) and vice versa.[3)]

For simplicity, by a *UL$\triangle$-algebra*, we ambiguously express UL$\triangle_{\top}$-, UL$\triangle_x$-, and UL$\triangle_{\top t}$-algebras all together if we do not need distinguish them; and by an *L$\triangle$-algebra*, L$\triangle_{\top}$-, L$\triangle_x$-, and L$\triangle_{\top t}$-algebras. L$\triangle$-algebra is here defined as L-algebra in Definition 3.5.

Note that since the class of L-algebras forms a variety (see Proposition 3.6) and each condition for $\triangle$ operation has a form of equation or can be defined in equation, the class of all L$\triangle$-algebras forms a variety of algebras.

As in section 4.1, we can provide algebraic completeness for UL$\triangle$. The notion of evaluations and tautology easily

3) As an example we prove (Nec.) using $\triangle$t: let $\top_t \leq x$. Then, $\top_t \wedge x = \top_t$. Furthermore, $\triangle\top_t \leq x$ by $\triangle$t and so $\triangle\top_t \wedge x = \triangle\top_t$. By $\triangle$t and $\triangle$4 (and its converse), $\top_t = \triangle\top_t = \triangle(\top_t \wedge x) = \triangle\top_t \wedge \triangle x$. Hence, $\triangle\top_t \leq \triangle x$, and so by $\triangle$t, $\top_t \leq \triangle x$, as desired..

generalize to UL$\triangle$ and algebras. We just note that in case L$\triangle$ is a UL$\triangle$, the truth function for $\triangle$ (denoted by ($\triangle$)) can be given as follows:

($\triangle_\top$) $\triangle x = \top$ if $\top_t \leq x$,
$\bot$ otherwise.
($\triangle_x$) $\triangle x = x$ if $\top_t \leq x$,
$\bot$ otherwise.
($\triangle_{\top t}$) $\triangle x = \top_t$ if $\top_t \leq x$,
$\bot$ otherwise.

($\triangle_\top$) is for L$\triangle_\top$, ($\triangle_x$) for L$\triangle_x$, and ($\triangle_{\top t}$) for L$\triangle_{\top t}$. A$\triangle$1, Proposition 6.2 (i-1), and (ii-3) hold in L$\triangle_\top$; A$\triangle$1, (i-1), and (iii-2) in L$\triangle_x$; and A$\triangle$1, (i-1), and (iv-2) in L$\triangle_{\top t}$ (see Proposition 6.2). Thus we can ensure that in linearly ordered L$\triangle$-algebras, ($\triangle$) holds.[4] Moreover, the decomposition of any L$\triangle$-algebra as a subdirect product of (linearly) ordered ones holds. Then, using this we can give

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[4] More exactly, ($\triangle_\top$) can be proved as follows: by $\triangle$1, $\top_t \leq \max\{\triangle x_{\top t}, \triangle x \to \bot\}$. Let $\top_t \leq \triangle x_{\top t}$. Then, since $\triangle x_{\top t} \leq \triangle x$, by $\triangle$2, $\top_t \leq (\top \to \triangle x)$. This implies that $\triangle x = \top$. Furthermore, since $\triangle x \wedge \top_t = \top_t$, by $\triangle$5, $\triangle x_{\top t} \leq x$, i.e., $\top_t \leq x$. Otherwise, i.e., if $x < \top_t$, $\triangle x_{\top t} \leq x < \top_t$ by $\triangle$5. Then, since $\triangle x_{\top t} < \top_t$, $\top_t \leq (\triangle x \to \bot)$. This implies that $\triangle x \leq \bot$, i.e., $\triangle x = \bot$, as wished.
($\triangle_x$): by $\triangle$1, $\top_t \leq \max\{\triangle x_{\top t}, \triangle x \to \bot\}$. Let $\top_t \leq \triangle x_{\top t}$. This implies that $\top_t \leq \triangle x$. Then, by $\triangle 5'$, $\top_t \leq \triangle x \leq x$ and so $\top_t \leq x$. Furthermore, by $\triangle 2'$, $\triangle x = x$. Otherwise, i.e., if $x < \top_t$, $\triangle x \leq x < \top_t$ by $\triangle 5'$. Then, since $\triangle x < \top_t$ and so $\triangle x_{\top t} < \top_t$, $\top_t \leq (\triangle x \to \bot)$. Thus $\triangle x = \bot$, as desired.
($\triangle_{\top t}$): its proof is almost the same as ($\triangle_x$).}

algebraic completeness of L△.

Now, as in section 5.1, we show that a UL△ is *fuzzy* in Cintula's sense. For this we first regard UL△- and L△-algebras as UL△- and L△-matrices as in 5.1. We moreover consider L△ as a *finitary* logic in the above sense. Notice that Cintula showed that L△ (as a finitary WIL with DT△) is a fuzzy logic iff it holds (△PL) (see Corollary 11 in [2]). In an analogy to it, we can show that

Proposition 6.6 Let L△ be a finitary UL△ logic with RDT△. Then L△ is a fuzzy logic iff it holds ($\triangle PL_t$), i.e., $\vdash_{L\triangle} \triangle(\phi \to \psi)_t \vee \triangle(\psi \to \phi)_t$.

Proof: Left to right is obvious. For right to left, we show that L△ has PP. Let T, $\phi \to \psi \vdash \chi$ and T, $\psi \to \phi \vdash \chi$. By RDT△, T $\vdash_{L\triangle} \triangle(\phi \to \psi)_t \to \chi$ and T $\vdash_{L\triangle} \triangle(\psi \to \phi)_t \to \chi$. Then by A5 (together with (adj) and (mp)), T $\vdash_{L\triangle} (\triangle(\phi \to \psi)_t \vee \triangle(\psi \to \phi)_t) \to \chi$. Thus, by ($\triangle PL_t$) and (mp), T $\vdash \chi$, as desired. □

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