열전도 방정식의 시간 불연속 유한요소법 적용

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An Application of Time Discontinuous Finite Element Method for Heat Conduction Problems

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Abstract : A finite element method which is discontinuous in time is developed for the solution of the classical parabolic model of heat conduction problems. The approximations are continuous with respect to the space variables for each fixed time, but they admit discontinuities with respect to the time variable at each time step. The method is superior to other well-known approaches to these problems in that it allows a wider range of moving boundary value problems to be dealt with, such as are encountered in complex engineering operations like ground freezing. The method is applied to one-dimensional and two-dimensional heat conduction problems in this paper, although it could be extended to more higher dimensional problems. Several example problems are discussed and illustrated, and comparisons are made with analytical approaches where these can also be used.

초 록 : 시간에 불연속성인 유한요소법이 열전도 방정식에 적용하였다. 근사값은 고정된 시간에 공간변수에는 연속이며 그러나 각 시간 구간에서는 시간변수에 불연속을 허용하였다. 이 유한요소법은 지금까지 많이 알려진 재래식 유한요소해석에 보다 해의 수렴속도가 빠르고 해를 쉽게 얻을 수 있으며 지반이 동결된 동상지반과 같이 복잡한 공학문제와 같은 동적 경계치 문제에 쉽게 접근할 수 있었다. 다차원 문제에도 적용이 가능하며 본 연구에서는 일차원, 이차원 열전도 문제에 적용하였다. 결과 치를 해석해와 비교 검토하였다.

Key Words : finite element method, discontinuous in time, parabolic equation, conduction, heat

1. Introduction

The solution of parabolic partial differential equations is of great importance in the modelling of many engineering system. With the large number of numerical methods for parabolic partial differential equations in vogue there is a need for comprehensive criteria to compare the accuracy of the various suitable methods. The most commonly used numerical method for the classical parabolic model of heat conduction problems is based on a Galerkin discretization in space followed by a difference approximation of the derivatives in the resulting semi-discrete system of time-dependent ordinary differential equations[1]. Procedures of this kind are widely used in practice and well understood numerically. However, these methods are less appropriate for some time-dependent problems, in particular for time-dependent free surface boundary value problems. In the solution of large scale transient problems that involve phase boundaries during nonequilibrium thermal processes or classical boundary layers, it might be hard to accurately capture the heat wave behavior. Recently, the time-discontinuous Galerkin method based upon using a finite element formulation in time has evolved. This method, working from the differential equation viewpoint, is different from those which have been previously studied in recent years[2,3]. The idea is to approximate each element in the solution of a system of partial differential equations by a discontinuous piecewise polynomial on one subinterval at a

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time. The approximations are continuous with respect to the space variables for each time step, but they admit discontinuities with respect to the time variable at each time step. In particular, the elements can be chosen arbitrarily at each time step with no connection with the elements corresponding to the previous step. Eriksson et al.[4] and Jamet[5] have performed well in practice and established a comprehensive set of mathematical results by using these methods. Structural dynamic problem by a time-discontinuous Galerkin method has been achieved by Li. and Wiberg[6]. Kim[7] also has solved two-dimensional heat equation by discontinuous time-space Galerkin method. However, most of the early works on the discontinuous Galerkin method for differential equations have been mathematically oriented and proved. In this paper, our consideration will be confined to problems of non-stationary heat conduction of finite regions[8]. The finite element method is described in detail, with emphasis placed on the formulation of a finite element for the general linear parabolic equations in a given time-dependent domain based the time-discontinuous Galerkin approach. By using the discontinuous discretization in time, we are able to match sequentially through time and solve for only a fraction of the total solution at one time. The mathematical properties of the time-discontinuous Galerkin method for unsteady problems are completely analogous to those of the Galerkin method for the steady case, as observed by Bruch and Zyvoloski[9]. This is important because the construction of a good method based upon the Galerkin method for a steady problem readily extends to unsteady problems. Some numerical examples which confirm the stability analysis and in fact suggest that even more general results may be obtainable. Based on the numerical success of the time-discontinuous Galerkin method for parabolic differential equations, this research can be extended the method to problems of multi-dimensional transient general problems.

2. Governing Equations

The problem which is considered here is the numerical solution of the quasilinear parabolic equation typical of unsteady thermal fields in substances where the context of application will be that of a three-dimensional heat conduction problem.

$$\alpha c_p \dot{T}(t) - \nabla \cdot (k \nabla T(x, y, z, t)) = G(x, y, z, t) \text{ in } V \times I \quad (1)$$

subjected to boundary conditions:

$$T(x, y, z, t) = T_b \text{ for } t > 0 \text{ on } S_1 \quad (2)$$

$$k \nabla T(x, y, z) = Q_b(t) \text{ for } t > 0 \text{ on } S_2 \quad (3)$$

$$k \nabla T(x, y, z) + h_o(T - T_\infty) = 0 \text{ for } t > 0 \text{ on } S_3 \quad (4)$$

and initial condition as follow.

$$T(x, y, z, 0) = T_o(x, y, z) \text{ in } V(0),\ x, y, z \in V \quad (5)$$

The equations refer to unsteady thermal fields in substances with thermophysical properties dependent on temperature, where k is the thermal conductivity of materials(W/m · ℃); α is the density of material (kg/m^3); c_p is the specific heat of material(W · h/kg · ℃); G is the rate of internal heat generation(W/m^3); T is the temperature field(℃) for x, y, $z \subseteq V$ at time $t \in I$ and where I is the time domain; ∇ is the spatial gradient operator; T_b is the surface prescribed temperature(℃); $Q_b(t)$ is the surface prescribed heat flux(W/m^2); h_o is the heat transfer coefficient(W/m^2 · ℃); T_∞ is the surrounding temperature(℃); T_o is the initial temperature field(℃); $\dot{T}$ is the temperature first-order time derivative($\partial T / \partial t$). S_1, S_2 and S_3 are the boundary on which temperature is specified and heat flux is specified and convection heat loss is specified, respectively.

3. Discontinuous Approximations in Time

With the traditional finite element method, equation (1) can be discretized in spatial domain and expressed as

$$K_t \dot{T}(t) + K_c T(t) = Q(t),\ t \in I \quad (6)$$

Each component matrix is given as

$$K_t = \alpha c_p \int_V N^T N dV, \qquad (7)$$

$$K_c = \int_V B^T DB dV + \int_S h_o N^T N dS, \qquad (8)$$

$$Q(t) = \int_V GN^T dV - \int_V Q_b N^T dS + \int_V h_o T_\infty N^T dS \qquad (9)$$

where K_c can be regarded as the element thermal diffusivity matrix, K_t is the element matrix related to time dependence, $Q(t)$ is the element nodal vector of forcing parameters, which can be time dependent. Consider a partition of the time domain, having the form $t_0 < t_1 < t_2 < \ldots < T_N$. Let $\{t_n ; 0 \leq n \leq N\}$ be a finite sequence of real numbers with $t_0 = 0$, $t_n < t_{n+1}$ and denote by $I_n = [t_n, t_{n+1}]$ the n-th time interval. For the typical time instant t_n, a temporal jump operator of T is introduced with the notation

$$[T(t_n)] = T(t_n^+) - T(t_n^-) \qquad (10)$$

where

$$T(t_n^\pm) = \lim_{\lambda \to 0^\pm} T(t_n - \lambda)$$

Using the usual weighted residual approach to develop an integral equivalent to equation(6), the time-discontinuous Galerkin approximation to T on each interval I_n can now be formulated as

$$\int_{t_n}^{t_{n+1}} (K_t \dot{T} + K_c T)\, W dt = \int_{t_n^-}^{t_{n+1}^-} Q(t)\, W dt$$

after arranging the terms

$$\int_{t_n^-}^{t_n^+} K_t \dot{T}\, W dt + \int_{t_n^-}^{t_n^+} K_c T W dt + \int_{t_n^+}^{t_{n+1}^-} K_t \dot{T} + K_c T)\, W dt = \int_{t_n^+}^{t_{n+1}^-} Q(t)\, W dt \qquad (11)$$

This equation is decomposed into discontinuous and continuous time intervals. The value of the approximation at t_n, a point of discontinuity in the approximating polynomial T, is given by $(T(t_n^-) + T(t_n^+))/2$. This is an average across the jump. On the other hand, the derivative of T with respect to time using the delta function $\delta(t)$ is defined as

$$\begin{aligned} \dot{T}(t_n) &= \lim_{\lambda \to 0} \frac{T(t_n + \lambda/2) - T(t_n - \lambda/2)}{\lambda} \\ &= (T(t_n^+) - T(t_n^-))\, \delta(t - t_n) \\ &= (T(t_n)\, \delta(t - t_n) \end{aligned}$$

Substituting of delta function and average mean value into equation(11) leads to

$$K_t(T(t_n^+) - T(t_n^-)) \int_{t_n^-}^{t_n^+} K_t \delta(t - t_n) W dt + \lambda \frac{K_c}{2}(T(t_n^+) + T(t_n^-))\, W(t_n^+) + \int_{t_n^-}^{t_{n+1}^-} (K_t \dot{T} + K_c T)\, W dt = \int_{t_n^+}^{t_{n+1}^-} Q(t)\, W dt \qquad (12)$$

As λ approaches zero, the second term of equation (12) vanishes. Integrating the last term of Equation (12) by parts leads to

$$\int_{t_n^+}^{t_{n+1}^-} (-K_t T \dot{W} + K_c T W) dt + K_t T$$

$$(t_{n+1}^-)\, W(t_{n+1}^-) - K_t T(t_n^-)\, W(t_n^+) = \int_{t_n^+}^{t_{n+1}^-} Q(t)\, W dt \qquad (13)$$

with the initial condition

$$T(t_o^-) = T_o \qquad (14)$$

The integral in equation[13] resulting from integration-by-parts of the time flux constitutes the discontinuous Galerkin formulation.

4. Finite Element Formulation

A typical weighted residual equation can now be written assuming that the full domain of investigation corresponds with that of one element. Proceeding in the usual manner of discretization, with time as the independent variable, we can write the finite element functions for the n-th mesh

$$T = \Psi_{n+o}(t)\, T_{n+o} + \Psi_{n+1}(t)\, T_{n+1} \text{ for } t \in I_n \qquad (15)$$

where T_{n+0} are the nodal values of T at nodal points t_n^+, T_{n+1} at t_{n+1}^-, Ψ_{n+o} and Ψ_{n+1} are the piecewise polynomial temporal interpolation functions for node n. As only the first derivatives are involved in equation(6), first-order polynomials are sufficient to represent the interpolation functions Ψ_{n+1} in this approximation. Consider now a finite element of length Δt with T taking on nodal values T_{n+0} and T_{n+1}. These interpolation functions are assumed to be piecewise linear and are defined as

$$\Psi_{n+o}(t) = \frac{t_{n+1} - t}{\Delta t} \tag{16}$$

$$\Psi_{n+1}(t) = \frac{t - t_n}{\Delta t} \tag{17}$$

In terms of the local coordinate η, these interpolation functions and their derivatives can be written as

$$0 \le \eta \le 1 \quad \eta = \frac{t}{\Delta t} \tag{18}$$

$$\Psi_{n+o}(\eta) = 1 - \eta \; \dot{\Psi}_{n+o} = \frac{-1}{\Delta t} \tag{19}$$

$$\Psi_{n+1}(\eta) = \eta \; \dot{\Psi}_{n+1} = \frac{1}{\Delta t} \tag{20}$$

Insertion of the approximation for the function (18-20) into the variational equation(13) can be written as

$$\int_{t_n^+}^{t_{n+1}^-} (-\mathrm{Kt}(\Psi_n T_{n+o} + \Psi_{n+1} T_{n+1}) \dot{W}_i + K_t(\Psi_n T_{n+o} + \Psi_{n+1} T_{n+1}) W_i) dt + T(t_{n+1}^-)\, W(t_{n+1}^-) - T(t_n^-)\, W_i(t_n^+) = \int_{t_n^+}^{t_{n+1}^-} Q W_i dt \quad \text{for } i = n+o, n+1 \tag{21}$$

Using $W_i = \Psi_n = 1 - \eta$, and substituting the above functions of equation(18-20) into equation(21), we get as

$$(\frac{1}{2}K_t + \frac{1}{3}K_c \Delta t)\, T_{n+o} + (\frac{1}{2}K_t + \frac{1}{6}K_c \Delta t) T_{n+1} - K_t T_n = \frac{1}{2}\Delta t Q_{n+o} \quad \text{for } i = n+o \tag{22}$$

$$(-\frac{1}{2}K_t + \frac{1}{6}K_c \Delta t)\, T_{n+o} + (\frac{1}{2}K_t + \frac{1}{3}K_c \Delta t) T_{n+1} - K_t T_n = \frac{1}{2}\Delta t Q_{n+o} \quad \text{for } i = n+1 \tag{23}$$

The above equation is a recursive formulae, containing unknown values linking two times $n+0$ and $n+1$ spaced Δt apart. It is remarked that continuity of the nodal temperature vector T_n at any time level t_n in the time domain is automatically ensured in the present method. Because the initial values of the unknowns are specified, unknown values as successive times can be calculated. These linear algebraic equations are solved recursively, with iteration at each time step if necessary.

5. Numerical Implementations and Discussions

5.1. One-dimensional transient heat conduction

To illustrate the formulation of the method and its efficiency the solution of a simple problem(α = 1, k = 1, h_∞ = 1, c_p = 1) is presented. The problem is stated as

$$\frac{k}{\alpha c_p} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in } 0 < x < 1$$

subject to

$$u(0, t) = 10, \; u(1, t) + u_x(1, t) = 2, \; u(x, 0) = 3$$

The analytical solution for this problem is derived as

$$u(x, t) = 10 + \sum_{n=1}^{\infty} c_n \sin(p_n x) e^{-a^2 p_n^2 t}$$

$$c_n = \frac{2}{1 + \cos^2 p_n} [(-7)(1 - \frac{\cos p_n}{p_n}) + 8 (\frac{\sin p_n}{p_n^2} - \frac{\cos p_n}{p_n})], \; \sin p_n + p_n \cos p_n = 0, \; a^2 = 1$$

Fig. 1 presents profiles of temperature at five different times. These results in Fig. 1 show how the methods compare to the analytical solution with Δt

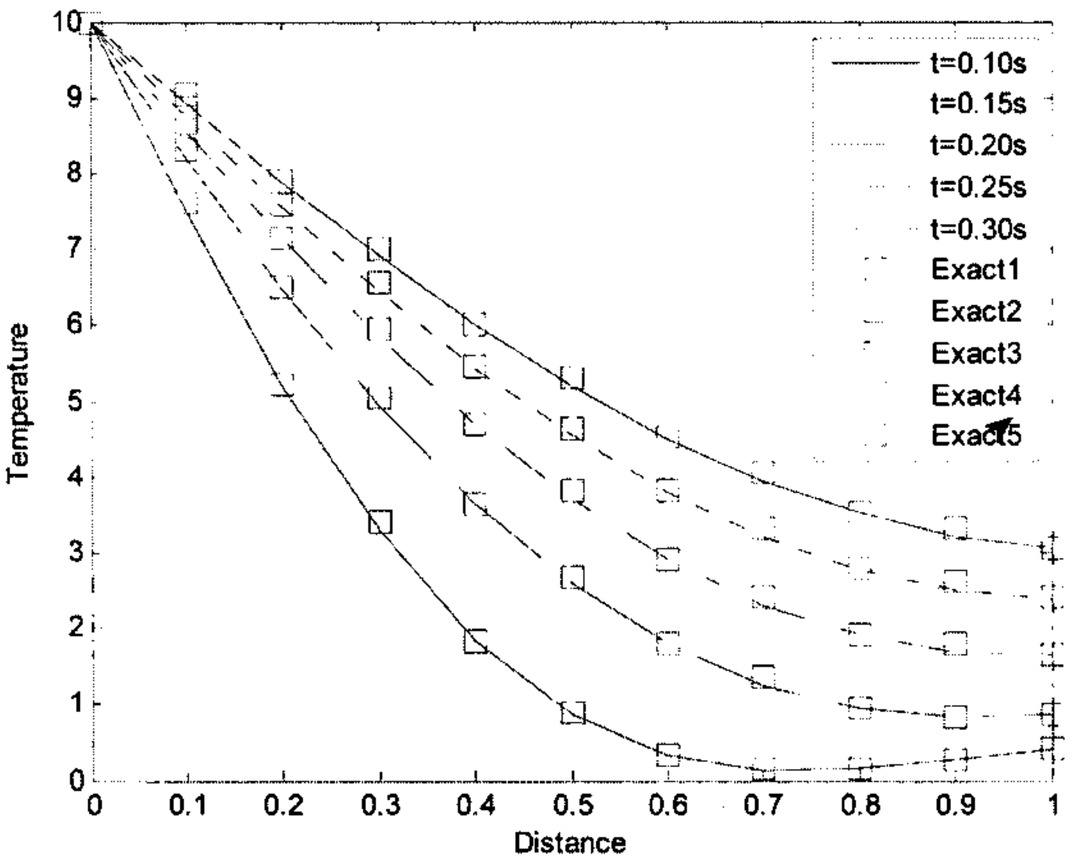

Fig. 1. Temperature distributions at various times versus distance.

= 0.01 sec. There is no visual difference between the analytical solution and the results of the discontinuous finite element method. The agreement of both solutions is quite good.

5.2. Two-dimensional transient heat conduction

The problem is stated as

$$k_x \frac{\partial^2 u}{\partial x^2} + k_y \frac{\partial^2 u}{\partial y^2} = \alpha c_p \frac{\partial u}{\partial t}$$

in $0 < x < 1,\ 0 < y < 1,\ t > 0$

subject to

$$u(0, y, t) = 0,\ u(1, y, t) = 0,\ u(x, 0, t) = 0,$$
$$u(x, 1, t) = 0,\ u(x, y, 0) = 1$$

The Fourier solution is expressed as

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, e^{-[m^2\pi^2 + n^2\pi^2]t}$$
$$(\sin m\pi x)\ (\sin n\pi y)$$
$$Amn = \frac{1}{mn\pi^2}(\cos m\pi - 1)(\cos n\pi - 1)$$

For simplicity, the bar is considered to be of a unit length and has been made for ten elements for each side. Hence, the employed finite element mesh consists of 10×10 rectangles. The thermophysical properties used for computation are also simple values

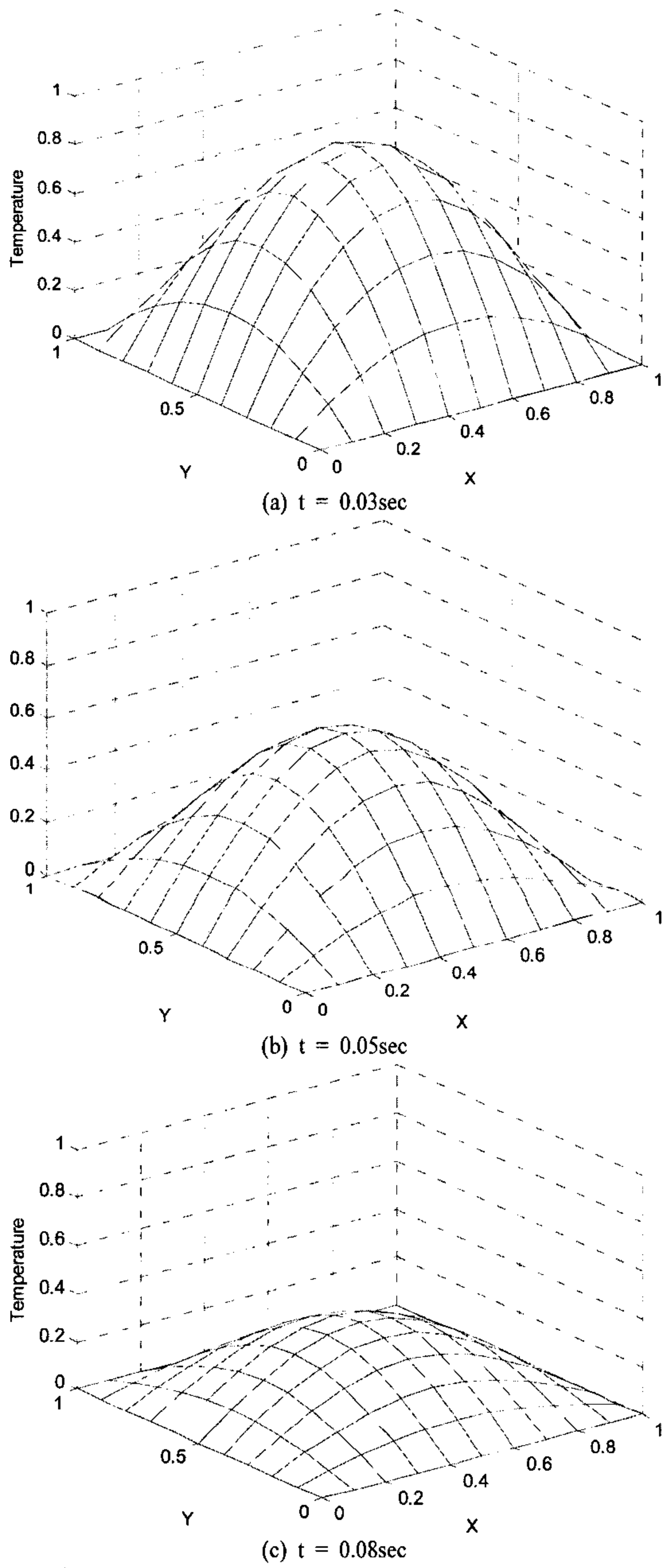

Fig. 2. Temperature distributions at various time levels.

as $k = 1$, $\alpha = 1$ and $c_p = 1$. The basic time increment was chosen to be 0.01sec., which corresponds to $\Delta t < \alpha c_p(\Delta x) < 2k$. Compared with the solutions obtained by using the analytical method, the accuracy of the present results is favorable. The large time increment $\Delta t = 0.15$ sec is taken for numerical stability considerations. Fig. 2 presents profiles of temperature at three different time levels(t = 0.03

sec, t = 0.05 sec, t = 0.08 sec) with Δt = 0.01 sec. It is noted that in all cases the solution is very accurately predicted, even with the five element model and the large time step. However, for an accurate prediction of the temperature, a finer finite element discretization and smaller time step need to be employed.

6. Conclusions

We have demonstrated that some of the parabolic type problems encountered in such branches of engineering as heat conductions can be analyzed successfully by means of the time-discontinuous Galerkin finite element method. A program has been developed for the case of one-dimensional and two-dimensional rectangular elements, and the solutions of some examples have been given in this paper. In comparison with the traditional finite element method, the present method possesses a number of usual advantages that made solutions very divergent. We have shown that the time-discontinuous method can provide very accurate solutions to the unsteady heat conduction problems. No significant instability problems and much more rapid convergence to the analytical solution were experienced in this approach than the standard semi-discrete method. It is generally said that the disadvantage of time continuous finite element methods is that a small time increment should be taken to insure numerical stability. This is particularly serious when the moving heat source produces steep gradients of the temperature within and near the region of moving source. We have demonstrated the efficiency of using finite elements by allowing discontinuities with the respect to the time variables at each time step. It may use fewer elements and larger time step size. Furthermore, we showed that the time-discontinuous methods lead to a unconditional stable higher accurate ordinary differential equations solver. This is in contrast to the conditional stability of some time-continuous methods. Furthermore, the time-discontinuous method seems conducive to the establishment of problems for capturing oscillations due to the steep gradients.

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