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STRONG LAWS OF LARGE NUMBERS FOR LINEAR PROCESSES GENERATED BY ASSOCIATED RANDOM VARIABLES IN A HILBERT SPACE

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Abstract. Let $\{\xi_k, k \in \mathbb{Z}\}$ be an associated sequence of H-valued random variables with $E\xi_k = 0$, $E\|\xi_k\| < \infty$ and $E\|\xi_k\|^2 < \infty$ and $\{a_k, k \in \mathbb{Z}\}$ a sequence of bounded linear operators such that $\sum_{j=0}^{\infty} j\|a_j\|_{L(H)} < \infty$. We define the stationary Hilbert space process by $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$ and prove that $n^{-1}\sum_{k=1}^{n} X_k$ converges to zero.

1. Introduction

Let H be a separable real Hilbert space with the norm $\|\cdot\|_H$ generated by an inner product, $< \cdot, \cdot >_H$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H. Let $L(H)$ be the class of bounded linear operators from H to H and denote by $\|\cdot\|_{L(H)}$ its usual norm. Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary sequence of H-valued random variables and $\{a_k, k \in \mathbb{Z}\}$ a sequence of operators with $a_k \in L(H)$. We define the stationary Hilbert space process by:

$$X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}, k \in \mathbb{Z}. \tag{1.1}$$

The sequence $\{X_k, k \in \mathbb{Z}\}$ is a natural extension of the multivariate linear processes(Brockwell and Davis[3], Chap. 11). We define

$$S_n = \sum_{k=1}^{n} X_k. \tag{1.2}$$

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Notice that if $\sum_{j=0}^{\infty} \|a_j\|_{L(H)} < \infty$ and $\{\xi_k, k \in \mathbb{Z}\}$ is a sequence of H-valued i.i.d. random variables centered in $L_2(H)$, then it is well known that the series in (1.1) converges almost surely([1], Chap. 3.2).

Lehmann[7] introduced the notion of positive quadrant dependence: Two random variables ξ_1 and ξ_2 are called positively quadrant dependent if for all real $\alpha_1,\ \alpha_2$

$$P(\xi_1 > \alpha_1, \xi_2 > \alpha_2) \geq P(\xi_1 > \alpha_1)P(\xi_2 > \alpha_2).$$

A finite family $\{\xi_i, 1 \leq i \leq n\}$ of real-valued random variables is said to be associated if for any coordinatewise increasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$

$$Cov(f(\xi_1, \cdots, \xi_n), g(\xi_1, \cdots, \xi_n)) \geq 0 \tag{1.3}$$

whenever this covariance exists. An infinite family of random variables is negatively associated if every finite subfamily is associated. This concept of dependence was introduced by Esary, Proschan and Walkup[5].

Let us remark that associated random variables are always pairwise positive quadrant dependent and pairwise independent random variables are always pairwise positive quadrant dependent and associated.

Newman[9] studied strong law of large numbers for strictly stationary associated sequences. Birkel[2] also studied the strong law of large number for the non stationary associated sequences.

As Burton, Dabrowski and Dehling[4] introduced the definition of weak association for random vectors we can give the definition of association for random vectors with values in $\mathbb{R}^d$:

Let $\{\xi_1, \cdots, \xi_m\}$ be a sequence of $\mathbb{R}^d$-valued random vectors. $\{\xi_1, \cdots, \xi_m\}$ is said to be associated if $Cov(f(\xi_1, \cdots, \xi_m), g(\xi_1, \cdots, \xi_m)) \geq 0$ for any nondecreasing functions f and g on $\mathbb{R}^{md}$, such that the covariance exists.

We also extend the concept of association for random vectors with values in $\mathbb{R}^d$ to random vectors with values in a separable Hilbert space as follows. Let $\{\xi_n, n \geq 1\}$ be a sequence of random variables taking values in a Hilbert space $(H. < \cdot, \cdot >)$. $\{\xi_n, n \geq 1\}$ is said to be associated if for some orthonormal basis $\{e_k, k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $(< \xi_i, e_1 >, \cdots, < \xi_i, e_d >), i \geq 1$, is associated(see [6]).

In Section 2 we will study the strong law of large number for associated random variables in a Hilbert space and in Section 3 we prove the strong law of large numbers for a strictly stationary linear process generated by associated random variables in a Hilbert space by applying this result.

2. Preliminaries

The following results are obtained by Ko et al.[6]. For completeness we repeat them here.

Lemma 2.1. Let $\{\xi_1, \cdots, \xi_m\}$ be a sequence of associated random variables with $E(\xi_i^2) < \infty$ and $E\xi_i = 0$, $i \geq 1$, and let $S_m = \xi_1 + \cdots, \xi_m$. Then

$$E(\max(|S_1|, \cdots, |S_m|))^2 \leq 2E(S_m^2). \tag{2.1}$$

Proof. First note that $\max(0, S_1, \cdots, S_m)$ is a nondecreasing function of $\xi_i^{'}$s and $\max(0, -S_1, \cdots, -S_m)$ is a nonincreasing function of $\xi_i^{'}$s, so $\max(0, S_1, \cdots, S_m)$ and $\max(0, -S_1, \cdots, -S_m)$ are negatively correlated by the definition of association(see (1.3) and Esary et al.[5]) and that

$$\max(|S_1|, \cdots, |S_m|) \leq \max(0, S_1, \cdots, S_m) + \max(0, -S_1, \cdots, -S_m).$$

Thus, by Theorem 2 in Newman and Wright[8] we have

$$\begin{aligned}
&E(\max(|S_1|, \cdots, |S_m|))^2 \\
&\leq E(\max(0, S_1, \cdots, S_m))^2 + E(\max(0, -S_1, \cdots, -S_m))^2 \\
&\ +2E(\max(0, S_1, \cdots, S_m) \cdot \max(0, -S_1, \cdots, -S_m)) \\
&\leq E(\max(0, S_1, \cdots, S_m))^2 + E(\max(0, -S_1, \cdots, -S_m))^2 \\
&\leq E(\max(S_1, \cdots, S_m))^2 + E(\max(-S_1, \cdots, -S_m))^2 \\
&\leq E(S_m^2) + E((-S_m)^2) \\
&= 2E(S_m^2).
\end{aligned}$$

We extend Lemma 2.1 to a Hilbert space:

Lemma 2.2. Let $\{\xi_i,\ i \geq 1\}$ be a strictly stationary associated sequence of H-valued random variables with $E\xi_1 = 0$ and $E\|\xi_1\|^2 < \infty$. Then

$$E \max_{1\leq k\leq n} \| \sum_{j=1}^{k} \xi_j \|^2 \leq 2E\| \sum_{j=1}^{n} \xi_j \|^2. \tag{2.2}$$

Proof. From (2.1) of Lemma 2.1 we obtain:

$$
\begin{aligned}
(2.3)\ E \max_{1\le k\le n}\Big(\sum_{j=1}^{k} <\xi_j, e_i>\Big)^2 &= E \max_{1\le k\le n}\Big(\Big|\sum_{j=1}^{k} <\xi_j, e_i>\Big|\Big)^2 \\
&\le E\Big(\max_{1\le k\le n}\Big|\sum_{j=1}^{k} <\xi_j, e_i>\Big|\Big)^2 \\
&\le 2E\Big(\sum_{j=1}^{n} <\xi_j, e_i>\Big)^2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
E \max_{1\le k\le n}\Big\|\sum_{j=1}^{k}\xi_j\Big\|^2 &= E \max_{1\le k\le n}\sum_{i=1}^{\infty}\Big(<\sum_{j=1}^{k}\xi_j, e_i>\Big)^2 \\
&\le \sum_{i=1}^{\infty} E \max_{1\le k\le n}\Big(\sum_{j=1}^{k} <\xi_j, e_i>\Big)^2 \\
&\le 2\sum_{i=1}^{\infty} E\Big(\sum_{j=1}^{n} <\xi_j, e_i>\Big)^2 \text{ by (2.1)} \\
&= 2E\sum_{i=1}^{\infty}\Big(<\sum_{j=1}^{n}\xi_j, e_i>\Big)^2 \\
&= 2E\Big\|\sum_{j=1}^{n}\xi_j\Big\|^2.
\end{aligned}
$$

Remark. From Lemma 2.2 we have the following Kolmogorv type inequality for H-valued associated random variables :

$$P\Big(\max_{1\le k\le n}\|S_k\| > \epsilon\Big) \le \epsilon^{-2}E\|S_n\|^2.$$

Theorem 2.3. Let $\{\xi_n,\ n\ge 1\}$ be an associated sequence of H-valued random variables with $E\xi_n = 0$ and $E\|\xi_n\|^2 < \infty, n \ge 1$.

Assume

$$(2.4)\qquad \sum_{j=1}^{\infty} j^{-2}E(<\xi_j, S_j>) < \infty.$$

Then, as $n \to \infty$ we have $n^{-1}S_n \to 0$ almost surely, where $S_n = \xi_1 + \cdots + \xi_n$.

Proof. Let $\epsilon > 0$. Then by Chebyshev's inequality

$$
\begin{aligned}
(2.5) \sum_{n=1}^{\infty} P\{2^{-n}\|S_{2^n}\| > \epsilon\} &\leq \epsilon^{-2}\sum_{n=1}^{\infty} 4^{-n}E\|S_{2^n}\|^2 \\
&= \epsilon^{-2}\sum_{n=1}^{\infty} 4^{-n}\sum_{k=1}^{\infty} E(< S_{2^n}, e_k >)^2 \\
&\leq 2\epsilon^{-2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(\sum_{n:2^n \geq j} 4^{-n}E(< \xi_j, e_k >< S_j, e_k > \\
&= 2\epsilon^{-2}\sum_{j=1}^{\infty}(\sum_{n:2^n \geq j} 4^{-n}E(< \xi_j, S_j >)) \\
&\leq (\frac{8}{3})\epsilon^{-2}\sum_{j=1}^{\infty} j^{-2}E(< \xi_j, S_j >) < \infty.
\end{aligned}
$$

Thus the Borel-Cantelli lemma implies that, as $n \to \infty$, $2^{-n}S_{2^n} \to 0$ almost surely. By standard arguments it suffices now to prove that, as $n \to \infty$,

$$
(2.6) \qquad 2^{-n}\max_{1 \leq k \leq 2^n}\|S_k\| \to 0 \text{ almost surely.}
$$

Using Chebyshev's inequality and applying Lemma 2.2 we obtain, for a given $n \geq 1$

$$
\begin{aligned}
(2.7) \qquad & P\{2^{-n}\max_{1 \leq k \leq 2^n}\|S_k\| > \epsilon\} \\
& \leq \epsilon^{-2}4^{-n}E(\max_{1 \leq k \leq 2^n}\|S_k\|)^2 \\
& \leq \epsilon^{-2}4^{-n}E\|S_{2^n}\|^2,
\end{aligned}
$$

and hence

$$\sum_{n=1}^{\infty} P\{2^{-n} \max_{1\le k\le 2^n} \|S_k\| > \epsilon\}$$

$$\le \epsilon^{-2} \sum_{n=1}^{\infty} 4^{-n} E\|S_{2^n}\|^2$$

$$\le \quad (\frac{8}{3})\epsilon^{-2} \sum_{j=1}^{\infty} j^{-2} E(<\xi_j, S_j>) < \infty,$$

according to the consideration of (2.5).

Again applying the Borel-Cantelli lemma, we obtain (2.6) which completes the proof of Theorem 2.3.

Corollary 2.4. Let $\{\xi_n, n \ge 1\}$ be an associated sequence of $\mathbb{R}^d$-valued random vectors, centered at expectations and with $E\|\xi_n\|^2 < \infty$, $n \ge 1$. Assume

$$\sum_{j=1}^{\infty} \sum_{i=1}^{d} j^{-2} E(\xi_{ji} S_{ji}) < \infty,$$

where ξ_{ji} is the i-th component of ξ_j and S_{ji} is the i-th component of S_j. Then, as $n \to \infty$ we have $n^{-1} S_n \to 0$ almost surely, where $S_n = \xi_1 + \cdots + \xi_n$.

3. Main results

Lemma 3.1. Let $\{a_k,\ k \in \mathbb{Z}\}$ be a sequence of bounded linear operators and let $\widetilde{a_i} = \sum_{j=i+1}^{\infty} a_j$. If $\sum_{j=0}^{\infty} j\|a_j\| < \infty$ then $\sum_{j=0}^{\infty} \|\widetilde{a_j}\| < \infty$.

Theorem 3.2. Let $\{\xi_k,\ k \in \mathbb{Z}\}$ be a strictly stationary associated sequence of H-valued random variables with $E\xi_n = 0$ and $E\|\xi_n\|^2 < \infty$. Let $\{a_k,\ k \in \mathbb{Z}\}$ be a sequence of bounded linear operators on H satisfying

$$\sum_{j=0}^{\infty} j\|a_j\|_{L(H)} < \infty. \tag{3.1}$$

Define the stationary Hilbert space by $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j},\ k \in \mathbb{Z}$. If (2.4) holds, then

$$n^{-1} \sum_{k=1}^{n} X_k \to 0 \ a.s. \tag{3.2}$$

Proof. Letting $\widetilde{a_j} = \sum_{i=j+1}^{\infty} a_i$ and $Y_k = \sum_{j=0}^{\infty} \widetilde{a_j} \xi_{k-j}$, which is well defined since $\sum_{j=0}^{\infty} \|\widetilde{a_j}\|_{L(H)} < \infty$ according to Lemma 3.1, we have

$$\begin{aligned} X_k &= \sum_{j=0}^{\infty} a_j \xi_{k-j} \\ &= a_0 \xi_k + \sum_{j=1}^{\infty} a_j \xi_{k-j} \\ &= (\sum_{j=0}^{\infty} a_j) \xi_k - \widetilde{a}_0 \xi_k + \sum_{j=1}^{\infty} (\widetilde{a_j} - \widetilde{a}_{j-1}) \xi_{k-j} \\ &= (\sum_{j=0}^{\infty} a_j) \xi_k + Y_{k-1} - Y_k, \end{aligned}$$

which implies that

$$\sum_{k=1}^{n} X_k = (\sum_{j=0}^{\infty} a_j) \sum_{k=1}^{n} \xi_k + Y_0 - Y_n.$$

Using Theorem 2.3 on $(\sum_{j=0}^{\infty} a_j) \sum_{k=1}^{n} \xi_k$, the theorem is proved if

$$n^{-1} Y_0 \to 0 \ a.s. \text{ and } n^{-1} Y_n \to 0 \ a.s., \ as\ n \to \infty. \tag{3.3}$$

To prove (3.3) we note that

$$E\|Y_k\| \le \sum_{j=0}^{\infty} \|\widetilde{a_j}\| E\|\xi_{k-j}\| = E\|\xi_{k-j}\| \sum_{j=0}^{\infty} \|\widetilde{a_j}\|_{L(H)} < \infty$$

and the fact that $\sum_{n=1}^{\infty} P(\|X\| > n) < \infty \Leftrightarrow E\|X\| < \infty$.

Hence

$$\sum_{n=1}^{\infty} P\{n^{-1}\|Y_n\| > \epsilon\} = \sum_{n=1}^{\infty} P\{n^{-1}\|Y_0\| > \epsilon\} < \infty$$

for any $\epsilon > 0$, which yields (3.3).

Corollary 3.3. Let $\{X_k,\ k = 0, \pm 1, \pm 2, \cdots,\}$ be a d-dimensional linear process of the form $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$, where $\{\xi_k, k \in \mathbb{Z}\}$ is a strictly stationary sequence of d-dimensional associated random vectors with $E\xi_n = 0$, $E\|\xi_n\| < \infty$ and $E\|\xi_n\|^2 < \infty$ and $\{a_k,\ k \in \mathbb{Z}\}$ is a sequence of $d \times d$ matrix with $\sum_{j=0}^{\infty} j\|a_j\| < \infty$. Assume that $\sum_{j=1}^{\infty} j^{-2} E < \xi_j, S_j >< \infty$. Then, as $n \to \infty$

$$n^{-1} \sum_{k=1}^{n} X_k \to 0 \ a.s.$$

Theorem 3.4. Let $\{\xi_k,\ k \in \mathbb{Z}\}$ be a strictly stationary associated sequence of H-valued random variables with $E\xi_1 = 0$, $E\|\xi_1\| < \infty$ and let $\{a_k,\ k \in \mathbb{Z}\}$ be a sequence of bounded linear operators on H satisfying (3.1). Then

$$n^{-1} \sum_{k=1}^{n} \xi_k \to 0 \ a.s. \text{ implies } n^{-1} \sum_{k=1}^{n} X_k \to 0 \ a.s.$$

The proof of Theorem 3.4 is similar to that of Theorem 3.2.

Corollary 3.5. Let $\{X_k,\ k \in \mathbb{Z}\}$ be a d-dimensional linear process of the form $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$, where $\{\xi_k, k \in \mathbb{Z}\}$ is a strictly stationary associated sequence of d-dimensional random vectors with $E\xi_1 = 0$, $E\|\xi_1\| < \infty$ and $\{a_k,\ k \in \mathbb{Z}\}$ is a sequence of $d \times d$ matrix with $\sum_{j=0}^{\infty} j\|a_j\| < \infty$. Then

$$n^{-1} \sum_{k=1}^{n} \xi_k \to 0 \ a.s. \text{ implies } n^{-1} \sum_{k=1}^{n} X_k \to 0 \ a.s.$$

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