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PROPERTIES OF DUAL RIESZ-NÁGY-TAKÁCS DISTRIBUTIONS

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Abstract. We give the relation between the Riemann-Stieltjes integrals with respect to the Riesz-Nágy-Takács(RNT) distribution $H_{a,p}$ satisfying the equation $a = (1-a)^m$ and those with respect to the dual RNT distribution $H_{1-a,1-p}$, which leads to a generalization of recent results for $a = \frac{1}{2}$.

1. Introduction

Recently moments of the Cantor distribution([7, 8, 9]) were investigated. More recently we([3]) also studied the moments of the RNT distribution([10]) which is a strictly increasing singular function using the so-called $(\tau, \tau-1)$-expansion([10]) of the unit interval. We note that the unit interval $[0,1]$ is the attractor of the iterated function system of some similarities with the open set condition([1, 4, 5, 6]). Further we([2]) also discussed the Riemann-Stieltjes integrals with respect to the RNT distribution $H_{a,p}$([3]) satisfying the equation $1-a = a^m$ where m is a positive integer over different intervals $[0, 1-a]$, $[a, 1]$ and $[0, 1]$ omitting $[1-a, 1]$. We note that the Riemann-Stieltjes integral with respect to the RNT distribution over $[0, a]$ was also shown in [3]. In this paper, we study the relation between the Riemann-Stieltjes integrals with respect to the RNT distribution $H_{a,p}$ satisfying the equation $a = (1-a)^m$ and the Riemann-Stieltjes integrals with respect to $H_{1-a,1-p}$ over $[0, a]$, $[1-a, 1]$, $[0, 1-a]$ and $[0, 1]$. This comparison gives the properties of the Riemann-Stieltjes integrals with respect to the RNT distribution $H_{a,p}$ satisfying the equation $a = (1-a)^m$ over $[1-a, 1]$, $[0, a]$, $[a, 1]$ and $[0, 1]$ omitting $[0, 1-a]$. This gives a generalization of recent results([7]) for $a = \frac{1}{2}$.

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2. Preliminaries

Let $\mathbb{N}$ be the set of the positive integers. Consider $a \in (0,1)$ and $p \in (0,1)$. We([2, 3]) recall the RNT distribution

$$H_{a,p}(x) = \sum_{j=1}^{\infty} \frac{(1-p)^{j-1}}{p^{j-1}} p^{a_j}$$

for

$$x = \sum_{j=1}^{\infty} \frac{(1-a)^{j-1}}{a^{j-1}} a^{a_j} \in (0,1]$$

with integers $1 \leq a_1 < a_2 < \cdots < a_j < \cdots$ and $H_{a,p}(0) = 0$. We note that if $a \neq p$ then it is a singular function([4, 10]) whereas it is the identity function if $a = p$.

3. Main results

We define $F(x) = H_{a,p}(x)$ and $G(x) = H_{1-a,1-p}(x)$. From now on, we consider the Riemann-Stieltjes integral $\int_\alpha^\beta \phi(t)dF(t)$ of a continuous function ϕ with respect to F over $[\alpha, \beta] \subset [0,1]$ and $\int_\alpha^\beta \phi(t)dG(t)$ similarly. We introduce the following propositions related to the RNT distribution $H_{1-a,1-p}$.

Proposition 1. ([3]) $G((1-a)x) = (1-p)G(x)$ *for* $x \in [0,1]$.

Proposition 2. ([3]) $G((1-a)+ax) = (1-p)+pG(x)$ *for* $x \in [0,1]$.

From now on, we assume that $a = (1-a)^m$ for some $m \in \mathbb{N}$. In this case, we note that $G(1-a) = 1-p$. We give some scaling and translation properties already obtained of the $(\tau, \tau-1)$-expansion of the unit interval.

Theorem 3. ([2]) $G(ax) = (1-p)^m G(x)$ *for* $x \in [0,1]$.

Theorem 4. ([2]) $G((1-a)+x) = (1-p) + \frac{p}{(1-p)^m} G(x)$ *for* $x \in [0,a]$.

Using the scaling and translation properties of the $(\tau, \tau-1)$-expansion of the unit interval, we had the following Riemann-Stieltjes integrals over different intervals.

Theorem 5. ([2])

$$\int_0^a \phi(t)dG(t) = (1-p)^m \int_0^1 \phi(at)dG(t).$$

Theorem 6. ([2])

$$\int_{1-a}^{1} \phi(t)dG(t) = \frac{p}{(1-p)^m} \int_0^a \phi((1-a)+t)dG(t).$$

We give some relation between two dual distributions.

Proposition 7.

$$F(y) - F(x) = G(1-x) - G(1-y)$$

for $0 \leq x < y \leq 1$.

Proof. It is clear to see that

$$F(x) = 1 - G(1-x)$$

from the dual graphs. □

Using the above property, we get the following relation between the Riemann-Stieltjes integrals with respect to two dual distributions.

Theorem 8.

$$\int_0^1 \phi(t)dF(t) = \int_0^1 \phi(1-t)dG(t).$$

Proof. Consider a partition

$$t_0 = 0 < t_1 = \frac{1}{n} < t_2 = \frac{2}{n} < \ldots < t_n = 1$$

of $[0,1]$. Putting $s_i = 1 - t_i$, we have a partition

$$s_n = 0 < s_{n-1} = \frac{1}{n} < s_{n-2} = \frac{2}{n} < \ldots < s_1 = 1 - \frac{1}{n} < s_0 = 1$$

of $[0,1]$. From Proposition 3, we have

$$\begin{aligned}
\int_0^1 \phi(t)dF(t) &= \lim_{n\to\infty} \sum_{i=1}^{n} \phi(t_i)[F(t_i) - F(t_{i-1})] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \phi(t_i)[G(1-t_{i-1}) - G(1-t_i)] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \phi(1-s_i)[G(s_{i-1}) - G(s_i)] \\
&= \int_0^1 \phi(1-s)dG(s).
\end{aligned}$$

□

Similarly we obtain the following relations between the Riemann-Stieltjes integrals with respect to two dual distributions F and G.

Theorem 9.

$$\int_{1-a}^{1} \phi(t)dF(t) = \int_{0}^{a} \phi(1-t)dG(t) = (1-p)^m \int_{0}^{1} \phi(1-at)dG(t).$$

Proof. Using similar arguments in the proof of Theorem 8, we get

$$\int_{1-a}^{1} \phi(t)dF(t) = \int_{0}^{a} \phi(1-t)dG(t).$$

It follows from Theorem 5. □

Theorem 10.

$$\int_{0}^{a} \phi(t)dF(t) = \int_{1-a}^{1} \phi(1-t)dG(t) = \frac{p}{(1-p)^m} \int_{0}^{a} \phi(a-t)dG(t).$$

Proof. Using similar arguments in the proof of Theorem 8, we get

$$\int_{0}^{a} \phi(t)dF(t) = \int_{1-a}^{1} \phi(1-t)dG(t).$$

It follows from Theorem 6. Precisely it follows from

$$\int_{1-a}^{1} \phi(1-t)dG(t) = \frac{p}{(1-p)^m} \int_{0}^{a} \phi(1-t-(1-a))dG(t).$$

□

Theorem 11.

$$\int_{a}^{1} \phi(t)dF(t) = \int_{0}^{1-a} \phi(1-t)dG(t) = (1-p) \int_{0}^{1} \phi(1-(1-a)t)dG(t).$$

Proof. Using similar arguments in the proof of Theorem 8, we get

$$\int_{a}^{1} \phi(t)dF(t) = \int_{0}^{1-a} \phi(1-t)dG(t).$$

It follows from Theorem 6. Precisely it follows from

$$\int_{0}^{1-a} \phi(t)dG(t) = (1-p) \int_{0}^{1} \phi((1-a)t)dG(t).$$

□

We obtain the following relations between the Riemann-Stieltjes integrals with respect to F over $[0,1]$ and different intervals.

Corollary 12.

$$\int_{1-a}^{1} \phi(t)dF(t) = (1-p)^m \int_0^1 \phi(at + (1-a))dF(t).$$

Proof. It follows from Theorem 9 and Theorem 8 with

$$\phi(1 - a(1-t)) = \phi(at + (1-a)).$$

□

We give proofs of the following two Corollaries for their relations between our Theorems, even though they were shown already.

Corollary 13. ([3])

$$\int_0^a \phi(t)dF(t) = p \int_0^1 \phi(at)dF(t).$$

Proof. From Theorem 10,

$$\int_0^a \phi(t)dF(t) = \frac{p}{(1-p)^m} \int_0^a \phi(a-t)dG(t).$$

Further from Theorem 5

$$\int_0^a \phi(a-t)dG(t) = (1-p)^m \int_0^1 \phi(a-at)dG(t).$$

From Theorem 8,

$$\int_0^1 \phi(a-at)dG(t) = \int_0^1 \phi(a - a(1-t))dF(t).$$

□

Corollary 14. ([3])

$$\int_a^1 \phi(t)dF(t) = (1-p) \int_0^1 \phi((1-a)t + a)dF(t).$$

Proof. It follows from Theorems 11 and 8, that is,

$$(1-p)\int_0^1 \phi(1-(1-a)t)dG(t) = (1-p)\int_0^1 \phi((1-a)t+a)dF(t).$$

□

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