Honam Mathematical J. **30** (2008), No. 4, pp. 617–629

REGULAR ENDOMORPHISM RINGS OF PROJECTIVE MODULES

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ABSTRACT. In this paper, the authors have found an equivalent condition of the endomorphism ring $End(M)$ of a projective module M being von Neumann regular (Theorem 1.14) and found an equivalent condition of any assocaitive ring R being von Neumann regular (Theorem 1.13).

0. Introduction

Assume that any ring R is an associative ring with identity. For left $R-$modules M and N, an $R-$homomorphism $f : M \to N$ is defined by $m \mapsto f(m)$, for each $m \in M$. The ring of all $R-$endomorphisms on a left $R-$module M is denoted by $End_R(M)$. And we have some materials to recall:

$$I^L = Hom_R(M, L) = \{f \in End_R(M) \mid Imf \leq L\},$$

for each submodule $L \leq M$ of M. For a subset J of $End_R(M) = S$, let $ImJ = JM = \sum_{f \in J} Imf = \sum_{f \in J} f(M)$ be the sum of images of endomorphisms in J. We call N an *open* submodule of M if

$$N = N^o, \text{ where } N^o = \sum_{f \in S,\ Imf \leq N} Imf \text{ is the sum of all images}$$

of endomorphisms in S contained in N.

Simply recall that the Jacobson Radical $Rad(M) = \cap M_\alpha$ for all maximal submodules $M_\alpha \lneq M$, for all $\alpha \in \Lambda$.

Received April 23, 2008. Revised October 1, 2008.
2000 Mathematics Subject Classification : 16D40, 16E50,16S50, 16S60, 16Nxx, 16W20.
Key words and phrases : semiprimitive, openly semiprimitive, maximal open submodule, subdirect product.
*: The first author was supported by Catholic University of Daegu.
**: The second author was supported by Chosun University.

A module M is said to be *semisimple* if it is a sum of (possibly infinitely many) simple modules (p.115, [5]).

A module ${}_RM$ is said to be *semiprimitive* if it has the zero Jacobson Radical of M, $Rad(M) = 0$, i.e., the zero intersection of all maximal submodules of M.

DEFINITION 0.1. *For any left $R-$module M,*

(1) *M is said to be openly simple if each image $Imf = M$, for each non-zero endomorphism $f : {}_RM \to {}_RM$.*
(2) *M is said to be openly semiprimitive if $Rad(M)^o = 0$.*
(3) *(p.94, [5]) Let $(M_\alpha)_{\alpha\in\Lambda}$ be an indexed set of modules and let $M \leq \prod_\Lambda M_\alpha$. Then M is said to be a subdirect product of $(M_\alpha)_{\alpha\in\Lambda}$ in case $(\pi_\alpha |_M) : M \to M_\alpha$ is an epimorphism, for each $\alpha \in \Lambda$.*

    Equivalently, (p.30, [6]) M is said to be a subdirect product of a class $\mathcal{U} = (M_\alpha)_{\alpha\in\Lambda}$ of left $R-$modules if there is a monomorphism $k : M \to \prod_\Lambda M_\alpha$ such that $\pi_\alpha \cdot k$ is an epimorphism for all $\alpha \in \Lambda$.

The Schur's lemma (p.152, [5] and p.52, [6]) says that each simple $R-$module M has a division endomorphism ring $End_R(M)$. A generalized Schur's lemma ([10]) is now introduced as follows.

PROPOSITION 0.2. [10]. *For any projective left $R-$module M, if M is openly simple, then its endomorphism ring $End(M)$ is a division ring.*

For example, taking the polynomial ring $R = \mathbb{Z}_p[x]$ (prime p) with an indeterminant x, then ${}_RR$ is a projective and openly simple $R-$module but not simple $R-$module.

An open submodule A of M is said to be *maximal open* in case $A \leq B \leq M$, for any open submodule B of M, implies that $A = B$ or $B = M$.

Clearly for any maximal submodule $N \leq M$, we have that N^o is maximal open in M

Easily it follows that for any left $R-$module ${}_RM$, if M is semiprimitive, then M is openly semiprimitive. But the converse does not hold, in general. See Example 0.3.

EXAMPLE 0.3. It is easy to find an openly semiprimitive module which is not semiprimitive. Consider an integral polynomial ring $\mathbb{Z}[x]$ with an indeterminant x, then its Jacobson Radical $Rad(\mathbb{Z}[x]) = x\mathbb{Z}[x] \neq 0$ but $Rad(\mathbb{Z}[x])^o = 0$ telling that $\mathbb{Z}[x]$ is openly semiprimitive but not semiprimitive.

In general, we mean a regular ring R by a von Neumann regular ring, that is, for any $a \in R$, there exists some element $b \in R$ such that $aba = a$.

The Proposition 4 (p.134, [6]) states that every left $R-$module is flat if and only if R is regular.

Also it is well-known that every commutative regular ring R is semiprimitve (p.33, [6]). In this paper we studied the openly semiprimitivity of projective modules whose endomorphism rings are regular in order to apply the free and projective module ${}_RR$ as a ring R to get regular ring R via a subdirect product of a class of openly simple modules.

It is well known that if a projective $R-$module P has a von Neumann regular (simply, regular) endomorphism ring $End_R(P) = S$, then the Jacobson Radical $Rad(S)$ is the zero ([3],[4]).

We also note that any ring R with identity can be regarded as a left (right, resp.)$R-$module ${}_RR$ (R_R, resp.) which is a module with $K = K^o$, for every submodule $K \leq R$ because every endomorphism $f : {}_RR \to {}_RR$ ($R_R \to R_R$, resp.) is completely determined by $f(1_R)$ in R.

$$\text{In detail, } K = \sum_{f(1_R)\in K} Imf = \sum_{f(1_R)\in K} Rf(1_R) = K^o, \quad \text{for each } K \leq M.$$

1. Results

We are to study endomorphism rings and homomorphism abelain groups of semiprimitve projective modules and openly semiprimitive projective modules.

PROPOSITION 1.1. *For any projective $R-$module M, we have the following:*

(1) *For any maximal open submodule N of M, we have a regular endomorphism ring $End_R(M/N)$.*

(2) *For any distinct maximal open submodules N_1, N_2 of M, we have that*

$$Hom_R(M/N_1, M/N_2) = \begin{cases} 0 \\ \{f \mid fgf = f, \text{ for some } g \in Hom_R(M/N_2, M/N_1)\}. \end{cases}$$

Proof. (1): For any non-zero $R-$homomorphism $f : M/N \to M/N$, considering the following diagram:

$$\begin{array}{ccccc} & & M & & \\ & & \big\downarrow \pi & & \\ & \swarrow \exists g & M/N & & \\ & & \big\downarrow f & & \\ M & \xrightarrow[\pi]{} & M/N & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccc} M & \overset{\exists g}{\underset{\leftarrow \exists g'}{\longrightarrow}} & M \\ \pi \big\downarrow & & \big\downarrow \pi \\ M/N & \xrightarrow{f} & M/N \\ \big\downarrow & & \big\downarrow \\ 0 & & 0, \end{array}$$

where $\pi : M \to M/N$ is a natural projection, then there exists an endomorphism $g : M \to M$ such that $\pi g = f\pi$, for any non-zero endomorphism $f : M/N \to M/N$ and a natural projection $\pi : M \to M/N$. Thus it follows immediately that g is an epimorphism from $f \neq 0$ (otherwise if g is not an epimorphism, then $Img \lneq M$ is an open proper submodule of M and $Img = g(M) \leq N$ inducing $f = 0$, which is contradicted). Since M is projective, g has a right inverse, say g', such that $gg' = 1_M$. Clearly we have that f is an epimorphism from the epimorphism $\pi g = f\pi$. Moreover the induced endomorphism, denoted by $g' : M/N \to M/N$, g' is a right inverse of f from the fact of $g'(N) \leq N$.

Therefore we have a regular endomorphism ring

$$\begin{aligned} End_R(M/N) &= \{f : M/N \to M/N \mid f \text{ is an epimorphism } \} \cup \{0\} \\ &= \{f : M/N \to M/N \mid f \text{ has a right inverse in } End_R(M/N)\} \cup \{0\} \\ &= \{f : M/N \to M/N \mid fg'f = f \text{ for some } g' \in End_R(M/N)\} \cup \{0\} \end{aligned}$$

(2): Let N_1 and N_2 be distinct maximal open submodules of M. Suppose that $f : M/N_1 \to M/N_2$ is any non-zero homomorphism. Then for an epimorphism $\pi_2 : M \to M/N_2$ and for a homomorphism $f\pi_1 : M \to M/N_2$ there exists an $R-$homomorphism $k : M \to M$ such that $f\pi_1 = k\pi_2$, since M is projective.

$$\begin{array}{ccccc} & & M & & \\ & & \big\downarrow{\scriptstyle \pi_1} & & \\ \exists_k \swarrow & & M/N_i & & \\ & & {\scriptstyle f}\big\downarrow & & \\ M & \xrightarrow[\pi_2]{} & M/N_2 & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccc} M & \underset{\leftarrow h}{\xrightarrow{\exists k}} & M & & \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} & & \\ M/N_1 & \xrightarrow{f \neq 0} & M/N_2 & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow & & \\ 0 & & 0 & & \end{array}$$

Since N_1 and N_2 are distinct maxiaml open submodules of M we have that

$$N_1 + N_2 = M.$$

By an easy calculation:

$$\begin{aligned} f(M/N_1) = f\pi_1(M) &= f\pi_1(N_1 + N_2) \\ &= f\pi_1(N_1) + f\pi_1(N_2) \\ &= \pi_2 k(N_1) + \pi_2 k(N_2) \\ &\neq 0, \end{aligned}$$

implying that $N_2 \lneq k(N_2) \leq k(M) \leq N_1 + N_2 = M$. Since $k(M)$ is an open submodule of M such that an open maximal submodule $N_2 \lneq k(M)$(otherwise, if $N_2 = k(M)$, which contradicts to $f \neq 0$) we have that $k(M) = M$ telling that k is an epimorphism. Since M is projective there is an $R-$homomorphism $h : M \to M$ such that $kh = 1_M$ the identity mapping on M.

$$\begin{array}{ccccc} & & M & & \\ \exists h \swarrow & & \big\downarrow{\scriptstyle 1_M} & & \\ M & \xrightarrow{k} & M & \longrightarrow & 0 \end{array}$$

Since $h(N_2) \leq N_1$ the induced homomorphism by h, denoted by g,

$$g : M/N_2 \to M/N_1, \text{ satisfies } fg = 1_{M/N_2}.$$

In addition to this, we have $fgf = f$.

It is easy to check that

$$Hom_R(M/N_1, M/N_2) \neq 0 \Longleftrightarrow Hom_R(M/N_2, M/N_1) \neq 0.$$

It completes the proof.

□

COROLLARY 1.2. *For any projective $R-$module M, we have the following:*

(1) *For any maximal submodule N of M, we have a regular endomorphism ring $End_R(M/N)$.*

(2) *For any distinct maximal submodules N_1, N_2 of M, we have that*

$$Hom_R(M/N_1, M/N_2) = \begin{cases} 0 \\ \{f \mid fgf = f, \text{ for some } g \in Hom_R(M/N_2, M/N_1)\}. \end{cases}$$

Proof. Because that every maximal submodule K induces an open maximal submodule K^o, precisely speaking, for any open submodule U of M with $K^o \leq U \leq M$, clearly we have that $K^o = U$ or $U = M$. Thus the proof is completed easily by the proof of the above Proposition 1.1. □

The following remark is a motivation of getting a new definition of a fully invariant subdirect product of a class of modules.

REMARK 1.3.

(1) *For any submodule $N \leq M$ of a module M, any restriction $f\mid_N : N \to M$ of an endomorphism $f : M \to M$ to the submodule N need not to have a one-side inverse function such as a left inverse or a right inverse of $f\mid_N$. For example, in a field $\mathbb{Q}$, forget the usual multiplication and regard $\mathbb{Q}$ as a left $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Q}$. Then we have a submodule $\mathbb{Z}$ of $\mathbb{Q}$, that is ${}_{\mathbb{Z}}\mathbb{Z} \leq {}_{\mathbb{Z}}\mathbb{Q}$ is a non-fully invariant submodule of ${}_{\mathbb{Z}}\mathbb{Q}$. Consider a left multiplication $\rho(2) : \mathbb{Z} \to \mathbb{Z} : k \mapsto 2k$ ($k \in \mathbb{Z}$), there is a right inverse $\rho(1/2) : \mathbb{Q} \to \mathbb{Q} : q \mapsto q/2$ ($q \in \mathbb{Q}$) in $End_{\mathbb{Z}}(\mathbb{Q})$ which is not in $End_{\mathbb{Z}}(\mathbb{Z})$.*

(2) *Consider a subdirect product* $\mathbb{Z}$ *of a class of simple modules* $\mathbb{Z}_p$ *(prime p). The monomorphism* $k : \mathbb{Z} \to \prod_{\text{prime } p} \mathbb{Z}_p$ *has a non-fully invariant submodule* $k(\mathbb{Z}) \leq \prod_{\text{prime } p} \mathbb{Z}_p$ *of* $\prod_{\text{prime } p} \mathbb{Z}_p$ *which induce the same trouble of any restricted endomorphisms having neither left inverse nor right inverse of the restricted endomorphism* $f\mid_{k(\mathbb{Z})}$ *of any endomorphism* $f : \prod_{\text{prime } p} \mathbb{Z}_p \to \prod_{\text{prime } p} \mathbb{Z}_p$.

For any left $R-$module M, M is said to be a subdirect product of a class $\{M_\alpha\}_{\alpha\in\Lambda}$ if there is a monomoprhism $k : M \to \prod_\Lambda M_\alpha$ such that $\pi_\alpha k : M \to M_\alpha$ is an epimorphism, for each $\alpha \in \Lambda$.

For example, a direct sum $\oplus_\Lambda L_\alpha$ and a direct product $\prod_\Lambda L_\alpha$ of modules $(L_\alpha)_{\alpha\in\Lambda}$ are subdirect products of modules $(L_\alpha)_{\alpha\in\Lambda}$ ([5]).

In order to have some criteria of modules having regular endomorphism rings of projective modules with their bases, we need the following definition.

Definition 1.4. *For any left* $R-$*module* M, M *is said to be a fully invariant subdirect product of a class of* $R-$*modules* $(M_\alpha)_{\alpha\in\Lambda}$ *if there is a monomoprhism*

$$k : M \to \prod_\Lambda M_\alpha \ \text{ such that}$$

(1) $\pi_\alpha k : M \to M_\alpha$ *is an epimorphism, for each* $\alpha \in \Lambda$,

(2) $k(M)$ *is fully invariant in* $\prod_\Lambda M_\alpha$.

Therefore the structures of the endomorphism ring $End_R(M)$ of M and the endomorphism ring $End_R(k(M))$ of $k(M)$ are isomorphic because of a monomorphism $k : M \to \prod_\Lambda M_\alpha$.

Recall that for any direct sum $\oplus_\Lambda M_\alpha$ of modules M_α ($\alpha \in \Lambda$) we have its endomorphism ring

$$End(\oplus_\Lambda M_\alpha) = Hom_R(\oplus_\Lambda M_\alpha, \oplus_\Lambda M_\alpha) = \prod_{(\alpha,\beta)\in\Lambda\times\Lambda} Hom_R(M_\alpha, M_\beta)$$

is expressed easily by 20.2 Proposition (p.236, [5]).

Recall Theorem 3.15 (Projective Basis) (p. 64, [8]): A module A is projective if and only if there exist elements $\{a_k | k \in K\} \subset A$ and $R-$maps (i.e., $R-$module homomorphisms) $\{\phi_k : A \to R \mid k \in K\}$ such that

(1) if $x \in A$, then almost all $\phi_k x = 0$;

(2) if $x \in A$, then $x = \sum_{k\in K} (\phi_k x) a_k$.

Moreover, A is generated by $\{a_k | k \in K\}$.

Recall that the Proposition 7 in [9] states that any left $R-$module M is a subdirect product of a class of modules $\mathcal{U} = (M_\alpha)_{\alpha\in\Lambda}$ if and only if the $\mathcal{P}-$reject of M in $U = (M_\alpha)_{\alpha\in\Lambda}$ is the zero, where the $\mathcal{P}-$reject is defined by the intersection

$$\cap\{\ker h |\ h \text{ is an epimorphism in } Hom_R(M, M_\alpha), \text{ for each } M_\alpha \in \mathcal{U}\ (\alpha \in \Lambda)\}.$$

PROPOSITION 1.5. *For any openly semiprimitive module M, if $Rad(M)^o = \cap_\Lambda M_\alpha = 0$, where each M_α is a maximal open submodule of M $(\alpha \in \Lambda)$, then M is a subdirect product of openly simple modules $(M/M_\alpha)_{\alpha\in\Lambda}$.*

Proof. It is sufficient to show that there is a monomorphism $k : M \to M/M_\alpha$. In fact, define $k : M \to M/M_\alpha$ by $m \mapsto (m + M_\alpha)$, for every $m \in M$. Then it follows that k is a monomorphism, inducing M is a subdirect product of openly simple modules $(M/M_\alpha)_{\alpha\in\Lambda}$. □

Now we can apply the above Proposition 1.5 to semiprimitive projective modules as follows:

COROLLARY 1.6. *For any semiprimitive $R-$module M, if $Rad(M) = \cap_\Lambda M_\alpha = 0$, where each M_α is a maximal submodule of M, then M is a subdirect product of simple modules $(M/M_\alpha)_{\alpha\in\Lambda}$.*

Proof. Similar method of the proof of Proposition 1.5 completes it. □

PROPOSITION 1.7. *For any projective $R-$module M with $Rad(M)^o = \cap M_\alpha = 0$, where each M_α is a maximal open submodule of M $(\alpha \in \Lambda)$, if M is a fully invariant subdirect product of $(M/M_\alpha)_{\alpha\in\Lambda}$, then its endomorphism ring $End_R(M)$ is a regular ring.*

Proof. Since M is projective, there is a basis $\mathcal{B} = \{a_k \mid k \in K\}$ and each canonical projection $\pi_\alpha : M \to M/M_\alpha$ is an epimorphism such that

$$\{\ \pi_\alpha(a_k) = a_k + M_\alpha \mid a_k \in \mathcal{B}\ \}$$

is a set of generators of M/M_α, for each maximal open submodule M_α and for each $\alpha \in \Lambda$.

By the Proposition 1.5 M is a subdirect product of openly simple modules (M/M_α).

Since we obtained a monomorphism

$$k : M \to \prod_{\Lambda} M/M_\alpha \text{ defined by } k(m) = (m + M_\alpha) \in \prod M/M_\alpha,$$

we have a fact of $k(m) = 0 + M_\alpha$ almost all $\alpha \in \Lambda$, which implies that the product $\prod_\Lambda M/M_\alpha$ of openly simple modules (M/M_α) can be replaced by the weak product $\prod^w M/M_\alpha$ of openly simple modules (M/M_α).

Furthermore $k(M) \leq \prod^w M/M_\alpha \leq \prod_\Lambda M/M_\alpha$ is fully invariant in $\prod^w M/M_\alpha$.

On the other hand, any weak direct product $\prod^w M/M_\alpha$ is a submodule of the direct sum $\oplus_\Lambda M/M_\alpha$ of exterior factor $(0, \cdots, M/M_\alpha, 0 \cdots, 0)$. Now we can apply the fact of the calculation $End(\oplus_\Lambda M/M_\alpha) = \prod_{\alpha,\, \beta} Hom_R(M_\alpha, M_\beta)$.

By the hypothesis of fully invariant submodule $k(M)$ in $\oplus_\Lambda M/M_\alpha$ and the fact that $k(M)$ has the endomorphism subring of a regular ring $End_R(\oplus(M/M_\alpha)) = \prod_{\alpha,\, \beta} Hom_R(M_\alpha, M_\beta)$ by the Proposition 1.1, it concludes that $End(k(M)) \simeq End(M)$ is a regular ring, completing the proof.

□

PROPOSITION 1.8. *For any projective $R-$module M with $Rad(M) = \cap M_\alpha = 0$, where each M_α is a maximal submodule of M, if M is a fully invariant subdirect product of $(M/M_\alpha)_{\alpha\in\Lambda}$, then its endomorphism ring $End_R(M)$ is regular ring.*

Proof. Replacing maximal open submodules of a projective module M by maximal submodules M_α of M in the proof of the Proposition 1.7 the proof is easily completed. □

REMARK 1.9. *The condition of fully invariantness of the image $k(M)$ in $\prod M/M_\alpha$ is essential in both the* Proposition 1.7 *and the* Proposition 1.8. *In the ring $\mathbb{Z}$ of integers, it is a subdirect product of simple modules $\mathbb{Z}_p$ (prime p) which is not fully invariant. But the endomorphism ring $End_{\mathbb{Z}}(\mathbb{Z})$ is not regular.*

We have noticed a fact that each openly simple module M has the properties of $Rad(M) = 0$ and $Rad(M)^o = 0$. And hence the next corollaries are followed.

COROLLARY 1. 10. *Every direct sum $\oplus_\Lambda M_\alpha$ of openly simple projective $R-$modules M_α $(\alpha \in \Lambda)$ has a regular endomorphism ring $End(\oplus_\Lambda M_\alpha)$.*

Proof. Since each direct sum $\oplus_\Lambda M_\alpha$ of projective $R-$modules M_α $(\alpha \in \Lambda)$ is a projective $R-$module and since

$$(Rad(\oplus M_\alpha))^o = (\oplus_\Lambda(Rad(M_\alpha)))^o = \oplus_\Lambda 0 = 0,$$

the endomorphism ring $End(\oplus_\Lambda M_\alpha$ is regular by the Proposition 1.7. □

Note here that every simple module M is a projective $R-$module. For each simple module M, it follows immediately that $M = Rx$ for any non-zero element $x \in M$. For this, consider any exact sequence

$$_RK \xrightarrow{\ \phi\ } {}_RL \longrightarrow 0.$$

For any given $R-$homomorphism $f : {}_RM = Rx \to {}_RL$, consider the following diagram:

$$\begin{array}{ccccc} & & {}_RM = Rx & & \\ & \exists h \swarrow & \downarrow f & & \\ {}_RK & \xrightarrow{\ \phi\ } & {}_RL & \longrightarrow & 0. \end{array}$$

Then there exists an $R-$homomorphism $h : {}_RM \to {}_RK : x \mapsto k$ such that $\phi h = f$ by selecting one element $k \in \phi^{-1}(f(x)) \leq K$. It says that every simple module is projective and hence every semisimple module is also projective.

COROLLARY 1. 11.

(1) *Every direct sum $\oplus M_\alpha$ of simple $R-$modules M_α $(\alpha \in \Lambda)$ has a regular endomorphism ring.*
(2) *Every semisimple $R-$module has a regular endomorphism ring.*

Proof. (1): Since each simple module M_α is openly simple and projective, the proof is completed immediately by the above Corollary 1.10.

(2): Since every semisimple module is a direct sum of simple modules, the proof is completed by (1). □

For any associative ring R with identity, consider R as a left $R-$module R, every submodule of $_RR$ is an open submodule of $_RR$.

In the ring theories, it is well-known that every regular ring R is semiprimitive .

THEOREM 1.12. *For any ring R with identity, if R is a fully invariant subdirect product of a class of simple modules M_α, then R is regular.*

Proof. By the Proposition 1.7, the proof is completed immediately since ${}_RR$ is free and projective. □

THEOREM 1.13. *For any ring R with identity, if R is a fully invariant subdirect product of a class of openly simple modules. Then we have the following: R is regular if and only if R is (openly) semiprimitive.*

Proof. Since every ring R is free and projective, if R is openly semiprimitive, then R is regular by the Proposition 1.5 with the hypothesis of a fully invariant subdirect product of a class of openly simple modules.

Assume that the a regular ring ${}_RR$ (as a left R–module) is not (openly) semiprimitive. Then

$$Rad({}_RR) = \{r = f(1_R) \mid Imf \text{ is small in } M,\ f \in End_R(R)\} \neq 0$$

induces a contradiction of a direct summand Imf of ${}_RR$ (see Theorem 1.14) . Therefore ${}_RR = R$ is (openly) semiprimitive.

□

It was studied that for any module M, the endomorphism ring $End(M)$ is von Neumann regular if and only if Imf and $\ker f$ are direct summands of M, for every $f \in End_R(M)$ (Lemma 3.1, [3]). The autors found a new result for projective module as follows:

THEOREM 1. 14. *For any projective R–module M, the following are equivalent:*

(1) *$End_R(M)$ is regular;*

(2) *Imf is a direct summand of M, for each $f \in End_R(M)$.*

Proof. (1) $\Longrightarrow$ (2): It is done in the Corollary 3.2 in [3].

(2) $\Longrightarrow$ (1): Assume that Imf is a direct summand of a projective module M. Then Imf is also a projective module. Consider the following diagram:

$$\begin{array}{ccccc} & & Imf & & \\ & \swarrow \exists g & \downarrow 1_{Imf} & & \\ M & \xrightarrow{f} & Imf & \longrightarrow & 0. \end{array}$$

Then there exists an R−homomorphism $g : Imf \to M$ such that $fg = 1_{Imf}$ since Imf is projective. Thus we have $g \oplus 0 : M = Imf \oplus K \to M$ in the endomorphism ring $End(M)$, for some submodule $K \leq M$. Therefore we have $f(g \oplus 0)f = f$, for every $f \in End(M)$. □

REMARK 1.15. *Easily a non-von Neumann regularity of the ring* $\mathbb{Z}$ *considered as a module* ${}_{\mathbb{Z}}\mathbb{Z}$ *follows from Theorem 1.12 since the endomorphism* $\rho(k) : \mathbb{Z} \to \mathbb{Z} : z \mapsto kz$ *defined by the multiplication by* $k \in \mathbb{N}$ *has its image* $Im\rho(k) = k\mathbb{Z}$ *being not a direct summand of* $\mathbb{Z}$.

References

[1] S. Lang, *Linear Algebra*, Forth Printing 1969, Addison-Wesley Publishing Company, 1966.

[2] I.N.Herstein, *Noncommutative rings*, The Math. Assoc. of America, 1968.

[3] Roger Ware, *Endomorphism Rings of Projective Modules*, Trans. of the AMS **55** (1971), 233–256.

[4] J. Zelmanowitz, *Regular Modules*, Trans. of the AMS **163** (1972), 341–355.

[5] Frank W.Anderson and Kent R.Fuller, *Rings and Categories of Modules*, New York ; Spring -Verlag, 1973.

[6] Joachim Lambek, *Lecture on Rings and Modules*, Chelsea Publ. Comp. New York, 1976.

[7] V. P. Camillo and K. R. Fuller, *Rings whose faithful modules and flat over their endomorphism rings*, Arch. Math. **27** (1976), 522–525.

[8] Joseph J. Rotman, *An introduction to homological Algebra*, Academic Press, Inc., 1979.

[9] S.-S. Bae, *On Reject of Subdirect Product*, Collection of thesis, Kyungnam University **14** (1987), 7–10.

[10] S.-S. Bae, *Generalized Schur's Lemmas*, Journal of Graduate School, Kyungnam University **16 No. 2** (2001), 7–11.

[11] S.-S. Bae, *Openly Semiprimitive Projective Modules*, Commun. Korean Math. Soc. **19** (2004), 619–637.

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