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ON THE $^*g$-ME-CONNECTION AND THE $^*g$-ME-VECTOR IN $^*g$-MEX_n

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ABSTRACT. A generalized n-dimensional Riemannian manifold X_n on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}$, satisfying certain conditions, through the $^*g$-ME-connection which is both Einstein's equation and of the form (3.1) is called $^*g$-ME-manifold and we denote it by $^*g$-MEX_n. In this paper, we prove a necessary and sufficient condition for the existence of $^*g$-ME-connection and derive a surveyable tensorial representation of the $^*g$-ME-connection and the $^*g$-ME-vector in $^*g$-MEX_n.

1. Introduction

In Appendix II to his last book, " The meaning of relativity ", Einstein [5] proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intend of this theory is physical, its exposition is mainly geometrical.

It may be characterized as a set of geometrical postulates for the space time X_4. However, the geometrical consequences of these postulates are not developed very far by Einstein. Characterizing Einstein's unified field theory as a set of geometrical postulates in X_4 Hlavatý [8] gave its mathematical foundation for the first time. Since then the geometrical consequence of these postulates are developed very far by numbers of mathematicians and theoretical physicists.

Generalizing X_4 to n-dimensional generalized Riemannian manifold X_n, n-dimensional generalization of this theory, so called *Einstein's n-dimensional unified field theory*(denoted by n-g-UFT), has been attempted by Wrede [12] and Mishra [11].

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Corresponding to n-g-UFT, Chung [1] introduced a new unified field theory, called *the Einstein's n-dimensional *g-unified field theory*(denoted by n-$^*g$-UFT), which is more useful than n-g-UFT in some physical aspects.

Friedmann and Schouten [6] introduced the idea of semi-symmetric connection in a differentiable manifold, and Hayden [7] the concept of metric connection. Yano [13] and Imai [9,10] assigned a semi-symmetric metric connection to an n-dimensional Riemannian manifold and found many results concerning this manifold. Recently, Chung [3] introduced a new concept of n-dimensional SE-manifold in n-g-UFT, imposing the semi-symmetri condition to X_n, and found a unique representation of n-dimensional Einstein's connection in a beautiful and surveyable form.

The purpose of the present paper is to study a necessary and sufficient condition for the existence of $^*g$-ME-connection and investigate the properties of the $^*g$-ME-connection and the $^*g$-ME-vector in the n-dimensional $^*g$-ME-manifold(denoted by $^*g$-MEX_n).

2. Preliminaries

This section is a brief collection of the basic concepts, notations, and results which are needed in our subsequent considerations in the present paper.

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys coordinate transformations $x^\nu \longrightarrow \bar{x}^\nu$ for which

$$Det\left(\frac{\partial \bar{x}}{\partial x}\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed which a general real non-symmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \tag{2.2}$$

where

$$Det(g_{\lambda\mu}) \neq 0, \qquad Det(h_{\lambda\mu}) \neq 0. \tag{2.3}$$

Hlavatý characterized Einstein's 4-dimensional unified field theory(4-g-UFT) as a set of geometrical postulates in a space-time X_4 for the first time and gave its mathematical foundation. Generalizing this theory, we may consider Einstein's n-dimensional unified field theory. Similarly, our n-dimensional $^*g$-unified field field theory(n-$^*g$-UFT), initiated by Chung [1] and originally suggested by Hlavatý[8], is based on the following three principles.

Principle A. The algebraic structure in n-$^*g$-UFT is imposed on X_n by the basic real tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta^\nu_\mu. \tag{2.4}$$

It may be decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}, \tag{2.5}$$

Since $Det(^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor $^*h_{\lambda\mu}$ by

$$^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.6}$$

In n-$^*g$-UFT, *we use both* $^*h^{\lambda\nu}$ *and* $^*h_{\lambda\mu}$ *as a tensors for raising and/or lowering indices of all tensor defined in* X_n *in the usual manner.* We then have, for example,

$$^*k_{\lambda\mu} = {}^*k^{\alpha\beta}{}^*h_{\lambda\alpha}{}^*h_{\mu\beta}, \quad {}^*g_{\lambda\mu} = {}^*g^{\alpha\beta}{}^*h_{\lambda\alpha}{}^*h_{\mu\beta}, \tag{2.7a}$$

so that

$$^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu}. \tag{2.7b}$$

Principle B. The differential geometric structure is imposed on X_n by the tensor $^*g^{\lambda\nu}$ by means of the connection $\Gamma^\nu_{\lambda\mu}$ defined by a system of Einstein's equations

$$D_\omega{}^*g^{\lambda\mu} = -2S_{\omega\alpha}{}^{\mu}{}^*g^{\lambda\alpha}. \tag{2.8}$$

Here D_ω denotes the symbol of the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$, and $S_{\lambda\mu}{}^\nu$ is *the torsion tensor* of $\Gamma^\nu_{\lambda\mu}$. The connection $\Gamma^\nu_{\lambda\mu}$ satisfying (2.8) is called *an Einstein's connection.* In virtue of (2.4), the system (2.8) is equivalent to the system of the original Einstein's equations

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}. \tag{2.9}$$

Principle C. In order to obtain ${}^*g^{\lambda\nu}$ involved in the solution for $\Gamma^\nu_{\lambda\mu}$, certain conditions are imposed. This conditions may be condensed to

$$S_\lambda = S_{\lambda\alpha}{}^{\alpha} = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} X_{\lambda]}, \quad R_{(\mu\lambda)} = \frac{1}{2}\left(R_{\mu\lambda} + R_{\lambda\mu}\right) = 0, \tag{2.10}$$

where X_λ is an arbitrary vector, S_λ is *the torsion vector*, and

$$R_{\omega\mu\lambda}{}^{\nu} = 2\left(\partial_{[\mu}\Gamma^{\nu}_{|\lambda|\omega]} + \Gamma^{\nu}_{\alpha[\mu}\Gamma^{\alpha}_{|\lambda|\omega]}\right), \tag{2.11}$$

$$R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^{\alpha}, \quad V_{\omega\mu} = R_{\omega\mu\alpha}{}^{\alpha} \tag{2.12}$$

are curvature tensors of X_n.

REMARK 2.1. *In* $\left\{\begin{array}{l} n - {}^*g - UFT \\ n - g - UFT \end{array}\right.$, *the algebraic structure is imposed on* X_n *by the tensor* $\left\{\begin{array}{l} {}^*g^{\lambda\nu} \\ g_{\lambda\mu} \end{array}\right.$. *On the other hand, in* $\left\{\begin{array}{l} n - {}^*g - UFT \\ n - g - UFT \end{array}\right.$, *the differential geometrical structure is imposed on* X_n *by* $\left\{\begin{array}{l} {}^*g^{\lambda\nu} \\ g_{\lambda\mu} \end{array}\right.$ *through* $\Gamma^\nu_{\lambda\mu}$ *satisfying* $\left\{\begin{array}{l} (2.8) \\ (2.9) \end{array}\right.$. *Hence, if* $\left\{\begin{array}{l} (2.8) \\ (2.9) \end{array}\right.$ *admits a solution* $\Gamma^\nu_{\lambda\mu}$, *it will be expressed in terms of* $\left\{\begin{array}{l} {}^*g^{\lambda\nu} \\ g_{\lambda\mu} \end{array}\right.$ *in* $\left\{\begin{array}{l} n - {}^*g - UFT \\ n - g - UFT \end{array}\right.$.

The following quantities will be used in our further considerations:

$${}^*\mathfrak{g} = Det({}^*g_{\lambda\mu}) \neq 0, \quad {}^*\mathfrak{h} = Det({}^*h_{\lambda\mu}) \neq 0, \quad {}^*\mathfrak{k} = Det({}^*k_{\lambda\mu}). \tag{2.13a}$$

$${}^*g = \frac{{}^*\mathfrak{g}}{{}^*\mathfrak{h}} \quad {}^*k = \frac{{}^*\mathfrak{k}}{{}^*\mathfrak{h}}, \tag{2.13b}$$

$$\sigma = \left\{\begin{array}{ll} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{array}\right. \tag{2.13c}$$

(2.13d) $$^{(0)*}k_{\lambda^{\nu}} = \delta^{\nu}_{\lambda}, \quad ^{(p)*}k_{\lambda}{}^{\nu} = {}^{(p-1)*}k_{\lambda}{}^{\alpha*}k_{\alpha}{}^{\nu},$$

(2.13e) $$K_0 = 1, \quad K_p = {}^*k_{[\alpha_1}{}^{\alpha_1*}k_{\alpha_2}{}^{\alpha_2}\cdots{}^*k_{\alpha_p]}{}^{\alpha_p},$$

(2.13f) $$K_{\omega\mu\nu} = \nabla_{\omega}{}^*k_{\nu\mu} + \nabla_{\mu}{}^*k_{\omega\nu} + \nabla_{\nu}{}^*k_{\omega\mu},$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbol $^*\left\{ {\nu \atop \lambda\mu} \right\}$ defined by $^*h_{\lambda\mu}$.

The following relations have been proved already in a X_n [1], [4]:

(2.14a) $$K_p = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ {}^*k & \text{if } p \text{ is even,} \end{cases}$$

(2.14b) $$\sum_{s=0}^{n-\sigma} K_s{}^{(n-s)*}k_{\lambda}{}^{\nu} = 0.$$

Here and in what follows, the index s is assumed to take the value $0, 2, 4, 6\cdots$ in the specified range.

If the equations (2.8) admit a solution $\Gamma^{\nu}_{\lambda\mu}$, the symmetric part of (2.8) implies that it must be of the form

(2.15) $$\Gamma^{\nu}_{\lambda\mu} = {}^*\left\{ {\nu \atop \lambda\mu} \right\} + S_{\lambda\mu}{}^{\nu} + {}^*U^{\nu}{}_{\lambda\mu},$$

where

(2.16) $$^*U^{\nu}{}_{\lambda\mu} = S_{\beta(\lambda}{}^{\nu*}k_{\mu)}{}^{\beta} + S^{\nu}{}_{\beta(\lambda}{}^*k_{\mu)}{}^{\beta} - S^{\beta}{}_{(\lambda\mu)}{}^*k_{\beta}{}^{\nu}.$$

The skew-symmetric part of (2.8) gives the following relations satisfied by the torsion tensor $S_{\omega\mu\nu}$:

(2.17a) $$B_{\omega\mu\nu} = S_{\omega\mu\nu} + \overset{101}{S}{}_{\omega\mu\nu} + \overset{011}{S}{}_{\omega\mu\nu} + \overset{110}{S}{}_{\omega\mu\nu},$$

where

(2.17b) $$B_{\omega\mu\nu} = \frac{1}{2}\left(K_{\omega\mu\nu} + 3K_{[\alpha\beta\gamma]}{}^*k_{\omega}{}^{\alpha*}k_{\mu}{}^{\beta*}k_{\nu}{}^{\gamma}\right),$$

(2.17c) $$\overset{pqr}{S}{}_{\omega\mu\nu} = S_{\alpha\beta\gamma}{}^{(p)*}k_{\omega}{}^{\alpha(q)*}k_{\mu}{}^{\beta(r)*}k_{\nu}{}^{\gamma}.$$

3. The $^*g$-ME-connection in n-$^*g$-UFT

In this section, we introduce the concept of $^*g$-ME-connection and devote mainly to the proof of a necessary and sufficient condition for the existence of $^*g$-ME-connection and derive an useful representation of $^*g$-ME-connection in n-$^*g$-UFT.

DEFINITION 3.1. *The Einstein's connection* $\Gamma^\nu_{\lambda\mu}$ *which take the form*

$$\Gamma^\nu_{\lambda\mu} = {}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2\delta^\nu_\lambda X_\mu - 2{}^*g_{\lambda\mu}X^\nu \tag{3.1}$$

for a non-null vector X_λ *is called a* $^*g$-ME*-connection in* n-$^*g$-UFT, *and* X_λ *is the corresponding* $^*g$-ME*-vector.*

REMARK 3.2. *If there exists a* $^*g$-ME*-connection* $\Gamma^\nu_{\lambda\mu}$, *it must be of the form (2.15). Hence, comparing (2.15) and (3.1) we have the following relations :*

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda}X_{\mu]} - 2{}^*k_{\lambda\mu}X^\nu, \tag{3.2}$$

$${}^*U^\nu{}_{\lambda\mu} = 2\delta^\nu_{(\lambda}X_{\mu)} - 2{}^*h_{\lambda\mu}X^\nu. \tag{3.3}$$

REMARK 3.3. *The relation (3.2) and (3.3) show that our problem of determining a* $^*g$-ME*-connection* $\Gamma^\nu_{\lambda\mu}$ *becomes that of studying the* $^*g$-ME*-vector* X_λ.

THEOREM 3.4. *If there exists a* $^*g$-ME*-connection* $\Gamma^\nu_{\lambda\mu}$, *the* $^*g$-$ME-$*vector* X_λ *satisfies the relation*

$$2\delta^\nu_{(\lambda}X_{\mu)} - 2{}^*h_{\lambda\mu}X^\nu + 2^{(2)*}k_{\lambda\mu}X^\nu - {}^*h_{\lambda\mu}{}^*k_\alpha{}^\nu X^\alpha = 0, \tag{3.4a}$$

or equivalently

$$2{}^*h_{\nu(\lambda}X_{\mu)} - 2{}^*h_{\lambda\mu}X_\nu + 2^{(2)*}k_{\lambda\mu}X_\nu + 2{}^*h_{\lambda\mu}{}^*k_\nu{}^\alpha X_\alpha = 0. \tag{3.4b}$$

Proof. The relation (3.2) gives

$$S^\nu{}_{\lambda\mu} = \delta^\nu_\mu X_\lambda - {}^*h_{\lambda\mu}X^\nu + 2{}^*k_\lambda{}^\nu X_\mu. \tag{3.5}$$

Substituting (3.2) and (3.5) into (2.16) and using (2.13d), we have

$${}^*U^\nu{}_{\lambda\mu} = {}^*h_{\lambda\mu}{}^*k_\alpha{}^\nu X^\alpha - 2^{(2)*}k_{\lambda\mu}X^\nu. \tag{3.6}$$

Comparing (3.3) with (3.6), we have the relation (3.4a). The equivalence of (3.4a) and (3.4b) is obvious. □

As consequence of Theorem 3.4, we have the following theorem.

THEOREM 3.5. *The *g-ME-vector X^ν satisfies the following relation :*

$$\left((n-1){}^*h_{\lambda\nu} - {}^*k_{\lambda\nu} - 2^{(2)*}k_{\lambda\nu}\right)X^\nu = 0. \tag{3.7a}$$

Therefore, a necessary condition that exists a non-trivial solution X^ν is

$$Det(((n-1){}^*h_{\lambda\nu} - {}^*k_{\lambda\nu} - 2^{(2)*}k_{\lambda\nu})) = 0. \tag{3.7b}$$

Proof. Putting $\mu = \nu$ in (3.4a) and using (2.7a), we have the relation (3.7a). □

THEOREM 3.6. *A necessary and sufficient condition for the system (2.8) to admit a solution *g-ME-connection $\Gamma^\nu_{\lambda\mu}$ of the form (3.1) is that*

(3.8a)

$$\nabla_\omega{}^*k^{\lambda\mu} = 4\left(\delta^{[\lambda}_\omega X^{\mu]} + {}^*k_\omega{}^\lambda X^\mu + \delta^\mu_\omega{}^*k_\alpha{}^\lambda X^\alpha - {}^{(2)*}k_\omega{}^{(\mu}X^{\lambda)} - {}^{(2)*}k_\omega{}^\lambda X^\mu\right),$$

or equivalently

$$\nabla_\omega{}^*k_{\lambda\mu} = 4\left(3{}^*h_{\omega[\lambda}X_{\mu]} + 2{}^*h_{\mu[\lambda}X_{\omega]} + {}^*k_{\omega\lambda}X_\mu + 3^{(2)*}k_{\omega[\mu}X_{\lambda]}\right). \tag{3.8b}$$

Proof. Suppose that the system (2.8) admits a solution *g-ME-connection $\Gamma^\nu_{\lambda\mu}$ of the form (3.1). Then substituting (3.1) into the left-hand side of (2.8), we have

(3.9a)

$$\begin{aligned} D_\omega{}^*g^{\lambda\mu} &= \partial_\omega{}^*g^{\lambda\mu} + \Gamma^\lambda_{\alpha\omega}{}^*g^{\alpha\mu} + \Gamma^\mu_{\omega\alpha}{}^*g^{\lambda\alpha} \\ &= \nabla_\omega{}^*k_{\lambda\mu} - 2\delta^\lambda_\omega X_\omega + 2{}^*g^{\lambda\mu}X_\omega + 2\delta^\mu_\omega{}^*k^{\lambda\alpha}X_\alpha + 4^{(2)*}k_\omega{}^{(\mu}X^{\lambda)}. \end{aligned}$$

On the other hand, substitution of (3.2) into the right-hand side of (2.8) gives
(3.9b)
$$-2S_{\omega\alpha}{}^{\mu *}g^{\lambda\alpha} = -2\delta^{\mu}_{\omega}X^{\lambda} + 2^{*}g^{\lambda\mu}X_{\omega} + 4^{*}k_{\omega}{}^{\lambda}X^{\mu} - 2\delta^{\mu *}_{\omega}k^{\lambda\alpha}X_{\alpha} - 4^{(2)*}k_{\omega}{}^{\lambda}X^{\mu}.$$

Comparing (3.9a) with (3.9b), we have the relation (3.8a). Conversely, suppose that the statement (3.8a) is satisfied. Define a connection $\Gamma^{\nu}_{\lambda\mu}$ by (3.1), and substitute it into both sides of (2.8). Then we obtain the relation (3.8a), which is satisfied by our assumption. Hence a $^{*}g$-ME-connection of the form (3.1) is an Einstein's connection under the condition (3.8a). On the other hand, in order to show the equivalence of (3.8a) and (3.8b), consider the following alternative form of (3.8a)
(3.10)
$$\nabla_{\omega}{}^{*}k_{\lambda\mu} = 4\left(h_{\omega[\lambda}X_{\mu]} + {}^{*}k_{\omega\lambda}X_{\mu} - {}^{*}h_{\omega\mu}{}^{*}k_{\lambda}{}^{\alpha}X_{\alpha} - {}^{(2)*}k_{\omega(\mu}X_{\lambda)} - {}^{(2)*}k_{\omega\lambda}X_{\mu}\right).$$

The equivalence of (3.8a) and (3.8b) immediately follows from (3.10) using of (3.4b). □

In our further considerations we shall assume that the tensor

$$A_{\lambda\mu} = 10(1-n)^{*}h_{\lambda\mu} - 4^{*}k_{\lambda\mu} \tag{3.11}$$

is of rank n, so that we may define a unique tensor $A^{\lambda\nu}$ satisfying the condition

$$A_{\lambda\mu}A^{\lambda\nu} = A_{\mu\lambda}A^{\nu\lambda} = \delta^{\nu}_{\mu}. \tag{3.12}$$

THEOREM 3.7. *A necessary and sufficient condition for the system (2.8) to admit a* $^{*}g$-ME*-connection* $\Gamma^{\nu}_{\lambda\mu}$ *of the form (3.1) is that the tensor field* $^{*}g_{\lambda\mu}$ *satisfies the relation*
(3.13)
$$\nabla_{\omega}{}^{*}k_{\lambda\mu} = 4\left(3^{*}h_{\omega[\lambda}{}^{*}h_{\mu]\beta} + 2^{*}h_{\omega[\beta}{}^{*}h_{\mu]\lambda} + {}^{*}h_{\mu\beta}{}^{*}k_{\omega\lambda} + 3^{(2)*}k_{\omega[\mu}{}^{*}h_{\lambda]\beta}\right)C_{\alpha}A^{\alpha\beta}.$$

If this condition is satisfied, then

$$X^{\nu} = C_{\alpha}A^{\alpha\nu}, \tag{3.14}$$

where

$$C_{\lambda} = \nabla_{\alpha}{}^{*}k_{\lambda}{}^{\alpha}. \tag{3.15}$$

Proof. If the system (2.8) admits a solution $\Gamma^\nu_{\lambda\mu}$ of the form (3.1), the condition (3.8b) must hold in virtue of Theorem 3.6. Putting $\omega = \mu$ and raising the index μ in (3.8b) and using (3.11) and (3.15), we have

$$C_\lambda = A_{\lambda\alpha} X^\alpha. \tag{3.16}$$

Multiplication of $A^{\lambda\nu}$ to both sides of (3.16) gives (3.14). The condition (3.13) follows by substituting (3.14) into (3.8b). The proof of the converse statement is obvious in virtue of Theorem 3.6. □

THEOREM 3.8. *The $^*g$-ME-connection $\Gamma^\nu_{\lambda\mu}$ may be given by*

$$\Gamma^\nu_{\lambda\mu} = {}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2\left(\delta^\nu_\lambda {}^*h_{\mu\beta} - {}^*g_{\lambda\mu}\delta^\nu_\beta\right) C_\alpha A^{\alpha\beta}. \tag{3.17}$$

Proof. Making use of (3.1) and (3.14), we have the relation (3.17). □

DEFINITION 3.9. *An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by the tensor $^*g^{\lambda\nu}$ satisfying the condition of (3.13) by means of the $^*g$-ME-connection given by (3.17), is called an n-dimensional $^*g$-ME-manifold and denoted by $^*g$-MEX_n.*

4. A General Representation of the $^*g$-ME-vector in $^*g$-MEX_n

This section is concerned mainly with a general representation of the $^*g$-ME-vector which holds for a general n and all possible classes.

In our further considerations, we need a symmetric tensor

$$P_{\lambda\mu} = {}^{(2)*}k_{\lambda\mu} - {}^*h_{\lambda\mu}, \tag{4.1a}$$

and its unique inverse tensor $Q^{\lambda\nu}$ defined by

$$P_{\lambda\mu} Q^{\lambda\nu} = \delta^\nu_\mu. \tag{4.1b}$$

It was proved [3] that

$$Q^{\lambda\nu} = -\frac{1}{^*g} \sum_{s=0}^{n-1} \widetilde{K}_s{}^{(n-2+\sigma-s)*}k^{\lambda\nu}, \tag{4.1c}$$

where

$$\widetilde{K}_s = 1 + K_2 + \cdots + K_s. \tag{4.1d}$$

We use the following abbreviation for an arbitrary real vector A_λ :

$$^{(p)}A_\lambda = {}^{(p)*}k_\lambda{}^\alpha A_\alpha, \tag{4.2a}$$

$$^{(p)}A^\nu = (-1)^{p\,(p)*}k_\alpha{}^\nu A^\alpha, \qquad (p = 0, 1, 2, \cdots). \tag{4.2b}$$

By multiplying A_ν to both sides of (2.14b) and using (4.2a), every vector A_ω satisfies the following recurrence relation :

$$\sum_{s=0}^{n-\sigma} K_s{}^{(n-s)}A_\omega = 0, \tag{4.3a}$$

or equivalently

$$^{(n)}A_\omega + K_2{}^{(n-2)}A_\omega + \cdots + K_{n-\sigma-2}{}^{(\sigma+2)}A_\omega + K_{n-\sigma}{}^{(\sigma)}A_\omega = 0. \tag{4.3b}$$

THEOREM 4.1. *In $^*g$-MEX_n, the following relation holds :*

$$B_{\omega\mu\nu} = -2P_{\nu[\omega}X_{\mu]} + 2^*k_\omega{}^\alpha P_{\alpha\mu}X_\nu. \tag{4.4}$$

Proof. In virtue of (2.17a), (2.17b), (3.2), and (4.2a), we have

(4.5)

$$\begin{aligned}\overset{pqr}{S}{}_{\omega\mu\nu} &= ({}^*h_{\gamma\alpha}X_\beta - {}^*h_{\gamma\beta}X_\alpha - 2^*k_{\alpha\beta}X_\gamma)\,{}^{(p)*}k_\omega{}^{\alpha\,(q)*}k_\mu{}^{\beta\,(r)*}k_\nu{}^\gamma \\ &= (-1)^{r\,(p+r)*}k_{\omega\nu}{}^{(q)}X_\alpha - (-1)^{q\,(q+r)*}k_{\nu\mu}{}^{(q)}X_\omega - 2(-1)^{q\,(p+q+r)*}k_{\omega\mu}{}^{(r)}X_\nu.\end{aligned}$$

Consequently, using (4.5) the relation (2.17a) is reduced to (4.4) as in the following way :

$$\begin{aligned}B_{\omega\mu\nu} &= S_{\omega\mu\nu} + \overset{101}{S}{}_{\omega\mu\nu} + \overset{011}{S}{}_{\omega\mu\nu} + \overset{110}{S}{}_{\omega\mu\nu} \\ &= 2\left({}^*h_{\nu[\omega} - {}^{(2)*}k_{\nu[\omega}\right)X_{\mu]} + 2\left({}^{(3)*}k_{\omega\mu} - {}^*k_{\omega\mu}\right)X_\nu \\ &= -2P_{\nu[\omega}X_{\mu]} + 2^*k_\omega{}^\alpha P_{\alpha\mu}X_\nu.\end{aligned}$$

□

In our further considerations we use the following quantities :

(4.6a) $$N = \frac{1-n}{2},$$

(4.6b) $$\widehat{K}_s = \sum_{t=0}^{s} K_t N^{s-t},$$

(4.6c) $$Y_\omega = \frac{1}{2} Q^{\nu\mu} B_{\omega\mu\nu}.$$

In virtue of (4.6a) and (4.6b), direct calculations show that

(4.6d) $$\widehat{K}_s = K_s + \widehat{K}_{s-2} N^2.$$

THEOREM 4.2. *In* $*g$-MEX_n, *the following relation holds :*

(4.7) $$^{(p)}X_\omega = {}^{(p-1)}Y_\omega + N^{(p-2)}Y_\omega + N^{2(p-2)}X_\omega, \quad (p = 1, 2, 3, \cdots),$$

(4.8a) $$^{(p)}Y_\omega = -\frac{1}{4^*g} \sum_{s=0}^{n-1} \widetilde{K}_s \left(\overset{p\hat{s}0}{K}{}_{\omega\alpha\beta} + \overset{p'1\hat{s}}{K}{}_{\omega\alpha\beta} - \overset{p'\hat{s}1}{K}{}_{\omega\alpha\beta} + \overset{1\hat{s}p'}{K}{}_{\alpha\beta\omega} \right) {}^*h^{\alpha\beta},$$

where

(4.8b) $$p' = p+1, \quad \hat{s} = n + \sigma - 2 - s, \quad p = 0, 1, 2, \cdots .$$

Proof. Multiply $Q^{\nu\mu}$ to both sides of (4.4) and making use of (4.1b) to obtain

(4.9) $$Q^{\nu\mu} B_{\omega\mu\nu} = (n-1)X_\omega + 2^* k_\omega{}^\alpha X_\alpha = (n-1)X_\omega + 2^{(1)}X_\omega.$$

Hence, employing the notations introduced in (4.3), we have

(4.10) $$^{(1)}X_\omega = Y_\omega + NX_\omega.$$

The relation (4.7) immediately follows from (4.10) as in the following way :

$$\begin{aligned} ^{(p)}X_\omega &= {}^{(p-1)*}k_\omega{}^{\alpha(1)}X_\alpha \\ &= {}^{(p-1)}Y_\omega + N^{(p-1)}X_\omega \\ &= {}^{(p-1)}Y_\omega + N^{(p-2)}Y_\omega + N^{2(p-2)}X_\omega. \end{aligned}$$

On the other hand, in virtue of (2.17b), (2.17c), (4.1c), (4.6c), and (4.8b) the representation (4.8a) may be proved as in the following way :

$$\begin{aligned} ^{(p)}Y_\omega &= \frac{1}{2} Q^{\nu\mu} B_{\gamma\mu\nu}{}^{(p)*}k_\omega{}^\gamma \\ &= -\frac{1}{4^*g} \sum_{s=0}^{n-1} \widetilde{K}_s{}^{(\widehat{s})*}k^{\nu\mu(p)*}k_\omega{}^\gamma \left(K_{\gamma\mu\nu} + \overset{110}{K}{}_{\gamma\mu\nu} + \overset{011}{K}{}_{\nu\gamma\mu} + \overset{101}{K}{}_{\mu\nu\gamma} \right) \\ &= \text{ the right hand side of (4.8a).} \end{aligned}$$

□

THEOREM 4.3. *The* $^*g$-ME*-vector* X_ω *in* $^*g$-MEX_n *may be given by*

$$X_\omega = \frac{1}{\phi} \sum_{s=0}^{n-\sigma-2} \widehat{K}_s \left({}^{(n-s-1)}Y_\omega + N^{(n-s-2)}Y_\omega \right) + \theta Y_\omega, \tag{4.11}$$

where

$$\phi = (\sigma - 1 - \sigma N)\widehat{K}_{n-\sigma}, \tag{4.12a}$$

$$\theta = \frac{\sigma}{\sigma - 1 - \sigma N}. \tag{4.12b}$$

Proof. Substituting (4.7) into (4.3b) with A_ω replaced by X_ω and using (4.6b) and (4.6d), we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N^{(n-2)}Y_\omega \right) + (K_2 + N^2)^{(n-2)}X_\omega + K_4{}^{(n-4)}X_\omega \\ &\quad + \cdots + K_{(n-\sigma-2)}{}^{(\sigma+2)}X_\omega + K_{n-\sigma}{}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{4.13a}$$

Substituting again for ${}^{(n-2)}X_\omega$ into (4.13a) from (4.7), we have

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N^{(n-2)}Y_\omega \right) + \widehat{K}_2 \left({}^{(n-3)}Y_\omega + N^{(n-4)}Y_\omega \right) \\ &\quad + (K_4 + N^2)^{(n-4)}X_\omega + K_6{}^{(n-6)}X_\omega + \cdots + \widehat{K}_{(n-\sigma-2)}{}^{(\sigma+2)}X_\omega \\ &\quad + \widehat{K}_{n-\sigma}{}^{(\sigma)}X_\omega = 0. \end{aligned} \tag{4.13b}$$

After $\frac{n-\sigma}{2}$ steps of successive repeated substitutions for ${}^{(p)}X_\omega$, we have
(4.13c)

$$\begin{aligned} &\widehat{K}_0 \left({}^{(n-1)}Y_\omega + N^{(n-2)}Y_\omega \right) + \widehat{K}_2 \left({}^{(n-3)}Y_\omega + N^{(n-4)}Y_\omega \right) \\ &\quad + \widehat{K}_4 \left({}^{(n-5)}Y_\omega + N^{(n-6)}Y_\omega \right) + \cdots + \widehat{K}_{(n-\sigma-2)} \left({}^{(\sigma+1)}Y_\omega + N^{(\sigma)}Y_\omega \right) \\ &\quad + \widehat{K}_{n-\sigma}{}^{(\sigma)}X_\omega = 0. \end{aligned}$$

On the other hand, it follows from (4.2a), (4.10), and (4.12b) that

$${}^{(\sigma)}X_\omega = \sigma Y_\omega + 2\sigma\theta X_\omega. \tag{4.14}$$

Our representation (4.11) immediately follows by substituting (4.14) into (4.13c) and making use of (4.12). □

REMARK 4.4. *As a consequence of Theorem 4.3, we note that there exists a unique $^*g$-ME-vector and a unique $^*g$-ME-connection if and only if the following condition holds for $^*g_{\lambda\mu}$:*

$$\widehat{K}_{n-\sigma} \neq 0. \tag{4,15}$$

REMARK 4.5. *The representation (4.11) of the $^*g$-ME-vector X_ω is the most general one which holds for a general and for all possible classes.*

REMARK 4.6. *we illustrate the representation of X_ω for the lower dimensional cases, provided that the condition (4.15) holds:*

$$X_\omega = \frac{2Y_\omega - 4^{(1)}Y_\omega}{1 + 4^*k} \qquad \text{if } n = 2, \tag{4.16a}$$

$$X_\omega = \frac{(1 + K_2)Y_\omega - {}^{(1)}Y_\omega + {}^{(2)}Y_\omega}{1 + K_2} \qquad \text{if } n = 3, \tag{4.16b}$$

$$X_\omega = \frac{2(9 + 4K_2)(3Y_\omega - 2^{(1)}Y_\omega) + 24^{(2)}Y_\omega - 16^{(3)}Y_\omega}{81 + 36K_2 + 16^*k} \qquad \text{if } n = 4. \tag{4.16c}$$

THEOREM 4.7. *The $^*g$-ME-connection of $^*g$-MEX_n may be given by*

(4.17)
$$\Gamma^\nu_{\lambda\mu} = {}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} - \frac{2}{\phi} \sum_{s=0}^{n-\sigma-2} \widehat{K}_s {}^*k_{\lambda\mu} \left({}^{(n-s-1)}Y^\nu + N^{(n-s-2)}Y^\nu \right) - 2\theta^* k_{\lambda\mu} Y^\nu.$$

Proof. Substituting (4.11) into (3.17) and making use of (2.7a) and (4.12b), we have the relation (4.17). □

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