Honam Mathematical J. **30** (2008), No. 3, pp. 557–566

ON IVF WEAKLY CONTINUOUS MAPPINGS ON THE IVF TOPOLOGICAL SPACES

WON KEUN MIN

Abstract. In this paper, we introduce the concept of IVF weakly continuity and investigate some characterizations for IVF weakly continuous mappings on the interval-valued fuzzy topological spaces. We introduce and study the concepts of almost IVF compactness and nearly IVF compactness.

1. Introduction

Zadeh [4] introduced the concept of fuzzy set and investigated basic properties. Gorzalczany [1] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. In [3], Mondal and Samanta introduced the concepts of interval-valued fuzzy topology, continuity and compactness and studied some topological properties. In [2], Jun et al. introduced the concepts of IVF semiopen sets, IVF α-open sets and IVF preopen sets and studied some results about them. Also they introduced the concepts of IVF semiopen mappings, IVF α-open mappings and IVF preopen mappings. In this paper, we introduce the concept of IVF weakly continuity and investigate some characterizations for IVF weakly continuous mappings on the interval-valued fuzzy topological spaces. We introduce and study the concepts of almost IVF compactness and nearly IVF compactness. In particular, we have the following: Let a mapping $f : X \to Y$ continuous and IVF open on two IVFTS's (X, τ_1) and (Y, τ_2). Then if A is a nearly IVF compact set, then $f(A)$ is also a nearly IVF compact set.

Received July 14, 2008. Accepted August 21, 2008.
2000 Mathematics Subject Classification: 52A01.
Key words and phrases: IVF weakly continuous, IVF continuous, IVF almost compact.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$ and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0], \mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for $a \in (0,1)$. We also note that

1. $(\forall M, N \in D[0,1])(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$.
2. $(\forall M, N \in D[0,1])(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D[0,1]$, the complement of M, denoted by M^c, is defined by $M^c = \mathbf{1} - M = [1 - M^U, 1 - M^L]$. Let X be a nonempty set. A mapping $A : X \to D[0,1]$ is called an interval-valued fuzzy set (simply, IVF set) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. In particular, for any $a \in [a,b]$, the IVF set whose value is $\mathbf{a} = [a,a]$ for all x X is denoted by simply $\widetilde{a}$. For a point $p \in X$ and for $[a,b] \in D[0,1]$ with $b > 0$, the IVF set which takes the value $[a,b]$ at p and $\mathbf{0}$ elsewhere in X is called an interval-valued fuzzy point (simply, IVF point) and is denoted by $[a,b]_p$. In particular, if $b = a$, then it is also denoted by a_p. Denoted by D^X the set of all IVF sets in X.

For every $A, B \in D^X$, we define

$$A = B \Leftrightarrow (\forall x \in X)([A(x)]^L = [B(x)]^L \text{ and } [A(x)]^U = [B(x)]^U),$$

$$A \subseteq B \Leftrightarrow (\forall x \in X)([A(x)]^L \subseteq [B(x)]^L \text{ and } [A(x)]^U \subseteq [B(x)]^U).$$

The complement A^c of A is defined by

$$[A^c(x)]^L = 1 - [A(x)]^U \text{ and } [A^c(x)]^U = 1 - [A(x)]^L$$

for all $x \in X$.

For a family of IVF sets $\{A_i : i \in J\}$ where J is an index set, the union $G = \cup_{i \in J} A_i$ and $F = \cap_{i \in J} A_i$ are defined by

$(\forall x \in X)$ $([G(x)]^L = \sup_{i \in J}[A_i(x)]^L, [G(x)]^U = \sup_{i \in J}[A_i(x)]^U)$,

$(\forall x \in X)$ $([F(x)]^L = \inf_{i \in J}[A_i(x)]^L, [F(x)]^U = \inf_{i \in J}[A_i(x)]^U)$, respectively.

Let $f : X \to Y$ be a mapping and let A be an IVF set in X. Then the image of A under f, denoted by $f(A)$, is defined as follows:

$$[f(A)(y)]^L = \begin{cases} \sup_{f(x)=y}[A(x)]^L, & \text{if } f^{-1}(y) \neq \emptyset,\ y \in Y \\ 0, & \text{otherwise} \ , \end{cases}$$

$$[f(A)(y)]^U = \begin{cases} \sup_{f(x)=y}[A(x)]^U, & \text{if } f^{-1}(y) \neq \emptyset,\ y \in Y \\ 0, & \text{otherwise} \ , \end{cases}$$

for all $y \in Y$.

Let B be an IVF set in Y. Then the inverse image of B under f, denoted by $f^{-1}(B)$, is defined as follows:

$$(\forall x \in X)([f^{-1}(B)(x)]^L = [B(f(x))]^L, [f^{-1}(B)(x)]^U = [B(f(x))]^U).$$

Definition 2.1 ([3]). A family τ of IVF sets in X is called an *interval-valued fuzzy topology* (simply, IVFT) on X if it satisfies:

1. $\mathbf{0}, \mathbf{1} \in \tau$.
2. $A, B \in \tau \Rightarrow A \cap B \in \tau$.
3. For $i \in J$, $A_i \in \tau \Rightarrow \cup_{i \in J} A_i \in \tau$.

Every member of τ is called an IVF open set. An IVF set A is called an IVF closed set if the complement of A is an IVF open set. And (X, τ) is called an *interval-valued fuzzy topological space* (simply, IVFTS).

In an IVF topological space (X, τ), for an IVF set A in X, the IVF closure and the IVF interior of A, denoted by $cl(A)$ and $int(A)$, respectively, are defined as

$$cl(A) = \cap\{B \in IVF(X) : B^c \in \tau \text{ and } A \subseteq B\},$$
$$int(A) = \cup\{B \in IVF(X) : B \in \tau \text{ and } B \subseteq A\},$$

respectively [3].

And an IVF set B in an IVF topological space (X, τ) is said to be a *neighborhood* (simply, nbd) of an IVF point M_x iff there exists an IVF open set O such that $M_x \in O \subseteq B$. An IVF set A in an IVF topological space X is said to be *IVF compact* [3] if every IVF open cover $\mathcal{A} = \{A_i : i \in J\}$ of B has a finite IVF subcover.

Theorem 2.2 ([3]). Let A be an IVF set in an IVF topological space (X, τ). Then $\mathbf{1} - cl(\mathbf{1} - A) = int(A)$

Definition 2.3 ([3]). Let (X, τ_1) and (Y, τ_2) be two IVFTS's. Then $f : X \to Y$ is said to be *continuous* if for every $A \in \tau_2$, $f^{-1}(A) \in \tau_1$.

Theorem 2.4 ([3]). Let $f : X \to Y$ be a mapping on two IFMS's (X, τ_1) and (Y, τ_2).

(1) f is continuous.

(2) $f^{-1}(B)$ is an IVF closed set in X for each IVF closed set B in Y.

(3) For each IVF point M_x in X, the inverse of every nbd V of $f(M_x)$ under f is an nbd of M_x.

(4) For each IVF point M_x in X and each nbd V of $f(M_x)$, there is an nbd W of M_x such that $f(W) \subseteq V$.

(5) $f(cl(A)) \subseteq cl(f(A))$ for $A \in D^X$.

3. IVF weakly continuity

Definition 3.1. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then f is said to be IVF weakly continuous if for every IVF point M_x and each IVF open set V containing $f(M_x)$, there exists IVF open set U containing M_x such that $f(U) \subseteq cl(V)$.

Theorem 3.2. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then the following statements are equivalent:

1. f is IVF weakly continuous.
2. $f^{-1}(B) \subseteq int(f^{-1}(cl(B)))$ for each IVF open set B of Y.
3. $cl(f^{-1}(int(F))) \subseteq f^{-1}(F)$ for each IVF closed set F in Y.
4. $cl(f^{-1}(int(cl(B)))) \subseteq f^{-1}(cl(B))$ for each $B \in D^Y$.
5. $f^{-1}(int(B)) \subseteq int(f^{-1}(cl(int(B))))$ for each $B \in D^Y$.
6. $cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for an IVF open set V in Y.

Proof. (1) $\Rightarrow$ (2) Let B be an IVF open set in Y. Since f is IVF weakly continuous, for each $M_x \in f^{-1}(B)$, there exists an IVF open set U_{M_x} of M_x such that $f(U_{M_x}) \subseteq cl(B)$. Now we can say for each $M_x \in f^{-1}(B)$, there exists an IVF open set U_{M_x} such that

$$M_x \in U_{M_x} \subseteq f^{-1}(f(U_{M_x})) \subseteq f^{-1}(cl(B)).$$

This implies $M_x \in int(f^{-1}(cl(B)))$. Hence $f^{-1}(B) \subseteq int(f^{-1}(cl(B)))$.

(2) $\Rightarrow$ (1) Let M_x be an IVF point in X and V an IVF open set containing $f(M_x)$. Then since $M_x \in f^{-1}(V) \subseteq int(f^{-1}(cl(V)))$, there exists an IVF open set U containing M_x such that $M_x \in U \subseteq f^{-1}(cl(V))$.

This implies $f(M_x) \in f(U) \subseteq f(f^{-1}(cl(V))) \subseteq cl(V)$. Hence f is IVF weakly continuous.

(1) $\Rightarrow$ (3) Let F be any IVF closed set of Y. Then $\mathbf{1} - F$ is an IVF open set in Y and

$$\begin{aligned} f^{-1}(\mathbf{1} - F) &\subseteq int(f^{-1}(cl(\mathbf{1} - F))) \\ &= int(f^{-1}(\mathbf{1} - int(F))) \\ &= int(\mathbf{1} - f^{-1}(int(F))) \\ &= \mathbf{1} - cl(f^{-1}(int(F))). \end{aligned}$$

Hence we have $cl(f^{-1}(int(F))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) Let B be any IVF set in Y. Since $cl(B)$ is an IVF closed set in Y, by (3),

$$cl(f^{-1}(int(cl(B))) \subseteq f^{-1}(cl(B)).$$

(4) $\Rightarrow$ (5) Let B be any IVF set of Y. Then,

$$\begin{aligned} f^{-1}(int(B) &= \mathbf{1} - (f^{-1}(cl(\mathbf{1} - B))) \\ &\subseteq \mathbf{1} - cl(f^{-1}(int(cl(\mathbf{1} - B)))) \\ &= int(f^{-1}(cl(int(B)))). \end{aligned}$$

Hence,

$$f^{-1}(int(B)) \subseteq int(f^{-1}(cl(int(B)))).$$

(5) $\Rightarrow$ (6) Let V be any IVF open set of Y. Then by (5),

$$\begin{aligned} \mathbf{1} - f^{-1}(cl(V)) &= f^{-1}(int(\mathbf{1} - V) \\ &\subseteq int(f^{-1}(cl(int(\mathbf{1} - V)))) \\ &= int(\mathbf{1} - (f^{-1}(int(cl(V))))) \\ &= \mathbf{1} - cl(f^{-1}(int(cl(V))))) \\ &\subseteq \mathbf{1} - cl(f^{-1}(V)). \end{aligned}$$

Hence we have

$$cl(f^{-1}(V)) \subseteq f^{-1}(cl(V)).$$

(6) $\Rightarrow$ (1) Let V be an IVF open set containing $f(M_x)$. By (6),

$$\begin{aligned} M_x \in f^{-1}(V) &\subseteq f^{-1}(int(cl(V))) \\ &= \mathbf{1} - f^{-1}(cl(\mathbf{1} - cl(V))) \\ &\subseteq \mathbf{1} - cl(f^{-1}(\mathbf{1} - cl(V))) \\ &= int(f^{-1}(cl(V))). \end{aligned}$$

Set $U = int(f^{-1}(cl(V)))$. Then U is an IVF open set satisfying $f(U) \subseteq cl(V)$.

□

Definition 3.3. Let A be an IVF set in an IVFTS (X, τ). Then A is said to be

1. *IVF semiopen* [2] if there is an IVF -open set B in X such that $B \subseteq A \subseteq cl(B)$,
2. *IVF preopen* [2] if $A \subseteq int(cl(A))$,
3. *IVF regular open* if $A = int(cl(A))$,
4. *IVF β-open* if $A \subseteq cl(int(cl(A)))$.

IVF-semiopen

↗ ↘

IVF regular open IVF β-open

↘ ↗

IVF preopen

Theorem 3.4. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and Y, τ_2. Then the following statements are equivalent:

1. f is IVF weakly continuous.
2. $cl(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for each IVF open set G in Y.
3. $cl(f^{-1}(int(cl(V)))) \subseteq f^{-1}(cl(V))$ for each IVF preopen set V in Y.
4. $cl(f^{-1}(int(K))) \subseteq f^{-1}(K)$ for each IVF regular closed set K in Y.
5. $cl(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for each IVF β-open set G in Y.
6. $cl(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for each IVF semiopen set G in Y.

Proof. (1) $\Rightarrow$ (2) Let G be an IVF open set of Y; then by Theorem 3.2 (3), we have $cl(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G)$.

(2) $\Rightarrow$ (3) Let V be an IVF preopen of Y. Then $V \subseteq int(cl(V))$. Set $A = int(cl(V))$. Then since A is an IVF open set, from (2), it follows

$$cl(f^{-1}(int(cl(A)))) \subseteq f^{-1}(cl(A)).$$

Since $cl(A) = cl(V)$, we have

$$cl(f^{-1}(int(cl(V)))) \subseteq f^{-1}(cl(V)).$$

(3) $\Rightarrow$ (4) Let K be an IVF regular closed set of Y. Then since $int(K)$ is an IVF preopen set, by (3),

$$cl(f^{-1}(int(cl(int(K))))) \subseteq f^{-1}(cl(int(K))).$$

Since $int(K) = int(cl(int(K)))$ and $K = cl(int(K))$, we have

$$cl(f^{-1}(int(K))) \subseteq f^{-1}(K).$$

(4) $\Rightarrow$ (5) Let G be an IVF β-open set. Then $G \subseteq (cl(int(cl(G)))$ and $cl(G)$ is an IVF regular closed set. Hence by (4), we have

$$cl(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G)).$$

(5) $\Rightarrow$ (6) It is obvious.

(6) $\Rightarrow$ (1) Let V be an IVF open set; then since V is an IVF semiopen set, by (6) and $V \subseteq int(cl(V))$, we have

$$cl(f^{-1}(V)) \subseteq cl(f^{-1}(int(cl(V)))) \subseteq f^{-1}(cl(V)).$$

Hence, f is an IVF weakly continuous mapping. □

Definition 3.5. Let (X, τ) be an IVFTS. An IVF set A in X is said to be *almost IVF compact* if for every IVF open cover $\mathcal{A} = \{A_i \in D^X : i \in J\}$ of A, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} cl(A_i)$.

Theorem 3.6. Let (X, τ) be an IVFTS. If an IVF set A in X is IVF compact, then it is also almost IVF compact.

Proof. Obvious. □

In Theorem 3.6, the converse is not always true as shown the next example.

Example 3.7. Let $X = \{a, b\}$. For each $n \in N$ let A_n be an IVF set defined as follows:

$$A_n(a) = [\frac{n}{1+n}, 1], A_n(b) = [1, 1].$$

Let B be an IVF set defined as follows:

$$B(a) = [\frac{3}{5}, \frac{4}{5}], B(b) = [0, 1].$$

Consider $\tau = \{\emptyset, A_n, B, A_1 \cup B, A_1 \cap B, X\}$ as an IVF topology on X. Let $\mathcal{C} = \{A_n : n \in N\}$ be an IVF open cover of X. Then there does not exist a finite subcover of $\mathcal{C}$. Thus X is not IVF compact but it is almost IVF compact.

Theorem 3.8. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then f is continuous if and only if $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for $B \in D^Y$.

Proof. Suppose f is continuous. Then for $B \in D^Y$, from Theorem 2.4 (5), it follows

$$f(cl(f^{-1}(B))) \subseteq (cl(f(f^{-1}(B)))) \subseteq cl(B).$$

Hence we have $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

The converse is obvious. □

Theorem 3.9. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be continuous on two IVFTS's. If A is an almost IVF compact set, then $f(A)$ is also an almost IVF compact set.

Proof. Let $\{B_i : i \in J\}$ be an IVF open cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) : i \in J\}$ is an IVF open cover of A in X. By definition of almost IVF compactness, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} cl(f^{-1}(B_i))$. From Theorem 3.8, it follows

$$\begin{aligned} \cup_{i \in J_0} cl(f^{-1}(B_i)) &\subseteq \cup_{i \in J_0} f^{-1}(cl(B_i)) \\ &= f^{-1}(\cup_{i \in J_0} cl(B_i)). \end{aligned}$$

Hence $f(A) \subseteq \cup_{i \in J_0} cl(B_i)$. It completes the proof. □

Theorem 3.10. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an IVF weakly continuous mapping on two IVFTS's. If A is an IVF compact set, then $f(A)$ is also an almost IVF compact set.

Proof. Let $\{B_i \in D^Y : i \in J\}$ be an IVF open cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) : i \in J\}$ is an IVF open cover of A in X. By definition of IVF compactness, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0}(f^{-1}(B_i))$. From Theorem 3.2 (6), it follows

$$\begin{aligned} \cup_{i \in J_0}(f^{-1}(B_i)) &\subseteq \cup_{i \in J_0} cl(f^{-1}(B_i)) \\ &\subseteq \cup_{i \in J_0} f^{-1}(cl(B_i)) \\ &= f^{-1}(\cup_{i \in J_0} cl(B_i)). \end{aligned}$$

Hence $f(A) \subseteq \cup_{i \in J_0} cl(B_i)$. □

Definition 3.11. Let (X, τ) be an IVFTS. An IVF set A in X is said to be *nearly IVF compact* if for every IVF open cover $\mathcal{A} = \{A_i \in D^X : i \in J\}$ of A, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} int(cl(A_i))$.

Remark 3.12. Let (X, τ) be an IVFTS. Then we have the following:

IVF compact $\Rightarrow$ nearly IVF compact $\Rightarrow$ almost IVF compact

Definition 3.13 ([2]). Let (X, τ_1) and (Y, τ_2) be two IVFTS's. Then a mapping $f : X \to Y$ is called *IVF open* if for every $A \in \tau_1$, $f(A)$ is in τ_2.

Theorem 3.14. Let $f : X \to Y$ be a mapping on IVFTS's (X, τ_1) and (Y, τ_2). The the following are equivalent:
(1) f is IVF open.
(2) $f(int(A)) \subseteq int(f(A))$ for $A \in D^X$.
(3) $int(f^{-1}(B)) \subseteq f^{-1}(int(B))$ for $B \in D^Y$.

Proof. (1) $\Rightarrow$ (2) For $A \in F(X)$,

$$\begin{aligned} f(int(A)) &= f(\cup\{B \in D^X : B \subseteq A, B \in \tau_1\}) \\ &= \cup\{f(B) \in D^Y : f(B) \subseteq f(A), f(B) \in \tau_2\} \\ &\subseteq \cup\{U \in D^Y : U \subseteq f(A), U \in \tau_2\} \\ &= int(f(A)) \end{aligned}$$

Hence $f(int(A)) \subseteq int(f(A))$.

(2) $\Rightarrow$ (1) Obvious.

(2) $\Leftrightarrow$ (3) For $B \in D^Y$, from (2) it follows that

$$f(int(f^{-1}(B))) \subseteq int(f(f^{-1}(B))) \subseteq int(B).$$

Hence $int(f^{-1}(B)) \subseteq f^{-1}(int(B))$. Similarly, we have (3) $\Rightarrow$ (2). □

Theorem 3.15. Let (X, τ_1) and (Y, τ_2) be IVFTS's and a mapping $f : X \to Y$ continuous and IVF open. If A is a nearly IVF compact set, then $f(A)$ is also a nearly IVF compact set.

Proof. Let $\{B_i \in D^Y : i \in J\}$ be an IVF open cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) : i \in J\}$ is an IVF open cover of A in X. By definition of nearly IVF compactness, there exists $J_0 = \{1, 2, \cdots n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} int(cl(f^{-1}(B_i)))$. From Theorem 3.8 and Theorem 3.14, it follows

$$\begin{aligned} f(A) &\subseteq \cup_{i \in J_0} f(int(cl(f^{-1}(B_i)))) \\ &\subseteq \cup_{i \in J_0} int(f(cl(f^{-1}(B_i)))) \\ &\subseteq \cup_{i \in J_0} int(f(f^{-1}(cl(B_i)))) \\ &\subseteq \cup_{i \in J_0} int(cl(B_i)). \end{aligned}$$

Hence $f(A)$ is a nearly IVF compact set.

Acknowledgements

I thank the referee for some useful comments on the paper.

References

[1] M. B. Gorzalczany, *A method of inference in approximate reasoning based on interval-valued fuzzy sets*, J. Fuzzy Math. **21** (1987), 1–17.

[2] Y. B. Jun, G. C. Kang and M.A. Ozturk *Interval-valued fuzzy semiopen, preopen and α-open mappings*, Honam Math. J., **28**(2), (2006) pp. 241–259.

[3] T. K. Mondal and S. K. Samanta, *Topology of interval-valued fuzzy sets* , Indian J. Pure Appl. Math., **30**(1), (1999) pp. 23–38.

[4] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

Won Keun Min
Department of Mathematics,
Kangwon National University,
Chuncheon, 200-701, Korea
E-mail: wkmin@kangwon.ac.kr