Honam Mathematical J. **30** (2008), No. 3, pp. 535–550

CHARACTERIZATIONS OF REAL HYPERSURFACES OF TYPE A IN A COMPLEX SPACE FORM USED BY THE ξ-PARALLEL STRUCTURE JACOBI OPERATOR

Nam-Gil Kim*, U-Hang Ki, and Hiroyuki Kurihara

Abstract. Let M be a real hypersurface of a complex space form with almost contact metric structure (ϕ, ξ, η, g). In this paper, we study real hypersurfaces in a complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is ξ-parallel. In particular, we prove that the condition $\nabla_\xi R_\xi = 0$ characterize the homogeneous real hypersurfaces of type A in a complex projective space $P_n\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ when $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant and not equal to $-c/24$ on M, where c is a constant holomorphic sectional curvature of a complex space form.

1. Introduction

Let $(M_n(c), J, \tilde{g})$ be a complex n-dimensional complex space form with Kähler structure $(J, \tilde{g})$ of constant holomorphic sectional curvature c and let M be an orientable real hypersurface in $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from $(J, \tilde{g})$.

It is known that there are no real hypersurfaces with parallel Ricci tensors in a nonflat complex space form (see [6], [9]). This result say that there does not exist locally symmetric real hypersurfaces in a nonflat complex space form. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Cho and the second author started the study on real hypersurfaces in a complex space form by using the operater R_ξ in [3], [4] and [5]. Recently Ortega, Pérez and Santos [13] have proved that there are no real hypersurfaces in a complex projective space $P_n\mathbb{C}, n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [14].

Received July 7, 2008. Accepted September 1, 2008.
2000 Mathematics Subject Classification: 53B20, 53C15, 53C25.
Key words and phrases: complex space form, real hypersurface, structure Jacobi operator.
* This study was supported by the research funds Chosun University, 2007.

Now in this paper, motivated by results mentioned above we consider the parallelism of the structure Jacobi operator R_ξ in the direction of the structure vector field, that is $\nabla_\xi R_\xi = 0$.

In 1970's, Takagi ([15], [16]) classified the homogeneous real hypersurfaces of $P_n\mathbb{C}$ into six types. On the other hand, Cecil and Ryan [2] extensively studied a Hopf hypersurface (whose structure vector field ξ is principal), which is realized as tubes over certain submanifolds in $P_n\mathbb{C}$, by using its focal map. By making use of those results and the mentioned work of Takagi, Kimura [10] proved the local classification theorem for Hopf hypersurfaces of $P_n\mathbb{C}$ whose all principal curvatures are constant. For the case of a complex hyperbolic space $H_n\mathbb{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant. Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's list, a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ $(0 \leq k \leq n-1)$ adding a horosphere in $H_n\mathbb{C}$, which is called type A, has a lot of nice geometric properties. For example, Okumura [12] (resp. Montiel and Romero [11]) shows that a real hypersurface of type A if and only if the structure operator ϕ commutes with the shape operator $(A\phi = \phi A)$.

In this paper we study a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant and not equal to $-c/24$ on M. We give another characterization of real hypersurfaces of type A in $M_n(c)$ by above two conditions. In particular, in the case of $P_n\mathbb{C}$, it is not necessary to the condition $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is not equal to $-c/24$. The main purpose of the present paper is to establish Thoerem 2 stated in section 5. We note that the condition $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant on M is a much weaker condition. Indeed, every Hopf hypersurface always satisfies this condition.

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

2. Preliminaries

Let M be a real hypersurface of a nonflat complex space form $M_n(c)$, $c \neq 0$ and C be a unit normal vector on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Kähler metric $\tilde{g}$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)C, \quad \tilde{\nabla}_X C = -AX$$

for any vector fields X and Y on M, where g denotes the Riemannian metric of M induced from $\tilde{g}$ and A is the shape operator of M in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we may see that M induces an almost contact metric structure (ϕ, ξ, η, g), namely

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M.

Since J is parallel, we verify, using the Gauss and Weingarten formulas, that

$$\nabla_X \xi = \phi AX, \tag{2.1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.2}$$

Since the ambient space is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively:

(2.3)
$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.4}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

In the sequel, to write our formulas in convention forms, we denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, and for a function f we denote by ∇f the gradient vector field of f.

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector ξ. From (2.1), we get

$$\phi U = -A\xi + \alpha\xi, \tag{2.5}$$

which enables us to obtain $g(U, U) = \beta - \alpha^2$. If we put

$$A\xi = \alpha\xi + \mu W, \tag{2.6}$$

where W is a unit vector field orthogonal to ξ. Then we get $U = \mu\phi W$, which shows that W is also orthogonal to U. Further we have

$$\mu^2 = \beta - \alpha^2. \tag{2.7}$$

Thus we see that ξ is principal curvature vector, that is $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$.

In this paper, we basically use the technical computations with the orthogonal triplet $\{\xi, U, W\}$ and their associated scalars α, β and μ.

Because of (2.1), (2.5) and (2.6), it is seen that

$$g(\nabla_X \xi, U) = \mu g(AW, X) \tag{2.8}$$

and

$$\mu g(\nabla_X W, \xi) = g(AU, X) \tag{2.9}$$

for any vector field X on M.

Differentiating (2.5) covariantly along M and making use of (2.1) and (2.2), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.10}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha, \tag{2.11}$$

where we have used (2.4). From (2.1) and (2.10), it is verified that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{2.12}$$

The curvature equation (2.3) gives the structure Jacobi operator R_ξ:

$$R_\xi(X) = R(X, \xi)\xi = \frac{c}{4}\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{2.13}$$

for any vector field X on M.

3. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$

We set $\Omega = \{p \in M; \mu(p) \neq 0\}$ and suppose that Ω is non-empty, that is, ξ is not a principal curvature vector on M. Hereafter, unless otherwise stated, we discuss our arguments on the open subset Ω of M.

Differentiating (2.13) covariantly, and using (2.11), we find

$$\begin{aligned} g((\nabla_\xi R_\xi)Y, Z) = &-\frac{c}{4}\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z) \\ &+ \alpha g((\nabla_\xi A)Y, Z) - \eta(AZ)\{3g(AU, Y) + Y\alpha\} \\ &- \eta(AY)\{3g(AU, Z) + Z\alpha\}, \end{aligned} \tag{3.1}$$

where u is a 1-form dual to U with respect to g, that is $u(X) = g(U, X)$.

We assume that $\nabla_\xi R_\xi = 0$. Then from (3.1) we have

$$\alpha(\nabla_\xi A)X+(\xi\alpha)AX = \frac{c}{4}\{u(X)\xi + \eta(X)U\} + \eta(AX)(3AU + \nabla\alpha) + \{3g(AU, X) + X\alpha\}A\xi. \tag{3.2}$$

Putting $X = \xi$ in this and using (2.11), we find

$$\alpha AU + \frac{c}{4}U = 0, \tag{3.3}$$

which shows that $\alpha \neq 0$ on Ω.

Differentiating (3.3) covariantly and using itself, we obtain

$$-\frac{c}{4}(X\alpha)U + \alpha^2(\nabla_X A)U + \alpha^2 A\nabla_X U + \frac{c}{4}\alpha\nabla_X U = 0, \tag{3.4}$$

or, using (2.4) and (2.5)

$$\frac{c}{4}\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + \frac{c}{4}\alpha^2\mu\{\eta(X)w(Y) - \eta(Y)w(X)\} + \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + \frac{c}{4}\alpha du(X, Y) = 0, \tag{3.5}$$

where w is a dual 1-form of W with respect to g, that is $w(X) = g(W, X)$. Here, du is the exterior derivative of a 1-form u given by

$$du(X, Y) = -Yu(X) + Xu(Y) - u([X, Y]).$$

If we replace X by U in (3.5), then it follows that

$$\frac{c}{4}\{\mu^2\nabla\alpha - (U\alpha)U\} + \alpha^2 A\nabla_U U + \frac{c}{4}\alpha\nabla_U U = 0, \tag{3.6}$$

because U and W are mutually orthogonal.

Combining (2.10) to (3.2) and using (2.4), we obtain

$$\begin{aligned}\alpha^2\phi\nabla_X U = {}& \alpha^2(X\alpha)\xi - \frac{c}{4}\alpha u(X)\xi + \alpha(\xi\alpha)AX + \frac{c}{4}\alpha^2\phi X \\ & - \eta(AX)\left(\alpha\nabla\alpha - \frac{3}{4}cU\right) - \left\{\alpha(X\alpha) - \frac{3}{4}cu(X)\right\}A\xi \\ & - \frac{c}{4}\alpha\{u(X)\xi + \eta(X)U\} - \alpha^2 A\phi AX + \alpha^3\phi AX.\end{aligned} \tag{3.7}$$

Applying ϕ to this and using (2.8), we have

$$\begin{aligned}&\alpha^2\nabla_X U + \alpha^2\mu g(AW, X)\xi - \alpha\eta(AX)\phi\nabla\alpha \\ &= -\alpha(\xi\alpha)\phi AX + \frac{c}{4}\alpha^2\{X - \eta(X)\xi\} + \frac{3}{4}c\mu\eta(AX)W + \alpha(X\alpha)U \\ &\quad - \frac{3}{4}cu(X)U + \alpha^3 AX - \frac{c}{4}\alpha\mu\eta(X)W - \alpha^3\eta(AX)\xi + \alpha^2\phi A\phi AX.\end{aligned} \tag{3.8}$$

On the other hand, differentiating (2.6) covariantly, and using (2.1), we find

$$(\nabla_X A)\xi - \frac{c}{4}\phi X + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W,$$

or using (2.4) and (3.2),

(3.9)

$$\alpha\mu\nabla_X W = \frac{c}{4}\{u(X)\xi + \eta(X)U\} + \eta(AX)(3AU + \nabla\alpha) + 3g(AU, X)A\xi$$
$$+ \mu(X\alpha)W - \frac{c}{4}\alpha\phi X + \alpha A\phi AX - \alpha^2\phi AX - (\xi\alpha)AX - \alpha(X\mu)W.$$

From (2.12) and (3.3) we have

$$\alpha\nabla_\xi U = \frac{3}{4}c\mu W + \alpha^2 A\xi - \alpha\beta\xi + \alpha\phi\nabla\alpha. \tag{3.10}$$

Putting $X = \alpha U$ in (3.2) and using (2.4) and (3.3), we get

$$\alpha^2(\nabla_\xi A)U - \frac{c}{4}(\xi\alpha)U = \frac{c}{4}\alpha\mu^2\xi + \left\{\alpha(U\alpha) - \frac{3}{4}c\mu^2\right\} A\xi. \tag{3.11}$$

If we put $X = \alpha\xi$ in (3.5) and make use of (3.10) and (3.11), then we have

(3.12)

$$\alpha A\phi\nabla\alpha + \frac{c}{4}\phi\nabla\alpha + (U\alpha)A\xi + \mu\left(\alpha^2 + \frac{3}{4}c\right)\left\{AW - \mu\xi - \frac{1}{\alpha}\left(\mu^2 - \frac{c}{4}\right)W\right\} = 0.$$

4. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ and $g(U,U)$ is constant

In this section we assume that $\nabla_\xi R_\xi = 0$ and at the same time $g(\nabla_\xi\xi, \nabla_\xi\xi)$ is constant, i.e. $g(U,U) = \mu^2$ is constant. Then we get

$$\nabla\mu = 0, \tag{4.1}$$

on Ω. Note that above equation implies that

$$g(\nabla_X U, U) = 0.$$

If we take a inner product U to (3.8), then we have

$$(W\alpha)A\xi = (\xi\alpha)AW + \frac{3}{4\alpha}c\mu U - \mu\nabla\alpha + \alpha A\phi AW, \tag{4.2}$$

which shows that $W\alpha = 0$ on Ω because of (3.3). Moreover we take a inner product W to (4.2), then we get

$$(\xi\alpha)g(AW, W) = 0. \tag{4.3}$$

Thus, (4.2) turns out to be

$$\mu\left(\nabla\alpha-\frac{3}{4\alpha}cU\right)=(\xi\alpha)AW+\alpha A\phi AW, \tag{4.4}$$

which implies that

$$\alpha(U\alpha)=\frac{3}{4}c\mu^2-\frac{c}{4}\alpha\lambda, \tag{4.5}$$

where we have put $\lambda=g(AW,W)$.

Putting $X=U$ in (3.8) and using (3.3), we also find

$$\alpha^2\nabla_U U=-\frac{c}{4}(\xi\alpha)\mu W+\left\{\alpha(U\alpha)-\frac{3}{4}c\mu^2\right\}U+\frac{c}{4}\alpha\mu\phi AW,$$

or using (4.5)

$$\alpha\nabla_U U=-\frac{c}{4\alpha}\mu(\xi\alpha)W-\frac{c}{4}\lambda U+\frac{c}{4}\mu\phi AW.$$

Hence we have

$$\alpha^2 A\nabla_U U=-\frac{c}{4}\mu(\xi\alpha)AW+\left(\frac{c}{4}\right)^2\lambda U+\frac{c}{4}\mu\alpha A\phi AW.$$

Combining last two equations, it follows that

$$\alpha^2 A\nabla_U U+\frac{c}{4}\alpha\nabla_U U=-\frac{c}{4\alpha}\mu(\xi\alpha)\left(\alpha AW+\frac{c}{4}\right)+\frac{c}{4}\mu\left(\alpha A\phi AW+\frac{c}{4}\phi AW\right),$$

which together with (3.6) yields

$$\mu^2\nabla\alpha-(U\alpha)U=\frac{\mu}{\alpha}(\xi\alpha)\left(\alpha AW+\frac{c}{4}W\right)-\mu\left(\alpha A\phi AW+\frac{c}{4}\phi AW\right). \tag{4.6}$$

If we take a inner product W to this and take account of (4.3), then we obtain $\xi\alpha=0$. So we see, using (4.5), that

$$\mu\nabla\alpha=\left(\frac{3}{4\alpha}c\mu-\frac{c\lambda}{4\mu}\right)U-\alpha A\phi AW-\frac{c}{4}\phi AW. \tag{4.7}$$

Combining this to (4.4), we have

$$\alpha A\phi AW=-\frac{c}{8}\left(\frac{\lambda}{\mu}U+\phi AW\right) \tag{4.8}$$

by virtue of $\xi\alpha=0$. Thus, (4.7) turns out to be

$$\mu\left(\alpha\nabla\alpha-\frac{3}{4}cU\right)=-\frac{c}{8}\left(\alpha\phi AW+\frac{\alpha\lambda}{\mu}U\right). \tag{4.9}$$

Putting $X=\xi$ in (3.9) and using (3.3) and (4.9), we obtain

$$\mu^2\nabla_\xi W=-\left(\mu\alpha+\frac{c\lambda}{8\mu}\right)U-\frac{c}{8}\phi AW, \tag{4.10}$$

which implies

$$g(AW, \nabla_\xi W) = 0.$$

We verify, using (3.2) and $W\alpha = 0$, that $g((\nabla_\xi A)W, W) = 0$. By the definition of λ, we see that

$$\xi\lambda = g((\nabla_\xi A)W, W) + 2g(AW, \nabla_\xi W).$$

Therefore we obtain $\xi\lambda = 0$. Summing up we have

Lemma 1. *$\xi\alpha = \xi\lambda = W\alpha = 0$ on Ω.*

Applying (4.8) by ϕ, we get

$$\alpha\phi A\phi AW = \frac{c}{8}(\lambda W + AW - \mu\xi).$$

If we apply (4.7) by ϕ and take account of the last equation, then we obtain

$$\mu\phi\nabla\alpha = \frac{c}{8}(AW - \mu\xi) + \frac{c}{8}\left(\lambda - \frac{6\mu^2}{\alpha}\right)W. \tag{4.11}$$

Accordingly we have

$$\mu\alpha A\phi\nabla\alpha = \frac{c}{8}\alpha(A^2W - \mu A\xi) + \frac{c}{8}(\alpha\lambda - 6\mu^2)AW.$$

Substituting (4.5), (4.11) and this into (3.12), we obtain

$$\begin{aligned} -\frac{c}{8}\alpha A^2W \;=\;& \left\{\mu^2\alpha^2 + \frac{c^2}{32} + \frac{c}{8}\alpha\lambda\right\}AW + \mu\left(\frac{3}{4\alpha}c\mu^2 - \frac{c}{4}\lambda - \frac{c}{8}\alpha\right)A\xi \\ & -\mu\left\{\frac{c^2}{32} + \mu^2\left(\alpha^2 + \frac{3}{4}c\right)\right\}\xi \\ & -\frac{1}{\alpha}\left\{\mu^2\left(\alpha^2 + \frac{3}{4}c\right)\left(\mu^2 - \frac{c}{4}\right) + \frac{3}{16}c^2\mu^2 - \frac{c^2}{32}\alpha\lambda\right\}W, \end{aligned} \tag{4.12}$$

which implies

$$\begin{aligned} &\frac{c}{8}\alpha g(A^2W, W) = \frac{c}{8}\mu^2\left(\alpha + \frac{3}{2\alpha}c\right) - \frac{3c}{4\alpha}\mu^4 \\ &\quad +\mu^2\left(\alpha + \frac{3}{4\alpha}c\right)\left(\mu^2 - \frac{c}{4}\right) + \lambda\left(\alpha^2\mu^2 - \frac{1}{4}c\mu^2 + \frac{c^2}{16}\right) - \frac{c}{8}\alpha\lambda^2. \end{aligned} \tag{4.13}$$

If we take account of Lemma 1, we can write the equation (3.8) as

$$\begin{aligned} &\alpha(\nabla_X u)(Y) + \alpha\mu g(AX, W)\eta(Y) - \eta(AX)g(\phi\nabla\alpha, Y) \\ =&\frac{c}{4}\alpha\{g(X, Y) - \eta(X)\eta(Y)\} + \frac{3}{4\alpha}c\mu\eta(AX)w(Y) + (X\alpha)u(Y) - \frac{3}{4\alpha}cu(X)u(Y) \\ &+ \alpha^2 g(AX, Y) - \frac{c}{4}\mu\eta(X)w(Y) - \alpha^2\eta(AX)\eta(Y) + \alpha g(\phi A\phi AX, Y). \end{aligned}$$

From this, we have a Codazzi-type for u:

$$\begin{aligned}&\alpha\{(\nabla_X u)(Y)-(\nabla_Y u)(X)\}+\alpha\mu\{\eta(Y)w(AX)-\eta(X)w(AY)\}\\&-\eta(AX)g(\phi\nabla\alpha,Y)+\eta(AY)g(\phi\nabla\alpha,X)\\=&\mu\left(\alpha^2+\frac{c}{2}\right)(\eta(X)w(Y)-\eta(Y)w(X))+(X\alpha)u(Y)-(Y\alpha)u(X)\\&+\alpha g((\phi A\phi A-A\phi A\phi)X,Y),\end{aligned}$$

where we have used (2.6). Putting $X=\xi$ in this and using (3.3), (4.10) and Lemma 1, we get

$$\begin{aligned}&(\nabla_\xi u)(Y)-(\nabla_Y u)(\xi)\\&=\left(\mu+\frac{c}{8\mu}\right)w(AY)-\left(\mu^2+\frac{c}{8}\right)\eta(Y)+\left(\mu\alpha+\frac{c\lambda}{8\mu}\right)w(Y).\end{aligned}\tag{4.14}$$

Applying (4.10) by αA and using (3.3) and (4.8), we have

$$\alpha\mu^2A\nabla_\xi W=\frac{c}{4}\left(\mu\alpha+\frac{3\lambda}{16\mu}c\right)U+\left(\frac{c}{8}\right)^2\phi AW,$$

which shows that

$$\alpha\mu^2\phi A\nabla_\xi W=-\frac{c}{4}\left(\alpha\mu^2+\frac{3}{16}c\lambda\right)W-\left(\frac{c}{8}\right)^2(AW-\mu\xi).\tag{4.15}$$

If we replace X by W in (3.2) and make use of (3.3) and Lemma 1, then we obtain

$$\alpha(\nabla_\xi A)W=\mu\left(\nabla\alpha-\frac{3}{4\alpha}cU\right),$$

which together with (4.9) yields

$$\alpha\phi(\nabla_\xi A)W=\frac{c}{8}(AW-\mu\xi+\lambda W).\tag{4.16}$$

5. Lemmas and theorems

We will continue our arguments under the hypotheses $\nabla_\xi R_\xi=0$ and at the same time $g(U,U)$ is constant. Then (4.9) is rewrittes as

$$\frac{4}{c}\mu^2(Y\alpha^2)=(6\mu^2-\alpha\lambda)u(Y)-\mu\alpha g(\phi AW,Y)\tag{5.1}$$

for any vector field Y. Since μ is constant, if we differentiate this with respect to a vector field X again, and take the skew-symmetric part for

X and Y, then we eventually have

(5.2)

$$
\begin{aligned}
0 =&Y(\alpha\lambda)u(X) - X(\alpha\lambda)u(Y) + (6\mu^2 - \alpha\lambda)\{(\nabla_X u)(Y) - (\nabla_Y u)(X)\} \\
&+ \frac{c}{8}\left(\frac{\lambda}{\mu} - \frac{6\mu}{\alpha}\right)\{u(X)g(\phi AW, Y) - u(Y)g(\phi AW, X)\} \\
&+ \mu\alpha\{g(A^2W, X)\eta(Y) - g(A^2W, Y)\eta(X) + g(\phi(\nabla_Y A)W, X) \\
&- g(\phi(\nabla_X A)W, Y) + g(\phi A\nabla_Y W, X) - g(\phi A\nabla_X W, Y)\}.
\end{aligned}
$$

Putting $X = \xi$ in this, and using (4.14), (4.15), (4.16) and Lemma 1, we have

(5.3)

$$
\begin{aligned}
\mu\alpha A^2W =& \left\{(6\mu^2 - \alpha\lambda)\left(\mu + \frac{c}{8\mu}\right) - \frac{c}{8}\mu + \frac{1}{\mu}\left(\frac{c}{8}\right)^2\right\}AW \\
&+ \left\{\alpha\mu^2(\alpha + \lambda) - (6\mu^2 - \alpha\lambda)\left(\mu^2 + \frac{c}{8}\right) + \frac{c}{8}\mu^2 - \left(\frac{c}{8}\right)^2\right\}\xi \\
&+ \left\{(6\mu^2 - \alpha\lambda)(\mu\alpha + \frac{c}{8\mu}\lambda) - \frac{c}{8}\mu\lambda + \frac{c}{4}\alpha\mu + 3\left(\frac{c}{8}\right)^2\frac{\lambda}{\mu}\right\}W,
\end{aligned}
$$

which implies that

(5.4)

$$
\mu\alpha g(A^2W, W) = (6\mu^2 - \alpha\lambda)\left(\mu\lambda + \mu\alpha + \frac{c\lambda}{4\mu}\right) + \frac{c}{4}\alpha\mu - \frac{c}{4}\mu\lambda + \left(\frac{c}{4}\right)^2\frac{\lambda}{\mu}.
$$

Combine (4.12) and (5.3), we find

$$
(5.5)\quad \left\{\alpha^2\mu^2 + \frac{3}{4}c\mu^2 + \frac{5}{64}c^2 - \left(\frac{c}{8}\right)^2\frac{\alpha\lambda}{\mu^2} - \left(\frac{c}{8}\right)^3\frac{1}{\mu^2}\right\}AW = f_1\xi + f_2W
$$

for some smooth functions f_1 and f_2 on Ω.

Now, we are going to prove the following:

Lemma 2. *$AW = \mu\xi + \lambda W$ on Ω.*

Proof. If not, then we have from (5.5)

$$
(5.6)\qquad \left(\frac{c}{8}\right)^2\alpha\lambda = \mu^4\alpha^2 + \frac{3}{4}c\mu^4 + \frac{5}{64}c^2\mu^2 - \left(\frac{c}{8}\right)^3.
$$

If we combine (4.13) to (5.4), then we get

$$
\begin{aligned}
&\frac{c}{8}\left\{(6\mu^2 - \alpha\lambda)\left(\mu\lambda + \mu\alpha + \frac{c\lambda}{4\mu}\right) + \frac{c}{4}\mu\alpha - \frac{c}{4}\mu\lambda + \left(\frac{c}{4}\right)^2\frac{\lambda}{\mu}\right\} \\
=&\mu^3\left(-\alpha\mu^2 + \frac{c}{8}\alpha - \frac{3}{4\alpha}c\mu^2 + \frac{3}{16\alpha}c^2 + \lambda\alpha^2 - \frac{c}{4}\lambda\right) + \frac{c}{8}\lambda\mu\left(\alpha\lambda + \frac{c}{2}\right).
\end{aligned}
$$

Comparing this with (5.6), we obtain

$$\left\{\mu^{10}+\frac{3c}{8}\mu^8+\left(\frac{c}{8}\right)^2\mu^6\right\}\alpha^4$$
$$+\left\{\frac{12c}{8}\mu^{10}+32\left(\frac{c}{8}\right)^2\mu^8-9\left(\frac{c}{8}\right)^4\mu^4-3\left(\frac{c}{8}\right)^5\mu^2\right\}\alpha^2+36\left(\frac{c}{8}\right)^2\mu^{10}$$
$$+42\left(\frac{c}{8}\right)^3\mu^8+27\left(\frac{c}{8}\right)^4\mu^6-38\left(\frac{c}{8}\right)^5\mu^4-29\left(\frac{c}{8}\right)^6\mu^2+6\left(\frac{c}{8}\right)^7=0,$$

which implies that α is a root of the algebraic equation with constant coefficient, because μ is constant. Consequently α is constant and hence $3\mu^2=\alpha\lambda$ by virtue of (4.5). Thus, (4.9) is reduced to

$$\mu\phi AW=\lambda U.$$

So we have $AW=\mu\xi+\lambda W$, a contradiction. Thus, Lemma 2 is proved. □

Using Lemma 2, we have

$$(\nabla_X A)W+A\nabla_X W=\mu\phi AX+(X\lambda)W+\lambda\nabla_X W, \tag{5.7}$$

which yields

$$g((\nabla_X A)W,Y)+g(A\nabla_X W,Y)=\mu g(\phi AX,Y)+(X\lambda)w(Y)+\lambda g(\nabla_X W,Y).$$

Putting $Y=W$ in this and using (2.9) and (3.3), we find

$$g((\nabla_X A)W,W)=\frac{c}{2\alpha}u(X)+X\lambda,$$

which together with (2.4) implies that

$$(\nabla_W A)W=\frac{c}{2\alpha}U+\nabla\lambda.$$

If we put $X=W$ in (5.7) and make use of Lemma 2 and the last equation, then we obtain

$$A\nabla_W W-\lambda\nabla_W W=\left(\lambda-\frac{c}{2\alpha}\right)U-\nabla\lambda. \tag{5.8}$$

Indeed, it is, using Lemma 2, seen that $g(A^2W,W)=\lambda^2+\mu^2$. Hence (4.13) becomes

$$\frac{c}{4}\alpha\lambda^2-\left(\frac{c}{2}\alpha\lambda-\alpha^2\mu^2+\frac{c}{4}\mu^2-\frac{c^2}{16}\right)\lambda-\alpha\mu^2\left(\mu^2-\frac{c}{8}\right)=0.$$

From this and Lemma 1 we verify that $W\lambda=0$. Thus, (5.8) is accomplished on Ω.

Because of Lemma 2, (4.9) turns out to be

$$\mu^2\alpha\nabla\alpha = \frac{c}{4}(3\mu^2 - \alpha\lambda)U. \tag{5.9}$$

If we take account of (2.6), (5.9), Lemma 1 and Lemma 2, then (3.9) is reduced to

$$\begin{aligned}\mu^2\alpha\nabla_X W = &-\frac{c}{4}\alpha\phi X + \alpha A\phi AX - \alpha^2\phi AX - \frac{c}{2}u(X)\xi \\ &+ \frac{c}{4\mu}(\mu^2 - \alpha\lambda)\eta(X)U - \frac{c\lambda}{4}(u(X)W + w(X)U).\end{aligned} \tag{5.10}$$

Putting $X = W$ in this and using (3.3) and Lemma 2, we have

$$\mu^2\alpha\nabla_W W = -\left(\frac{c}{2}\lambda + \frac{c}{4}\alpha + \alpha^2\lambda\right)U.$$

Combining this to (5.8), we obtain

$$\alpha\mu^2\nabla\lambda = \left\{\alpha\mu^2\left(\lambda - \frac{c}{2\alpha}\right) - (\lambda + \frac{c}{4\alpha})\left(\alpha^2\lambda + \frac{c}{2}\lambda + \frac{c}{4}\alpha\right)\right\}U. \tag{5.11}$$

In the nextplace, we prove

Lemma 3. *α and λ are constant on Ω.*

Proof. (5.9) is rewritten as

$$\mu^2\alpha(Y\alpha) = \frac{c}{4}(3\mu^2 - \alpha\lambda)u(Y) \tag{5.12}$$

for any vector field Y. Using the same method as that used to derive (5.2) from (5.1), we can deduce from the last equation the following:

$$Y(\alpha\lambda)u(X) - X(\alpha\lambda)u(Y) + (3\mu^2 - \alpha\lambda)((\nabla_X u)(Y) - (\nabla_Y u)(X)) = 0,$$

which together with Lemma 1 gives

$$(3\mu^2 - \alpha\lambda)((\nabla_\xi u)(Y) - (\nabla_Y u)(\xi)) = 0.$$

Now, we suppose that $3\mu^2 - \alpha\lambda \neq 0$ on Ω, and that we restrict the arguments on such a place. Then we have from the last equation

$$g(\nabla_\xi U, Y) + g(\nabla_Y \xi, U) = 0.$$

Further, we get from (4.14)

$$\left(\mu + \frac{c}{8\mu}\right)AW - \left(\mu^2 + \frac{c}{8}\right)\xi + \left(\mu\alpha + \frac{c}{8\mu}\lambda\right)W = 0$$

and hence

$$\mu^2(\lambda + \alpha) + \frac{c}{4}\lambda = 0 \tag{5.13}$$

with the aid of Lemma 2, which implies

$$(\mu^2 + \frac{c}{4})\nabla\lambda = -\mu^2\nabla\alpha.$$

From this and (5.12) we see that

$$\alpha(\mu^2 + \frac{c}{4})\nabla\lambda = \frac{c}{4}(\alpha\lambda - 3\mu^2)U, \tag{5.14}$$

which together with (5.11) yields

$$\begin{aligned}&\frac{c}{4}\mu^2(\alpha\lambda - 3\mu^2)\\ &= \alpha\mu^2\left(\mu^2 + \frac{c}{4}\right)\left(\lambda - \frac{c}{2\alpha}\right) - \left(\lambda + \frac{c}{4\alpha}\right)\left(\mu^2 + \frac{c}{4}\right)\left(\alpha^2\lambda + \frac{c}{2}\lambda + \frac{c}{4}\alpha\right).\end{aligned}$$

If we make use of (5.13), then we get

$$\mu^4\alpha^4 + \left(\mu^6 - \frac{c^2}{8}\mu^2\right)\alpha^2 + f(\mu) = 0,$$

where $f(\mu)$ is certain polynomial with respect to μ. Since μ is constant, the last equation tells us that α is constant and hence $3\mu^2 = \alpha\lambda$ because of (5.12), a contradiction. Therefore, we arrive at the conclusion. □

Lemma 4. *$\alpha^2 + (3/4)c = 0$ on Ω.*

Proof. Replacing X by U in (5.10) and making use of (3.3) and Lemma 2, we find

$$\alpha\nabla_U W = -\frac{c}{4}c\mu\xi. \tag{5.15}$$

If we take a inner product (5.10) with U and use (3.3), Lemma 2 and Lemma 3, then we also obtain

$$\alpha g(\nabla_X W, U) = -\mu\left(\alpha^2 + \frac{3}{4}c\right)\eta(X) - \left(\frac{c}{4}\alpha + \frac{c}{2}\lambda + \alpha^2\lambda\right)w(X). \tag{5.16}$$

From (5.7) we have a Codazzi-type formula for w:

$$\begin{aligned}\lambda((\nabla_X w)(Y) - (\nabla_Y w)(X)) &= \frac{c}{4\mu}(u(X)\eta(Y) - u(Y)\eta(X))\\ &+g(A\nabla_X W, Y) - g(A\nabla_Y W, X) - \mu g((\phi A + A\phi)X, Y),\end{aligned}$$

where we have used (2.4) and Lemma 3. If we replace X by U and take account of (3.3) and (5.15), then we obtain

$$\left(\lambda + \frac{c}{4\alpha}\right)g(\nabla_X W, U) = -\mu^2 g(AW, X) + \frac{c}{2\alpha}\mu^2 w(X) - \frac{c}{4\alpha}\lambda\mu\eta(X),$$

or make use of (5.16),

$$\left(\lambda+\frac{c}{4\alpha}\right)\left\{\mu\left(\alpha^2+\frac{3}{4}c\right)\xi+\left(\frac{c}{4}\alpha+\frac{c}{2}\lambda+\alpha^2\lambda\right)W\right\}$$
$$=\mu^2\alpha AW-\frac{c}{2}\mu^2W+\frac{c}{4}\lambda\mu\xi.$$

From this we have

$$\left(\lambda+\frac{c}{4\alpha}\right)\left(\alpha^2+\frac{3}{4}c\right)-\alpha\mu^2-\frac{c}{4}\lambda=0 \tag{5.17}$$

and

$$\left(\lambda+\frac{c}{4\alpha}\right)\left(\frac{c}{4}\alpha+\frac{c}{2}\lambda+\alpha^2\lambda\right)+\mu^2\left(\frac{c}{2}-\alpha\lambda\right)=0 \tag{5.18}$$

because of Lemma 2. Since we have

$$3\mu^2=\alpha\lambda \tag{5.19}$$

by virtue of (5.9) and Lemma 3, we can deduce (5.17) as

$$\left(\alpha^2+\frac{3}{4}c\right)\left(2\mu^2+\frac{c}{4}\right)=0.$$

Now, we suppose that $2\mu^2+c/4=0$ on Ω, then we have $c<0$. However combining this and (5.19) to (5.18) we get $\alpha^2=c/4$, which impplies that $c>0$, a contradiction. Therefore, Lemma 4 is proved. □

From (5.19) and Lemma 4, we have

$$\mu^2\alpha=-\frac{c}{4}\lambda, \tag{5.20}$$

which together with (5.18) implies that $6\lambda=\alpha$. So we see, using (5.20), that $6\mu^2+c/4=0$. Developed as above we conclude that $\mu=0$ or $6\mu^2+c/4=0$ on M because μ is constant. Thus we have

Proposition 1. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c\neq 0$ which satisfies $\nabla_\xi R_\xi=0$. If $g(\nabla_\xi\xi,\nabla_\xi\xi)=\mu^2$ is constant on M, then $\mu=0$, that is, $A\xi=\alpha\xi$ or $6\mu^2+c/4=0$.*

If $6\mu^2+c/4\neq 0$ holds on Proposition 1, then we have $A\xi=\alpha\xi$ on whole space M. So we verify that α is constant on M (see [8]). Thus it follows from (3.2) that $\alpha\nabla_\xi A=0$. Consequently, we see that $\alpha(A\phi-\phi A)=0$ by virtue of (2.1) and (2.4).

Here, we note the case $\alpha=0$ corresponds to the case of tube of radius $\pi/4$ in $P_n\mathbb{C}$ (see [2]). But, in the case of $H_n\mathbb{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf. [1]). Thus, owing to Okumura's work for $P_n\mathbb{C}$ or Montiel and Romero's work for $H_n\mathbb{C}$ we have

Theorem 2. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$ which satisfies $\mu^2 = g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant and $6\mu^2 + c/4 \neq 0$. Then M holds $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following:*

(I) *in case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*

(A_1) *a geodesic hypershpere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

(II) *in case that $M_n(c) = H_n\mathbb{C}$,*

(A_0) *a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbb{C}(1 \leq k \leq n-2)$.*

Corollary 3. *Let M be a real hypersurface of a complex projective space $P_n\mathbb{C}$ which satisfies $\mu^2 = g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant. Then M holds $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following:*

(A_1) *a geodesic hypershpere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*

(A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*

provided that $\eta(A\xi) \neq 0$.

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Nam-Gil Kim
Department of Mathematics,
Chosun University.
Kwangju 501-759, Korea
E-mail: `ngkim@chosun.ac.kr`

U-Hang Ki
Department of Mathematics,
Kyungpook National University,
Daegu 702-701, Korea
E-mail: `uhangki2005@yahoo.co.kr`

Hiroyuki Kurihara
Department of Computer and Media Science,
Saitama Junior College,
Hanasaki-ebashi, Kazo, Saitama 347-8503, Japan
E-mail: `kurihara@sjc.ac.jp`

Current address
5-15-5 Hanamigawaku-makuharihongou,
Chiba 262-0033, Japan
E-mail: `h-kuri@leo.nit.ac.jp`