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QUALITATIVE ANALYSIS OF A LOTKA-VOLTERRA TYPE IMPULSIVE PREDATOR-PREY SYSTEM WITH SEASONAL EFFECTS

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Abstract. We investigate a periodically forced Lotka-Volterra type predator-prey system with impulsive perturbations - seasonal effects on the prey, periodic releasing of natural enemies(predator) and spraying pesticide at the same fixed times. We show that the solutions of the system are bounded using the comparison theorems and find conditions for the stability of a stable prey-free solution and for the permanence of the system.

1. Introduction

The study of complex population dynamics is nearly as old as population ecology. In the 1920s, A. Lotka and V. Volterra independently suggested a simple model of interacting species named a Lotka-Volterra model. The principles of Lotka-Volterra models, conservation of mass and decomposition of the rates of change in birth and death processes, have remained valid until today. Many theoretical ecologists developed a lot of population models using Lotka-Volterra models. For these reasons, we mention a predator-prey model with a Lotka-Volterra functional response as the following form [2, 5]:

$$(1.1)\qquad \begin{cases} x'(t) = x(t)(a - bx(t) - cy(t)), \\ y'(t) = y(t)(-d + ex(t)). \end{cases}$$

It is necessary and important to consider models with periodic ecological parameters or perturbation which might be quite naturally exposed such as those due to seasonal effects of weather or food supply etc[3, 9, 10]. There are a number of ways to apply periodic perturbation

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in an ecological model[23, 30, 31]. In this paper, we consider the intrinsic growth rate a in the model (1.1) as periodically varying function of time due to seasonal variation. The seasonality is superimposed as follows:

$$a_0 = a(1 + \epsilon \sin(\omega t)),$$

where the parameter ϵ represents the degree of seasonality; $a\epsilon$ is the magnitude of the perturbation in a_0, ω is the angular frequency of the fluctuation caused by seasonality. Since a_0 is assumed to be positive, we have $0 \le \epsilon \le 1$. With this idea of periodic forcing, we consider the following predator-prey model with periodic variation in the intrinsic growth rate of the prey.

$$(1.2) \qquad \begin{cases} x'(t) = x(t)(a - bx(t) - cy(t)) + \lambda x(t)\sin(\omega t), \\ y'(t) = y(t)(-d + ex(t)), \end{cases}$$

where λ and ω represent the magnitude and frequency of the forcing term, respectively.

There are still some other perturbations such as fire, flood, mating habits or harvesting seasons etc, which are not suitable to be considered continually. When we consider pest outbreak, for example, there are many ways to beat agricultural pests. One way is biological control leading reduction in pest population from the actions of other living organisms, often called natural enemies or beneficial species. Pesticides are also a useful method to stamp out pest because they quickly kill a significant portion of pests. However, we can't use these ways continuously but impulsively to eliminate pests. Thus, we consider the following predator-prey model with periodic constant impulsive immigration of the predator, spraying pesticide at the same time and periodic variation in the intrinsic growth rate of the prey:

$$(1.3) \qquad \begin{cases} \left. \begin{array}{l} x'(t) = x(t)(a - bx(t) - cy(t) + \lambda \sin(\omega t)), \\ y'(t) = y(t)(-d + ex(t)), \end{array} \right\} t \neq nT, \\ \left. \begin{array}{l} \Delta x(t) = -p_1 x(t), \\ \Delta y(t) = -p_2 y(t) + q, \end{array} \right\} t = nT, \\ (x(0^+), y(0^+)) = (x_0, y_0). \end{cases}$$

where, $\Delta x(t) = x(t^+) - x(t), \Delta y(t) = y(t^+) - y(t)$ and $0 \le p_1, p_2 < 1$ present the fraction of prey and predator which die due to the harvesting or pesticides etc. T is the period of the impulsive immigration, q is the size of immigration. The parameters a, b, c, d and e are positive constants.

Many authors have studied population models with impulsive control terms similar to (1.3)[13, 14, 16, 17, 18, 21, 27, 29, 30, 31]. Actually, the authors in [18] studied a T-periodic competing Lotka-Volterra predator-prey system with impulsive effects and gave the conditions for the global stability of these solutions as a consequence of some abstract monotone iterative schemes and for the existence of at least one strictly positive (componentwise) periodic solution. The authors in [19] proposed the classical Lotka-Volterra predator-prey system with impulsive control strategies and proved the existence of a globally asymptotically stable pest-eradication periodic solution and the permanence of the system under some condition. Especially, in [20], the authors investigated the system (1.3) with no seasonal effects. i.e., $\lambda = 0$. They showed that the existence of predator-free periodic solutions and their stabilities when the pest-eradication lost its stability and gave the condition for the permanence or extinction in the system.

This paper is arranged as follows: In the next section, we make mention of Lemmas and notations used in this paper. We investigate mathematical properties of the systems (1.2) and (1.3) in section 3 where we prove that the solutions of the system (1.3) are bounded using the comparison theorem of impulsive differential equations. Moreover, we find conditions for the local stability of prey-free periodic solutions of the system (1.3) and for the permanence of the systems (1.2) and (1.3) by using the Floquet theory of the impulsive differential equation and small perturbation skills.

2. Preliminaries

Firstly, we give some notations, definitions and Lemmas which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^2_+ = \{\mathbf{x} = (x(t), y(t)) \in \mathbb{R}^2 : x(t), y(t) \geq 0\}$. Denote $\mathbb{N}$ the set of all of nonnegative integers and $f = (f_1, f_2)^T$ the right hand of (1.3). Let $V : \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+$. Then V is said to be in a class V_0 if

(1) V is continuous on $(nT, (n+1)T] \times \mathbb{R}^2_+$ and $\lim_{\substack{(t,\mathbf{y}) \to (nT, \mathbf{x}) \\ t > nT}} V(t, \mathbf{y}) = V(nT^+, \mathbf{x})$ exists.

(2) V is a locally Lipschitzian in $\mathbf{x}$.

Definition 2.1. Let $V \in V_0, (t, \mathbf{x}) \in (nT, (n+1)T] \times \mathbb{R}_+^2$. The upper right derivatives of $V(t, \mathbf{x})$ with respect to the impulsive differential system (1.3) is defined as

$$D^+V(t, \mathbf{x}) = \limsup_{h \to 0+} \frac{1}{h}[V(t+h, \mathbf{x} + hf(t, \mathbf{x})) - V(t, \mathbf{x})]$$

The solution of the system (1.3) is a piecewise continuous function $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}_+^2$, $\mathbf{x}(t)$ is continuous on $(nT, (n+1)T], n \in \mathbb{N}$ and $\mathbf{x}(nT^+) = \lim_{t \to nT^+} \mathbf{x}(t)$ exists. The smoothness properties of f guarantee the global existence and uniqueness of solution of the system (1.3). (See [15] for the details).

Note that $x'(t) = y'(t) = 0$ whenever $x(t) = y(t) = 0, t \neq nT$ and $x(nT^+) = (1 - p_1)x(nT)$ and $y(nT^+) = (1 - p_2)y(nT) + q(0 \leq p_i < 1, i = 1, 2, q \geq 0)$. Then we have the following lemma.

Lemma 2.2. *Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of (1.3).*
(1) If $\mathbf{x}(0^+) \geq 0$, then $\mathbf{x}(t) \geq 0$ for all $t \geq 0$.
(2) If $\mathbf{x}(0^+) > 0$, then $\mathbf{x}(t) > 0$ for all $t \geq 0$.

3. Qualitative Analysis

3.1. Boundedness

We show that all solutions of (1.3) are uniformly ultimately bounded.

Theorem 3.1. *There is an $M > 0$ such that $x(t) < M$ and $y(t) \leq M$ for all t large enough, where $(x(t), y(t))$ is a solution of (1.3).*

Proof. Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of (1.3) and let $V(t) \equiv V(t, \mathbf{x}) = ex(t) + cy(t)$. Then $V \in V_0$, if $t \neq nT$
(3.1)
$$\begin{aligned} D^+V + \beta V &= -ebx^2(t) + e(a + \beta + \lambda \sin(\omega t))x(t) + c(\beta - d)y(t) \\ &\leq -ebx^2(t) + e(a + \beta + \lambda)x(t) + c(\beta - d)y(t) \end{aligned}$$

Clearly, the right hand of (3.1) is bounded by $\frac{(a+\beta+\lambda)^2}{4b^2}$ when $0 < \beta < d$ and $V(nT^+) = ex(nT^+) + cy(nT^+) = (1-p_1)ex(nT) + (1-p_2)cy(nT) + cq$ $\leq V(nT) + cq$. So we can choose $0 < \beta_0 < d$ and $M_0 > 0$ such that

$$\begin{cases} D^+V \leq -\beta_0 V + M_0, t \neq nT, \\ V(n\tau^+) \leq V(n\tau) + cq, t = nT. \end{cases} \tag{3.2}$$

From the comparison theorem of impulsive differential inequalities [15], we can obtain that, for $t \in (nT, (n+1)T]$,

$$\begin{aligned} V(t) &\leq V(0^+)\exp(-\beta_0 t) + \int_{0^+}^{t} M_0 \exp(-\beta_0(t-s))ds \\ &\quad + \sum_{0<nT<t} cq \exp(-\beta_0(t-nT)) \\ &= V(0^+)\exp(-\beta_0 t) + \frac{M_0}{\beta_0} - \frac{M_0}{\beta_0}\exp(-\beta_0 t) \\ &\quad + \frac{cq(1-\exp(-(n+1)\beta_0 T))}{1-\exp(-\beta_0 T)}\exp(-\beta_0(t-nT)). \end{aligned}$$

Since the right hand side of the above inequality as $t \to \infty$ is $\frac{cq}{1-\exp(-\beta_0 T)}$ $\exp(-\beta_0 T)) + \frac{M_0}{\beta_0}$, $V(t)$ is bounded by a constant M for sufficiently large t. □

3.2. Stability of a prey-free periodic solution

First, we give the basic properties of the following impulsive differential equation considered the absence of prey.

$$(3.3) \qquad \begin{cases} y'(t) = -dy(t), \ t \neq nT \\ \triangle y(t) = -p_2 y(t) + q, \ t = nT, \\ y(0^+) = y_0. \end{cases}$$

Solving the first equation of (3.3) between pulses gives

$$(3.4) \qquad y(t) = y(nT) + \exp(-d(t-nT)), t \in (nT, (n+1)T].$$

Substituting it in the second equation of (3.3), the following difference equation is obtained:

$$(3.5) \qquad y((n+1)T) = (1-p_2)y(nT) + q + \exp(-DT).$$

Then a periodic solution $y^*(t)$ of (3.3) is given as

$$y^*(t) = \frac{q\exp(-d(t-nT))}{1-(1-p_2)\exp(-dT)}, t \in (nT, (n+1)T], n \in \mathbb{N}.$$

Thus we can easily obtain the following results from [20].

Lemma 3.2. [20] *(1)* $y^*(t) = \dfrac{q\exp(-d(t-nT))}{1-(1-p_2)\exp(-dT)}$, $t \in (nT, (n+1)T], n \in \mathbb{N}$ *and* $y^*(0^+) = \dfrac{q}{1-(1-p_2)\exp(-dT)}$ *is the positive periodic solution of (3.3).*

(2) $y(t) = (1-p_2)^{n+1}\left(y(0^+) - \frac{q\exp(-dT)}{1-(1-p_2)\exp(-dT)}\right)\exp(-dt) + y^*(t)$ *is the general solution of (3.3) with* $y_0 \geq 0$, $t \in (nT, (n+1)T]$ *and* $n \in \mathbb{N}$.
(3) All solutions $y(t)$ *of (1.3) with* $y_0 \geq 0$ *tend to* $y^*(t)$. *i.e.,* $|y(t) - y^*(t)| \to 0$ *as* $t \to \infty$.

It is from Lemma 3.2 that the general solution $y(t)$ of (3.3) can be synchronized with the positive periodic solution $y^*(t)$ of (3.3) for sufficiently large t and we can obtain the complete expression for the prey-free periodic solution of (1.3)

$$(0, y^*(t)) = \left(0, \frac{q\exp(-d(t-nT))}{1-(1-p_2)\exp(-dT)}\right) \text{ for } t \in (nT, (n+1)T].$$

Now, we present a condition which guarantees the local stability of the prey-free periodic solution $(0, y^*(t))$.

Theorem 3.3. *The periodic solution* $(0, y^*(t))$ *is locally asymptotically stable if*

$$(a+\lambda)T + \ln(1-p_1) < \frac{cq(1-\exp(-DT))}{d(1-(1-p_2)\exp(-DT))}. \tag{3.6}$$

Proof. Consider the following impulsive differential equation:

$$\begin{cases} \left.\begin{array}{l} x_1'(t) = x_1(t)(a - bx_1(t) - cy_1(t) + \lambda), \\ y_1'(t) = y_1(t)(-d + ex_1(t)), \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} \Delta x_1(t) = -p_1 x_1(t), \\ \Delta y_1(t) = -p_2 y_1(t) + q, \end{array}\right\} t = nT, \\ (x_1(0^+), y_1(0^+)) = (x_0, y_0). \end{cases} \tag{3.7}$$

Let $y_1^*(t) = y^*(t)$. Then $(0, y_1^*(t))$ is a periodic solution of (3.7). The local stability of the periodic solution $(0, y_1^*(t))$ can be determined by considering the behavior of small amplitude perturbations of the solution. Define $x_1(t) = u(t)$ and $y_1(t) = y_1^*(t) + v(t)$. Then they may be written as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, 0 \leq t \leq T, \tag{3.8}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi}{dt} = \begin{pmatrix} a + \lambda - cy_1^*(t) & 0 \\ ey_1^*(t) & -d \end{pmatrix}\Phi(t) \tag{3.9}$$

and $\Phi(0) = I$, where I is the identity matrix. The linearization of the third and fourth equation of the model (1.3) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix} \begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}. \tag{3.10}$$

Note that all eigenvalues of $S = \begin{pmatrix} 1-p_1 & 0 \\ 0 & 1-p_2 \end{pmatrix} \Phi(T)$ are $\mu_1 = (1 - p_1)\exp(\int_0^T a + \lambda - cy_1^*(t)dt)$ and $\mu_2 = (1-p_2)\exp(-dT) < 1$. Since $y_1^*(t) = \dfrac{q\exp(-dt)}{1-(1-p_2)\exp(-dT)}$ if $0 < t \leq T$, we obtain $\int_0^T y_1^*(t)dt = \dfrac{q(1-\exp(-dT))}{d(1-(1-p_2)\exp(-dT))}$. Thus, the condition $|\mu_1| < 1$ is equivalent to the equation (3.6). According to Floquet theory of impulsive differential equation[4], $(0, y^*(t))$ is locally asymptotically stable. Thus we complete the proof. □

Remark 3.4. Define $F(T) = (a+\lambda)T + \ln(1-p_1) - \frac{cq(1-\exp(-dT))}{d(1-(1-p_2)\exp(-dT))}$. Since $F(0) = \ln(1-p_1) - \frac{cq}{dp_2} < 0$, $\lim_{T\to\infty} F(T) = \infty$, and

$$F''(T) = \frac{cdp_2 q\exp(dT)(\exp(dT) + (1-p_2))}{(\exp(dT) - (1-p_2))^3} > 0 \tag{3.11}$$

so, $F(T)$ has a unique positive solution $T_{\max}$. It follows form Theorem 3.3 that if $T < T_{\max}$, then the prey-free solution is locally asymptotically stable.

3.3. Permanence

In the subsection, we investigate the permanence of the system (1.3). We mention the following definition before stating the permanent theorem.

Definition 3.5. The system (1.3) is permanent if there exist $M \geq m > 0$ such that, for any solution $(x(t), y(t))$ of (1.3) with $x_0 > 0$ and $y_0 > 0$,

$$m \leq \lim_{t\to\infty} \inf x(t) \leq \lim_{t\to\infty} \sup x(t) \leq M \text{ and } m \leq \lim_{t\to\infty} \inf y(t) \leq \lim_{t\to\infty} \sup y(t) \leq M.$$

Now, we can obtain a condition that the prey and the predator can coexist.

Theorem 3.6. *The system (1.3) is permanent if*

$$(a-\lambda)T + \ln(1-p_1) > \frac{cq(1-\exp(-DT))}{d(1-(1-p_2)\exp(-DT))}. \tag{3.12}$$

Proof. Let $(x(t), y(t))$ be any solution of the system (1.3) with $x_0 > 0$ and $y_0 > 0$. Consider the system

$$(3.13)\qquad \begin{cases} \left.\begin{array}{l} x_2'(t) = x_2(t)(a - bx_2(t) - cy_2(t) - \lambda), \\ y_2'(t) = y_2(t)(-d + ex_2(t)), \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} \Delta x_2(t) = -p_1 x_2(t), \\ \Delta y_2(t) = -p_2 y_2(t) + q, \end{array}\right\} t = nT, \\ (x_2(0^+), y_2(0^+)) = (x_0, y_0). \end{cases}$$

From the comparison theorem[15], we have $x(t) \geq x_2(t)$ and $y(t) \geq y_2(t)$. From Lemma 3.1, we may assume that $x(t) \leq M$, $y(t) \leq M$, $t > 0$ and $M > \frac{a-\lambda}{c}$. Clearly, $y_2'(t) \geq -Dy_2(t)$ if $t \neq nT$. By the comparison theorem[15], we obtain $y_2(t) \geq \tilde{y}_2(t)$, where $\tilde{y}_2(t)$ is a solution of (3.3). It follows from Lemma 3.2 that $\tilde{y}_2(t) \to \tilde{y}_2^*(t)$ as $t \to \infty$, where $\tilde{y}_2^*(t) = y^*(t)$, which implies that $y_2(t) \geq \frac{q\exp(-dT)}{1-(1-p_2)\exp(-dT)} - \epsilon \equiv m_2$ for sufficiently small $\epsilon > 0$. Thus we have $y(t) \geq m_2$. Now we shall find an $m_1 > 0$ such that $x(t) \geq m_1$ for all t large enough. We will do this in the following two steps.

(Step 1) It follows from (3.12) that there exist $m_3 > 0$ and $\epsilon_1 > 0$ small enough such that $em_3 < d$ and $R =$

$$(1-p_1)\exp\Big((a-\lambda-bm_3-c\epsilon_1)T - \frac{cq(1-\exp((-d+em_3)T))}{(-d+em_3)(1-(1-p_2)\exp((-d+em_3)T))}\Big) > 1$$

Suppose that $x_2(t) < m_3$ for all t. Then we get $y_2'(t) \leq y_2(t)(-d + em_3)$ from the second equation in (3.13). By the comparison theorem [15] and Lemma 3.2, we have $y_2(t) \leq u(t)$ and $u(t) \to u^*(t)$, $t \to \infty$, where $u(t)$ is the solution of

$$(3.14)\qquad \begin{cases} u'(t) = (-d + em_3)u(t),\ t \neq nT, \\ u(t^+) = (1 - p_2)u(t) + q,\ t = nT \\ u(0^+) = y_0 \end{cases}$$

and $u^*(t) = \dfrac{q\exp((-d + em_3)(t - nT))}{1 - (1 - p_2)\exp((-d + em_3)T)}$, $t \in (nT, (n + 1)T]$ is the periodic solution of (3.14). Then there exists $T_1 > 0$ such that $y_2(t) \leq u(t) \leq u^*(t) + \epsilon_1$ and

$$\begin{aligned} x_2'(t) &= x_2(t)(a - bx_2(t) - cy_2(t) - \lambda) \\ &\geq x_2(t)(a - bm_3 - \lambda - c(u^*(t) + \epsilon_1)) \text{ for } t \geq T_1 \text{ and } t \neq nT. \end{aligned}$$

Let $N_1 \in \mathbb{N}$ and $N_1 T \geq T_1$. We have, for $n \geq N_1$

$$(3.15)\qquad \begin{cases} x_2'(t) & \geq x_2(t)(a - bm_3 - \lambda - c(u^*(t) + \epsilon_1)), t \neq nT, \\ x_2(t^+) & = (1 - p_1)x_2(t), t = nT. \end{cases}$$

Integrating (3.15) on $(nT, (n+1)T](n \geq N_1)$, we obtain

$$x_2((n+1)T) \geq x_2(nT^+)\exp\Big(\int_{nT}^{(n+1)T} a - bm_3 - c(u^*(t) + \epsilon_1) - \lambda dt\Big).$$

Since $\int_{nT}^{(n+1)T} u^*(t)dt = \int_0^T \frac{q\exp((-d+em_3)t)}{1-(1-p_2)\exp((-d+em_3)T)}dt$, we get $x_2((n+1)T) \geq x_2(nT)R$. Then $x_2((N_1+k)T) \geq x_2(N_1T)R^k \to \infty$ as $k \to \infty$ which contradicts the boundedness of $x(t)$. Hence there exists a $t_1 > 0$ such that $x_2(t_1) \geq m_3$.

(Step 2) If $x_2(t) \geq m_3$ for all $t \geq t_1$, then we are done. If not, we may let $t^* = \inf_{t>t_1}\{x_2(t) < m_3\}$. Then $x_2(t) \geq m_3$ for $t \in [t_1, t^*]$ and, by the continuity of $x_2(t)$, we have $x_2(t^*) = m_3$. In this step, we have only to consider two possible cases.

Case 1) $t^* = n_1 T$ for some $n_1 \in \mathbb{N}$. Then $(1-p_1)m_3 \leq x_2(t^{*+}) = (1-p_1)x_2(t^*) < m_3$. Select $n_2, n_3 \in \mathbb{N}$ such that $(n_2 - 1)T > \dfrac{\ln(\frac{\epsilon_1}{M+q})}{-d + em_3}$ and $(1-p_1)^{n_2}R^{n_3}\exp(n_2\sigma T) > (1-p_1)^{n_2}R^{n_3}\exp((n_2+1)\sigma T) > 1$, where $\sigma = a - bm_3 - cM - \lambda < 0$. Let $T' = n_2T + n_3T$. In this case we will show that there exists $t_2 \in (t^*, t^* + T']$ such that $x_2(t_2) \geq m_3$. Otherwise, by (3.14) with $u(t^{*+}) = y(t^{*+})$, we have $u(t) =$

$$(1-p_2)^{n_1+1}\left(u(t^{*+}) - \frac{q\exp((-d+em_3)T)}{1-(1-p_2)\exp((-d+em_3)T)}\right)\exp((-d+em_3)(t-t^*)) + u^*(t)$$

for $(n-1)T < t \leq nT$ and $n_1 + 1 \leq n \leq n_1 + 1 + n_2 + n_3$. So we get $|u(t) - u^*(t)| \leq (M+q)\exp((-d+em_3)(t-t^*)) < \epsilon_1$ and $y(t) \leq u(t) \leq u^*(t) + \epsilon_1$ for $t^* + n_2T \leq t \leq t^* + T'$. Also, we get to know that

$$(3.16)\qquad \begin{cases} x_2'(t) \geq x_2(t)\big(a - bm_3 - c(u^* + \epsilon_1) - \lambda\big), t \neq nT, \\ x_2(t^+) = (1-p_1)x_2(t), t = nT, \end{cases}$$

for $t \in [t^* + n_2T, t^* + T']$. As in step 1, we have

$$x_2(t^* + T') \geq x_2(t^* + n_2T)R^{n_3}.$$

Since $y(t) \leq M$, we have

$$(3.17)\qquad \begin{cases} x_2'(t) \geq x_2(t)\big(a - bm_3 - bM - \lambda\big) = \sigma x_2(t), t \neq nT, \\ x_2(t^+) = (1-p_1)x_2(t), t = nT, \end{cases}$$

for $t \in [t^*, t^* + n_2T]$. Integrating (3.17) on $[t^*, t^* + n_2T]$ we have

$$\begin{aligned} x_2((t^* + n_2T)) &\geq x_2(t^*)(1 - p_1)^{n_2} \exp(\sigma n_2 T) \\ &\geq m_3(1 - p_1)^{n_2} \exp(\sigma n_2 T) > m_3. \end{aligned}$$

Thus $x(t^*+T') \geq m_3(1-p_1)^{n_2} \exp(\sigma n_2 T) R^{n_3} > m_3$ which is a contradiction. Now, let $\bar{t} = \inf_{t>t^*}\{x_2(t) \geq m_3\}$. Then $x_2(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x_2(\bar{t}) = m_3$. For fixed $t \in [t^*, \bar{t})$, suppose $t \in (t^* + (k-1)T, t^* + kT]$, for some $k \in \mathbb{N}$ and $k \leq n_2 + n_3$. Thus, from (3.17), we obtain $x_2(t) \geq x_2(t^{*+})(1 - p_1)^{k-1} \exp((k-1)\sigma T) \exp(\sigma(t - (t^* + (k-1)T))) \geq m_3(1 - p_1)^k \exp(k\sigma T) \geq m_3(1 - p_1)^{n_2+n_3} \exp(\sigma(n_2 + n_3)T) \equiv m_1'$ if $t \in [t^*, \bar{t})$.

Case (2) $t^* \neq nT, n \in \mathbb{N}$. Then $x_2(t) \geq m_3$ for $t \in [t_1, t^*)$ and $x_2(t^*) = m_3$. Suppose that $t^* \in (n_1'T, (n_1' + 1)T)$ for some $n_1' \in \mathbb{N}$. There are two possible cases.

Case(2(a)) $x_2(t) < m_3$ for all $t \in (t^*, (n_1' + 1)T]$. In this case we will show that there exists $t_2 \in [(n_1' + 1)T, (n_1' + 1)T + T']$ such that $x_2(t_2) \geq m_3$. Suppose not. i.e., $x_2(t) < m_3$, for all $t \in [(n_1' + 1)T, (n_1' + 1 + n_2 + n_3)T]$. Then $x_2(t) < m_3$ for all $t \in (t^*, (n_1' + 1 + n_2 + n_3)T]$. By (3.14) with $u((n_1' + 1)T^+) = y((n_1' + 1)T^+)$, we have $u(t) =$

$$\left(u((n_1'+1)T^+) - \frac{q \exp(-d + em_3)}{1 - (1 - p_2) \exp(-d + em_3)}\right) \exp((-d+em_3)(t-(n_1'+1)T)) + u^*(t)$$

for $t \in (nT, (n+1)T], n_1' + 1 \leq n \leq n_1' + n_2 + n_3$. As in step 1, we obtain that

$$x_2((n_1' + 1 + n_2 + n_3)T) \geq x_2((n_1' + 1 + n_2)T) R^{n_3}.$$

It follows from (3.17) that

$$x_2((n_1' + 1 + n_2)T) \geq m_3(1 - p_1)^{n_2+1} \exp(\sigma(n_2 + 1)T).$$

Thus $\geq m_3(1 - p_1)^{n_2+1} \exp(\sigma(n_2 + 1)T) R^{n_3}$. By the choice of n_2 and n_3, we obtain $x_2((n_1' + 1 + n_2 + n_3)T) > m_3$ which contradicts the boundedness of $x(t)$. Now, let $\bar{t} = \inf_{t>t^*}\{x_2(t) \geq m_3\}$. Then $x_2(t) \leq m_3$ for $t^* \leq t < \bar{t}$ and $x_2(\bar{t}) = m_3$. For fixed $t \in [t^*, \bar{t})$, take $k' \in \mathbb{N}$ such that $k' \leq 1 + n_2 + n_2$ and $t \in (n_1'T + (k' + 1)T, n_1'T + k'T]$. Then we have $x_2(t) \geq m_3(1 - p_1)^{1+n_2+n_3} \exp(\sigma(1 + n_2 + n_3)T) \equiv m_1$. Since $m_1 < m_1'$, so $x_2(t) \geq m_1$ for $t \in (t^*, \bar{t})$.

Case (2(b)) There is a $t' \in (t^*, (n_1' + 1)T]$ such that $x_2(t') \geq m_3$. Let $\hat{t} = \inf_{t>t^*}\{x_2(t) \geq m_3\}$. Then $x_2(t) \leq m_3$ for $t \in [t^*, \hat{t})$ and $x_2(\hat{t}) = m_3$. Also, the equation (3.17) holds for $t \in [t^*, \hat{t})$. Integrating the equation (3.17) on $[t^*, t)(t^* \leq t \leq \hat{t})$, we obtain that $x_2(t) \geq x_2(t^*) \exp(\sigma(t - t^*)) \geq m_3 \exp(\sigma T) \geq m_1$. Thus in both case the similar argument can be continued since $x_2(t) \geq m_1$ for some $t > t_1$. This completes the proof. $\square$

It is easily seen from Theorem 3.6 that the system (1.2) can be also permanent.

Corollary 3.7. *The system (1.2) is permanent if $a > \lambda$.*

Remark 3.8. (1) Let $q^* = \frac{d((a-\lambda)T+\ln(1-p_1))(1-(1-p_2)\exp(-dT))}{c(1-\exp(-dT))}$ for a fixed T. From Theorem 3.3 and Theorem 3.6, we know that the prey-free periodic solution is locally asymptotically stable if $q > q^*$ and otherwise, the prey and predator can coexist. Thus q^* plays a role in a critical value that discriminates between stability and permanence. Moreover, it is from Theorem 3.3 and Theorem 3.6 that $T_{\max}$ functions as a threshold. (2) One of noteworthy phenomena is that the frequency ω of the seasonality does not affect the permanence and stability of the system (1.3).

References

[1] R. Arditi and L. R. Ginzburg, Coupling in predator-prey dynamics:Ratio-dependence, *J. Theor. Biol.*, **139**(1989), 311-326.

[2] D. K. Arrowsmith and C. M. Place, *Dynamical systems : Differential equations, maps and chaotic behaviour*, Chapman and Hall(1992).

[3] M. Bardi, Predator-prey models in periodically fluctuatin environment, *J. Math. Biol.*, **12**(1981), 127-140.

[4] D.D. Bainov and P.S. Simeonov, Impulsive Differential Equations:Periodic Solutions and Applications, vol. 66, of Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Science & Technical, Harlo, UK, 1993.

[5] F. Brauer and C. Castillo-Chávez, *Mathematical models in population biology and epidemiology*, Texts in applied mathematics 40, Springer-Verlag, New York(2001).

[6] L. Chen and Z. Jing, The existence and uniqueness of predator-prey differential equations, *Chin. Sci. Bull.*, **9**(1984),521-523.

[7] Z. Cheng, Y. Lin and J. Cao, Dynamical behaviors of a partial-dependent predator-prey system, *Chaos, Solitons and Fractals*, **28**(2006), 67-75.

[8] J. B. Collings, The effects of the functional response on the bifurcation behavior of a mite predator-prey interaction model, *J. Math. Biol.*, **36**(1997), 149-168.

[9] J. M. Cushing, Periodic time-dependent predator-prey systems, *SIAM J. Appl. Math.*, **32**(1977), 82-95.

[10] J. M. Cushing, Periodic Kolmogorov systems, *SIAM J. Math. Anal.* , **13**(1987), 811-827.

[11] H.I. Freedman, Deterministic Mathematical Models in Population Ecology, Marcel Dekker, New York (1980)

[12] S.-B. Hsu and T.-W. Huang, Global stability for a class of predator-prey systems, *SIAM J. Appl. Math.*, **55(3)**(1995), 763-783.

[13] G. Jiang, Q. Lu and L. Qian, Complex dynamics of a Holling type II prey-predator system with state feedback coontrol, *Chaos, Solitons and Fractals*, **31**(2007), 448-461.

[14] A. Lakmeche and O. Arino, Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment, *Dynamics of Continuous, Discrete and Impulsive Systems*, **7**(2000), 265-287.

[15] V Lakshmikantham, D. Bainov, P.Simeonov, Theory of Impulsive Differential Equations, World Scientific Publisher, Singapore, 1989.

[16] X. Liu and L. Chen, Complex dynamics of Holling type II Lotka-Volterra predator-prey system with impulsive perturbations on the predator, *Chaos, Solitons and Fractals*, **16**(2003), 311-320.

[17] B. Liu, Z. Teng and L. Chen, Analsis of a predator-prey model with Holling II functional response concerning impulsive control strategy, *J. of Comp. and Appl. Math.*, **193(1)**(2006), 347-362

[18] B. Liu, Z. Teng and W. Liu, Dynamic behaviors of the periodic Lotka-Volterra competing system with impulsive perturbations, *Chaos, Solitons and Fractals*, **31**(2007),356-370.

[19] B. Liu, Y. Zhang and L. Chen, Dynamic complexities in a Lotka-Volterra predator-prey model concerning impulsive control strategy, *Int. J. of Bifur. and Chaos*, **15(2)**(2005), 517-531.

[20] B. Liu, Y. Zhang and L. Chen, The dynamical behaviors of a Lotka-Volterra predator-prey model concerning integrated pest mangement, *Nonlinear Analysis:Real World Applications*, **6**(2005), 227-243.

[21] Z. Lu, X. Chi and L. Chen, Impulsive control strategies in biological control and pesticide, *Theoretical Population Biology*, **64**(2003), 39-47.

[22] R. May, Limit cycles in predator prey communities, *Science*, **177**(1972), 900-902.

[23] G. C. W. Sabin and D. Summers, Chaos in a periodically forced predator-prey ecosystem model, *Math. Bioscience*, **113**(1993), 91-113.

[24] E, Saez and E. Gonzalez-Olivares, Dynamics of a predator-prey model, *SIAM J. Appl. Math.*, **59(5)**(1999), 1867-1878.

[25] G. T. Skalski and J. F. Gilliam, Functional response with predator interference: Viable alternatives to the holling type II model, *Ecology*, **82(11**(2001), 3083-3092.

[26] J. Sugie, R. Kohno and R. Miyazaki, On a predator-prey system of Holling type, *Proceedings of the A.M.S.*, **125(7)**(1997), 2041-2050.

[27] S. Tang, Y. Xiao, L. Chen and R.A. Cheke, Integrated pest management models and their dynamical behaviour, *Bulletin of Math. Biol.*, **67**(2005), 115-135.

[28] H. Wang and W. Wang, The dynamical complexity of a Ivlev-type prey-predator system with impulsive effect, *Chaos, Solitons and Fractals(2007)*, doi:10.1016/j.chaos.2007.02.008.

[29] S. Zhang, L. Dong and L. Chen, The study of predator-prey system with defensive ability of prey and impulsive perturbations on the predator, *Chaos, Solitons and Fractals*, **23**(2005), 631-643.

[30] S. Zhang, D. Tan and L. Chen, Chaos in periodically forced Holling type II predator-prey system with impulsive perturbations, *Chaos, Solitons and Fractals*, **28**(2006), 367-376.

[31] S. Zhang, D. Tan and L. Chen, Chaos in periodically forced Holling type IV predator-prey system with impulsive perturbations, *Chaos, Solitons and Fractals*, **27**(2006), 980-990.

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