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GENERALIZED LOCAL COHOMOLOGY AND MATLIS DUALITY

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, $E := E_R(R/\mathfrak{m})$ and let I be an ideal of R. Let M and N be finitely generated R-modules. It is shown that $H_I^n(M, (H_I^n(N)^\vee)) \cong (M \otimes_R N)^\vee$ where $\text{grade}(I, N) = n = \text{cd}_i(I, N)$. We also show that for $n = \text{grade}(I, R)$, one has $\text{End}_R(H_I^n(P, R)^\vee) \cong \text{Ext}_R^n(H_I^n(P, R), P^*)^\vee$.

1. Introduction

Let R be commutative Noetherian ring with nonzero identity and let I be an ideal of R. Recall that for an R-module N,

$$H_I^0(N) = \{x \in N \mid I^n x = 0 \text{ for some natural number } n\},$$

and that the right derived functors of $H_I^0(-)$ are the local cohomology functors $H_I^i(N)$, for all $i \in \mathbb{Z}$.

There is a generalization of this functor which introduced by J. Herzog in [10] in the local case and Bijan-Zadeh in [2] in the non-local as

$$H_I^i(M, N) = \lim_{k \geqslant 0} \text{Ext}_R^i(M/I^k M, N).$$

We are putting in mind that the cohomological dimension of an R-module N with respect to I is defined as

$$\text{cd}(I, N) = \max\{i \in \mathbb{Z} \mid H_I^i(N) \neq 0\}$$

A sequence $x_1, \ldots, x_k$ of elements of I is said to be an I-filter regular sequence on N if

$$\text{Supp}_R\left(\frac{(x_1, \ldots, x_i)N :_N x_i}{(x_1, \ldots, x_i)N}\right) \subset V(I)$$

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for all $i = 1, \cdots, k$, where $V(I)$ denotes the set of all prime ideals of R containing I. This concept is a generalization of the one of the filter regular sequence which has been studied in [24] and [25], and has led to some interesting results. Note that both concepts coincide if I is the maximal ideal of a local ring. Also, note that $x_1, \ldots, x_k$ is a weak sequence on N if and only if it is an R-filter regular sequence on N. It is easy to see that the analogue of [25, Appendix 2(ii)] holds for the ideal I whenever R is Noetherian and N is finitely generated; so that, if $x_1, \ldots, x_k$ is an I-filter regular sequence on N, then there is an element $x \in I$ such that $x_1, \ldots, x_k, x$ is an I-filter regular sequence on N. Thus, for a positive integer n, there exists an I-filter regular sequence on N of length n.

Let $(R, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$ and let $E := E_R(R/\mathfrak{m})$ denotes the injective envelope of the field $R/\mathfrak{m}$. We will use the notation $(-)^\vee$ for Matlis dual functor $Hom_R(-, E)$.

Recently there were some works on the module $(H_I^i(R))^\vee$, see for example [8], [9], [7], [5] and [6].

In [6], the author has proved that

Theorem 1.1. *Let (R,m) be a Noetherian local complete Cohen-Macaulay ring with coefficient field k and $x_1, \cdots, x_i \in R$ $(i \geq 1)$ a regular sequence in R. Set $I := (x_1, \cdots, x_i)R$ (I is a set-theoretic complete intersection ideal of R). Then $H_I^i(H_I^i(R))^\vee = E_R(R/\mathfrak{m})$.*

Khashyarmanesh in [12] has proved that $H_I^n(H_I^n(M)^\vee) \cong M^\vee$, where $n = \mathrm{grade}_M I = \mathrm{cd}(I, M) \geq 1$ and $IM \neq M$.

We use the following lemma (see Lemma2.3)

Lemma 1.2. *Let $(R, \mathfrak{m})$ be a local ring with an ideal I. Let $k > \mathrm{cd}(I, N)$ be an integer. Let $x_1, \ldots, x_k$ be an I-filter regular sequence on N. Then*

$$H_I^n(M, H^k_{(x_1,\ldots,x_k)}(N)^\vee) = 0$$

for all $n \in \mathbb{Z}$.

as we extend the main result of [12] to the generalized local cohomology modules (see Theorem 2.5).

Theorem 1.3. *Let $IN \neq N$ and $n \geq 1$ be as $\mathrm{grade}(I, N) = n = \mathrm{cd}_i(I, N)$. Then*

$$H_I^n(M, (H_I^n(N)^\vee)) \cong (M \otimes_R N)^\vee$$

Khashyarmanesh in [11] has showed that $\mathrm{End}_R(H_I^n(R)) \cong \mathrm{Ext}_R^n(H_I^n(R), R)$, where n is the grade of a proper ideal I of R. We extend this result to the following.

Theorem 1.4. *Let P be a finitely projective R-module. Then*

$$\operatorname{End}_R(H_I^n(P,R)^\vee) \cong \operatorname{Ext}_R^n(H_I^n(P,R),P^*)^\vee,$$

where $n = \operatorname{grade}(I,R)$ and $P^* = \operatorname{Hom}_R(P,R)$.

2. Main result

Throughout this notes we suppose that $(R,\mathfrak{m})$ is a commutative Noetherian local ring and I is an ideal of R. Let also, M and N be finitely generated R-modules. Moreover, we use $\mathbb{N}$ and $\mathbb{N}_0$ to denote the sets of positive and non-negative integers, respectively.

Proposition 2.1. *(See [13] and [22]) Suppose that $x_1,\dots,x_k$ is an I-filter regular sequence on N. Then $H_I^i(N) \cong H^i_{(x_1,\dots,x_k)}(N)$, for all $0 \le i < k$, and $H_I^i(N) \cong H_I^{i-k}(H^k_{(x_1,\dots,x_k)}(N))$ if $k \le i$.*

Remark 2.2. Let $x_1,\dots,x_k,x_{k+1}$ be an I-filter regular sequence on N. We set $W_0 := N$ and for all $i = 1,\dots,k$, $W_i := H^i_{(x_1,\dots,x_i)(N)}$. Then in the light of Proposition 2.1, one has

$$H^0_{(x_{k+1})}(W_i)) \cong H^0_{(x_1,\dots,x_{i+1})}(W_i) \cong H_I^i(N)$$

and

$$H^1_{(x_{i+1})}(W_1) \cong H^{i+1}_{(x_1,\dots,x_{i+1})}(N) = W_{i+1}$$

Now the following sequence is exact by [17, Remark 2.2.17]:

$$0 \longrightarrow H_I^i(N) \longrightarrow W_i \longrightarrow (W_i)_{x_{i+1}} \longrightarrow W_{i+1} \longrightarrow 0 \quad (*)$$

Theorem 2.3. *Let $(R,\mathfrak{m})$ be a local ring with an ideal I. Let $k > \operatorname{cd}(I,N)$ be an integer. Let $x_1,\dots,x_k$ be an I-filter regular sequence on N. Then*

$$H_I^n(M, H^k_{(x_1,\dots,x_k)}(N)^\vee) = 0$$

for all $n \in \mathbb{Z}$.

Proof. Consider the exact sequence

$$0 \longrightarrow H^k_{(x_1,\dots,x_k)}(N) \longrightarrow (H^k_{(x_1,\dots,x_k)}(N))_{x_{k+1}} \longrightarrow H^{k+1}_{(x_1,\dots,x_{k+1})}(N) \longrightarrow 0$$

So, we have the exact sequence

$$0 \longrightarrow (H^{k+1}_{(x_1,\dots,x_{k+1})}(N))^\vee \longrightarrow ((H^k_{(x_1,\dots,x_k)}(N))_{x_{k+1}})^\vee \longrightarrow (H^k_{(x_1,\dots,x_k)}(N))^\vee \longrightarrow 0$$

Now, for all $n \in \mathbb{Z}$, we conclude that the following long exact sequence of R-modules. ... $\longrightarrow H_I^n(M,((H^k_{(x_1,\dots,x_k)}(N))_{x_{k+1}})^\vee) \longrightarrow H_I^n(M,(H^k_{(x_1,\dots,x_k)}(N))^\vee) \longrightarrow$

$$H_I^{n+1}(M,(H^{k+1}_{(x_1,\dots,x_{k+1})}(N))^\vee) \longrightarrow H_I^{n+1}(M,(H^{k+1}_{(x_1,\dots,x_{k+1})}(N))^\vee) \longrightarrow \dots$$

The multiplication by x_{k+1} introduces an automorphism on $(H^k_{(x_1,\dots,x_k)}(N))_{x_{k+1}}$ and for all $\alpha \in H^n_I(M,((H^k_{(x_1,\dots,x_k)}(N))_{x_{k+1}})^\vee)$, there is an integer s such that $I^s\alpha = 0$. This implies that $H^n_I(M,((H^k_{(x_1,\dots,x_k)}(N))_{x_{k+1}})^\vee) = 0$. Hence,

$$H^n_I(M,(H^k_{(x_1,\dots,x_k)}(N))^\vee) \cong H^{n+1}_I(M,(H^{k+1}_{(x_1,\dots,x_{k+1})}(N))^\vee)$$

But There is an integer m such that $H^{n+m}_{(x_1,\dots,x_{n+m})}(N) = 0$. So, the recent equivalence implies that $H^n_I(M,(H^k_{(x_1,\dots,x_k)}(N))^\vee) = 0$ □

We need to define the ith cohomological dimension $\mathrm{cd}_i(I,N)$ with respect to an I-filter regular sequence of length i.

Definition 2.4. Let N be an R-module and I be an ideal of R. Then $\mathrm{cd}_i(I,N)$ denotes the maximum positive integer i such that there exists an I-filter regular sequence $x_1,\dots,x_i$ where $H^i_{(x_1,\dots,x_i)}(N) \neq 0$.

Theorem 2.5. *Let $(R,\mathfrak{m})$ be a local ring and I be an ideal of R. Let $IN \neq N$ and $n \geq 1$ be as* $\mathrm{grade}(I,N) = n = \mathrm{cd}_i(I,N)$. *Then*

$$H^n_I(M,(H^n_I(N)^\vee)) \cong (M \otimes_R N)^\vee$$

Proof. Let $x_1,\dots,x_{n+1}$ be an I-filter regular sequence which $x_1,\dots,x_n$ is an M-sequence. For all positive integer k, set $W_k := H^k_{(x_1,\dots,x_k)}(N)$. By Remark 2.2, the following sequence is exact.

$$0 \longrightarrow H^n_I(N) \longrightarrow W_n \xrightarrow{f} (W_n)_{(x_{n+1})} \longrightarrow 0$$

Applying $Hom_R(-,E)$, we obtain the following exact sequence,

$$0 \longrightarrow ((W_n)_{x_{n+1}})^\vee \longrightarrow (W_n)^\vee \longrightarrow (H^i_I(N))^\vee \longrightarrow 0,$$

Hence, the exact sequence

$$\begin{aligned} H^k_I(M,((W_n)_{x_{n+1}})^\vee) \longrightarrow\ & H^k_I(M,(W_n)^\vee) \longrightarrow \\ & H^k_I(M,(H^n_I(N))^\vee) \longrightarrow H^{k+1}_I(M,((W_n)_{x_{n+1}})^\vee) \end{aligned}$$

implies that

$$H^k_I(M,(W_n)^\vee) \cong H^k_I(M,(H^n_I(N))^\vee). \tag{1}$$

Assume that $1 \leq i \leq n-1$. Then one can has the exact sequence

$$0 \longrightarrow (W_{i+1})^\vee \longrightarrow ((W_i)_{x_{i+1}})^\vee \longrightarrow (W_i)^\vee \longrightarrow 0$$

So,

$$H^k_I(M,(W_i)^\vee) \cong H^{k+1}_I(M,(W_{i+1})^\vee),$$

and by continuing this method, one has the following isomorphism.

$$H^n_I(M,(H^n_{(x_1,\ldots,x_n)}(N))^\vee)\cong H^1_I(M,(H^1_{x_1})^\vee)\cong Hom_R(M\otimes_R N,E)$$

Now the result obtains from (1). □

Corollary 2.6. *With notations used theorem 2.5, the followings are true.*
(i) $H^k_I(M,(H^n_I(R))^\vee)\cong M^\vee$,
(ii) $H^k_I(R,(H^n_I(N))^\vee)\cong N^\vee$,
(iii) $H^k_I(R,(H^n_I(R))^\vee)\cong E$.

Lemma 2.7. (See [1, Corollary 3.4]) *Let P be a finitely generated projective R-module. Then for all $i\geqslant 0$ and all R-modules N there is an isomorphism*

$$H^i_I(P,N)\cong \operatorname{Hom}_R(P,H^i_I(N))\cong P^*\otimes H^i_I(N)$$

Theorem 2.8. *Let P be a finitely projective R-module. Then*

$$\operatorname{End}_R(H^n_I(P,R))\cong \operatorname{Ext}^n_R(H^n_I(P,R),P^*),$$

where $n:=\operatorname{grade}(I,R)$ and $P^*=\operatorname{Hom}_R(P,R)$.

Proof. By the Lemma 2.7 there is the following isomorphism.

$$\operatorname{End}_R(H^n_I(P,R))\cong \operatorname{Hom}_R(H^n_I(R),\operatorname{Hom}_R(P^*,P^*\otimes H^n_I(R)))$$

On the other hand, by [3, §4, no. 2],

$$\begin{aligned}\operatorname{Hom}_R(P^*,P^*\otimes H^n_I(R)) &\cong \operatorname{Hom}_R(P^*,\operatorname{Hom}_R(P,H^n_I(R)))\\ &\cong \operatorname{Hom}_R(P^*\otimes P,H^n_I(R)).\end{aligned}$$

So, it implies that

$$\operatorname{End}_R(H^n_I(P,R))\cong \operatorname{Hom}_R(\operatorname{End}_R(P),\operatorname{End}_R(H^n_I(R))).$$

Now, [11, Theorem 2.6], in the light of Lemma 2.7 implies that

$$\begin{aligned}\operatorname{End}_R(H^n_I(P,R)) &\cong \operatorname{Hom}(\operatorname{End}(P),\operatorname{Ext}^n(H^n_I(R),R))\\ &\cong \operatorname{Ext}^n(\operatorname{End}(P)\otimes H^n_I(R),R)\\ &\cong \operatorname{Ext}^n(\operatorname{Hom}(P,H^n_I(R)),\operatorname{Hom}(P,R))\\ &\cong \operatorname{Ext}^n(H^n_I(P,R),P^*).\end{aligned}$$

□

Corollary 2.9. *Let P be finitely generated projective R-module and let $n:=\operatorname{grade}(I,R)$. Then*

$$\operatorname{End}_R(H^n_I(P,R)^\vee)\cong \operatorname{Ext}^n_R(H^n_I(P,R),P^*)^\vee$$

Proof. We note that

$$\begin{aligned}\mathrm{End}(H_i^n(P,R)^\vee) &\cong \mathrm{Hom}(H_I^n(P,R)\otimes(H_I^n(P,R))^\vee, E)\\ &\cong \mathrm{End}_R(H_I^n(P,R))^\vee\\ &\cong \mathrm{Ext}_R^n(H_I^n(P,R),P^*)^\vee.\end{aligned}$$

□

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