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p-PRECONVEX SETS ON PRECONVEXITY SPACES

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Abstract. In this paper, we introduce the concept of p-preconvex sets on preconvexity spaces. We study some properties for p-preconvex sets by using the co-convexity hull and the convexity hull. Also we introduce and study the concepts of pc-convex function, p^*c-convex function, pI-convex function and p^*I-convex function.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a nonempty set X and investigated some properties. He showed that a preconvexity on a nonempty set yields a convexity space in the same manner as a proximity [6] yields a topological space. In [3], we introduced the concepts of co-convexity hull and co-convex sets on preconvexity spaces. And we characterized c-convex functions and c-concave functions by using the co-convexity hull and the convexity hull.

Semi-preconvex sets, sc-convex functions and s^*c-convex functions are introduced in [4]. In [5], we introduced the β-preconvex set on a preconvexity space and studied some properties. And we introduced the concepts of βc-convex functions and $\beta^* c$-convex functions which are defined by the β-preconvex sets.

In this paper, we introduce the concept of p-preconvex set on a preconvexity space and study some basic properties. And we introduce and study the concepts of pc-convex functions, p^*c-convex functions, pc-convex functions and p^*c-convex functions which are defined by the p-preconvex sets. In particular, for two preconvexity spaces $(X, \sigma), (Y, \mu)$, (a) if a function $f : (X, \sigma) \to (Y, \mu)$ is c-concave and pc-convex, then f is p^*c-convex;

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(b) if a function $f : (X, \sigma) \to (Y, \mu)$ is c-convext and pI-convex, then f is p^*I-convex.

2. Preliminaries

Definition 2.1 ([1]). Let X be a nonempty set. A binary relation σ on $P(X)$ is called a preconvexity on X if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:

1. If $A \subset B$, then $A\sigma B$.
2. If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.
3. If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.
4. If $A\sigma B$ and $x \in A$, then $x\sigma B$.

The pair (X, σ) is called a preconvexity space. Let (X, σ) be a preconvexity space and $A \subset X$. $G(A) = \{x \in X : x\sigma A\}$ is called the convexity hull of a subset A. A is called convex [1] if $G_\sigma(A) = A$ (simply, $G(A)$).

$I_\sigma(A) = \{x \in A : x \not\sigma (X - A)\}$ (simply, $I(A)$) is called the co-convexity hull [3] of a subset A. And A is called a co-convex set if $I(A) = A$ [3]. Let $\mathcal{I}(X) = \{A \subset X : I(A) = A\}$ and $\mathcal{G}(X) = \{A \subset X : G(A) = A\}$.

Theorem 2.2 ([1], [3]). *For a preconvexity space (X, σ),*

1. $G(\emptyset) = \emptyset$, $I(X) = X$.
2. $A \subset G(A)$, $I(A) \subset A$ *for all* $A \subset X$.
3. *If* $A \subset B$, *then* $G(A) \subset G(B)$, $I(A) \subset I(B)$.
4. $G(G(A)) = G(A)$, $I(I(A)) = I(A)$ *for* $A \subset X$.
5. $I(A) = X - G(X - A)$ *and* $G(A) = X - I(X - A)$.

Theorem 2.3 ([1], [3]). *Let σ be a preconvexity on X and $A, B \subset X$. Then*

1. $A\sigma B$ *iff* $A \subset G(B)$ *iff* $I(X - B) \subset X - A$.
2. $A\sigma B$ *iff* $G(A)\sigma G(B)$ *iff* $I(X - B)\sigma I(X - A)$.

Definition 2.4 ([4]). Let (X, σ) be a preconvexity space and $A \subset X$. A is called a semi-preconvex set if $A\sigma I(A)$. And A is called a cosemi-preconvex set if the complement of A is a semi-preconvex set.

Let $\mathcal{S}_\sigma(X)$(resp., $\mathcal{SC}_\sigma(X)$) denote the set of all semi-preconvex sets(resp., cosemi-preconvex sets) in a preconvexity space (X, σ).

Definition 2.5 ([5]). Let (X, σ) be a preconvexity space and $A \subset X$. A is called a β-preconvex set if $A\sigma I(G(A))$. And A is called a coβ-preconvex set if the complement of A is a β-preconvex set.

We recall that the notions of c-convex function and c-concave function: Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be c-concave [2] if for $C, D \subset Y$ whenever $C\mu D$, $f^{-1}(C)\sigma f^{-1}(D)$. A function $f : X \to Y$ is said to be c-convex [1] if $A\sigma B$ implies $f(A)\mu f(B)$. And f is c-convex iff for each $U \in \mathcal{I}(Y)$, $f^{-1}(U) \in \mathcal{I}(X)$ [3].

3. p-preconvex sets

Definition 3.1. Let (X, σ) be a preconvexity space and $A \subset X$. A is called a p-preconvex set if $A \subset I(G(A))$. And A is called a co$p$-preconvex set if the complement of A is a p-preconvex set.

Let $\mathcal{P}_\sigma(X)$ (resp., $\mathcal{PC}_\sigma(X)$) denote the set of all p-preconvex sets (resp., co$p$-preconvex sets) in a preconvexity space (X, σ).

Now we get the following implications but the converses are not true in general as shown in the next example:

semi-preconvex
$\nearrow$ $\searrow$
co-convex $\qquad$ β-preconvex
$\searrow$ $\nearrow$
p-preconvex

Example 3.2. (1) Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b, c\}\}$. Define $A\sigma B$ to mean $A \subset cl(B)$, the closure of B in X. Then σ is a preconvexity on X. In the preconvexity space (X, σ), $\mathcal{G}(X) = \{\emptyset, X, \{a\}\}$, $\mathcal{I}(X) = \{\emptyset, X, \{b, c\}\}$, $\mathcal{P}_\sigma(X) = \beta_\sigma(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $\mathcal{S}_\sigma(X) = \{\emptyset, X, \{b, c\}\}$. Hence we know that a p-convex set $\{a, b\}$ is neither co-convex nor semi-convex.

(2) Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}\}$. Define $A\sigma B$ to mean $A \subset cl(B)$, the closure of B in X. Then σ is a preconvexity on X. In the preconvexity space (X, σ), $\mathcal{G}(X) = \{\emptyset, X, \{b, c, d\}, \{a, b, c\}, \{b, c\}\}$, $\mathcal{I}(X) = \tau$. Then $\{a, b\}$ is semi-preconvex and β-preconvex but not preconvex.

From Theorem 2.2, we get the following:

Theorem 3.3. *Let (X, σ) be a preconvexity space and $A \subset X$. Then A is a cop-preconvex set if and only if $G(I(A)) \subset A$.*

Theorem 3.4. *Every p-preconvex set is a β-preconvex set in a preconvexity space (X, σ).*

Proof. Let A be a p-preconvex set; then by definition of p-preconvex sets, $A \subset I(G(A))$. By Definition 3.1 (1), $A\sigma I(G(A))$. □

Corollary 3.5. Every co$p$-preconvex set is coβ-preconvex in a preconvexity space (X, σ).

Proof. Obvious. □

Theorem 3.6. *In a preconvexity space (X, σ), X and $\emptyset$ are both p-preconvex and cop-preconvex.*

Proof. By Theorem 2.2, it is obvious. □

Theorem 3.7. *In a preconvexity space (X, σ), the arbitrary union of p-preconvex sets is a p-preconvex set.*

Proof. Let $\mathfrak{F} = \{A_\alpha : A_\alpha \in \mathcal{P}_\sigma(X)\}$ be any subfamily of $\mathcal{P}_\sigma(X)$ and $x \in \cup\mathfrak{F}$. Then there exists a p-preconvex set A_α containing x such that $x \in A_\alpha \subset I(G(A_\alpha))$. And from Theorem 2.2 and $A_\alpha \subset \cup\mathfrak{F}$, it follows $I(G(A_\alpha)) \subset I(G(\cup\mathfrak{F}))$ and so $x \in I(G(\cup\mathfrak{F}))$. Hence, $\cup\mathfrak{F} \subset I(G(\cup\mathfrak{F}))$. □

Theorem 3.8. *In a preconvexity space (X, σ), the arbitrary intersection of cop-preconvex sets is a cop-preconvex set.*

Proof. From Theorem 2.2 and Theorem 3.7, it is obvious. □

Definition 3.9. Let (X, σ) be a preconvexity space and $A \subset X$.

1. $pG(A) = \cap\{F : A \subset F, F^c \in \mathcal{P}_\sigma(X)\}$.
2. $pI(A) = \cup\{U : U \subset A, U \in \mathcal{P}_\sigma(X)\}$.

Theorem 3.10. *Let (X, σ) be a preconvexity space and $A, B \subset X$.*

1. $I(A) \subset pI(A) \subset A$.
2. $A \subset pG(A) \subset G(A)$.
3. *A is p-preconvex iff $A = pI(X)$.*
4. *A is cop-preconvex iff $A = pG(X)$.*

Proof. (1) and (2) are obvious from Theorem 3.4 and Corollary 3.5.
(3) It is obtained from Theorem 3.7.
(4) It is obtained from Theorem 3.8. □

Theorem 3.11. *Let (X, σ) be a preconvexity space and $A, B \subset X$.*

1. $pI(X) = X$.
2. $pI(A) \subset A$.

3. *If* $A \subset B$, *then* $pI(A) \subset pI(B)$.
4. $pI(pI(A)) = pI(A)$.

Proof. (1), (2) and (3) are obvious.

(4) Since $pI(A) \subset A$, by (3), $pI(pI(A)) \subset pI(A)$.

For the converse, let $x \in pI(A)$; then since $x \in pI(A) \subset pI(A)$ and $pI(A)$ is a p-preconvex set, we get $x \in pI(pI(A))$ by Definition 3.9. □

Theorem 3.12. *Let* (X, σ) *be a preconvexity space and* $A, B \subset X$.

1. $pG(\emptyset) = \emptyset$.
2. $A \subset pG(A)$.
3. *If* $A \subset B$, *then* $pG(A) \subset pG(B)$.
4. $pG(pG(A)) = pG(A)$.

Proof. It is similar to the proof of Theorem 3.11. □

4. pc-convex functions and pI-convex functions

Definition 4.1. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be pc-convex if for each $A \in \mathcal{I}(Y)$, $f^{-1}(A) \in \mathcal{P}_\sigma(X)$.

Every pc-convex function is βc-convex but the converse is not always true as follows:

Example 4.2. In Example 3.2 (2), consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as follows: $f(a) = f(b) = a$, $f(d) = b$ and $f(c) = c$. Then f is βc-convex but not pc-convex because for co-convex set $\{a\}$, $f^{-1}(\{a\}) = \{a, b\}$ is not p-preconvex.

Theorem 4.3. *Let* (X, σ) *and* (Y, μ) *be two preconvexity spaces and* $f : X \to Y$ *a function. Then the following things are equivalent:*

1. f *is pc-convex.*
2. $f^{-1}(I(B)) \subset I(G(f^{-1}(B)))$ *for all* $B \subset Y$.
3. $G(I(f^{-1}(B))) \subset f^{-1}(G(B))$ *for all* $B \subset Y$.
4. $f(G(I(A))) \subset G(f(A))$ *for all* $A \subset X$.
5. *For each* $U \in \mathcal{G}(Y)$, $f^{-1}(U) \in \mathcal{PC}_\sigma(X)$.

Proof. (1) ⇒ (2) Suppose that f is pc-convex and let $A \subset Y$; then since $I(A) \subset A$, by Theorem 2.2, we get $I(G(f^{-1}(I(A)))) \subset I(G(f^{-1}(A)))$. Since $I(A) \in \mathcal{I}(Y)$ and f is pc-convex, $f^{-1}(I(A)) \subset I(G(f^{-1}(I(A))))$. Hence, we have $f^{-1}(I(A)) \subset I(G(f^{-1}(A)))$.

(2) $\Rightarrow$ (3) Let $B \subset Y$; then by (2), we have
$X - f^{-1}(G(B)) = f^{-1}(I(Y - B)) \subset I(G(f^{-1}(Y - B))) = X - G(I(f^{-1}(B)))$. Hence (3) is obtained.

(3) $\Leftrightarrow$ (4) It is obvious.

(3) $\Rightarrow$ (5) It is obvious.

(5) $\Rightarrow$ (1) It is obvious by Theorem 2.2.

□

From Theorem 3.10 and Theorem 4.3, we get the following:

Theorem 4.4. *Let (X, σ) and (Y, μ) be two preconvexity spaces and $f : X \to Y$ a function. Then the following things are equivalent:*

1. *f is pc-convex.*
2. *$f^{-1}(I(B)) \subset pI(f^{-1}(B))$ for all $B \subset Y$.*
3. *$pG(f^{-1}(B)) \subset f^{-1}(G(B))$ for all $B \subset Y$.*
4. *$f(pG(A)) \subset G(f(A))$ for all $A \subset X$.*

Definition 4.5. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be p^*c-convex if for each $A \in \mathcal{P}_\mu(Y)$, $f^{-1}(A) \in \mathcal{P}_\sigma(X)$.

Every p^*c-convex function is *pc*-convex but the converse is not always true as shown in the next example:

Example 4.6. In Example 3.2 (1), consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as follows: $f(a) = b, f(b) = c$ and $f(c) = a$. Then f is *pc*-convex but not p^*c-convex because $f^{-1}(\{b\}) = \{a\}$ is not *p*-preconvex for a *p*-preconvex set $\{b\}$.

We have the following:

$$
\begin{array}{ccccc}
 & & \beta c\text{-convex} & \Leftarrow & \beta^* c\text{-convex} \\
 & & \Uparrow & & \Uparrow \\
c\text{-convex} & \Rightarrow & pc\text{-convex} & \Leftarrow & p^* c\text{-convex}
\end{array}
$$

Theorem 4.7. *Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is p^*c-convex iff for $A \subset Y$ whenever $A \subset I(G(A))$, $f^{-1}(A) \subset I(G(f^{-1}(A)))$.*

Proof. From Definition 4.5, it is obvious. □

Theorem 4.8 ([3]). *Let (X, σ) and (Y, μ) be two preconvexity spaces and $f : X \to Y$ a function. Then the following things are equivalent:*

1. f *is c-concave.*
2. $f^{-1}(G(A)) \subset G(f^{-1}(A))$ *for all* $A \subset Y$.
3. $I(f^{-1}(A)) \subset f^{-1}(I(A))$ *for all* $A \subset Y$.

Theorem 4.9 ([3]). *Let* (X, σ) *and* (Y, μ) *be two preconvexity spaces and* $f : X \to Y$ *a function. Then the following things are equivalent:*

1. f *is c-convex.*
2. $f(G(A)) \subset G(f(A))$ *for all* $A \subset X$.
3. $G(f^{-1}(B)) \subset f^{-1}(G(B))$ *for all* $B \subset Y$.
4. $f^{-1}(I(B)) \subset I(f^{-1}(B))$ *for all* $B \subset Y$.
5. *For each* $U \in \mathcal{I}(Y)$, $f^{-1}(U) \in \mathcal{I}(X)$.
6. *For each* $C \in \mathcal{G}(Y)$, $f^{-1}(U) \in \mathcal{G}(X)$.

Lemma 4.10. *Let* $f : (X, \sigma) \to (Y, \mu)$ *be a function on two preconvexity spaces. If f is c-convex and c-concave, then we have the following:*

1. $f^{-1}(I(B)) = I(f^{-1}(B))$ *for all* $B \subset Y$.
2. $G(f^{-1}(B)) = f^{-1}(G(B))$ *for all* $B \subset Y$.

Proof. From the above Theorem 4.8 and Theorem 4.9, the results are obtained. □

Theorem 4.11. *Let* $f : (X, \sigma) \to (Y, \mu)$ *be a function on two preconvexity spaces. If f is c-concave and c-convex, then f is* p^*c*-convex.*

Proof. Let $A \in \mathcal{P}_\mu(Y)$; then $A \subset I(G(A))$. Since f is c-concave and c-convex, from Lemma 4.10, it follows

$$f^{-1}(A) \subset f^{-1}(I(G(A))) = I(G(f^{-1}(A))).$$

Hence by Theorem 4.7, f is p^*c-convex. □

Corollary 4.12. Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. If f is c-concave and c-convex, then f is pc-convex.

Proof. Since every p^*c-convex function is p-convex, f is pI-convex. □

Theorem 4.13. *Let* $f : (X, \sigma) \to (Y, \mu)$ *be a function on two preconvexity spaces. If f is c-concave and pc-convex, then f is* p^*c*-convex.*

Proof. Suppose f is c-concave and pc-convex and let $A \in \mathcal{P}_\mu(Y)$; then $A \subset I(G(A))$. From Theorem 4.3 (2) and Theorem 4.8 (2), it follows $f^{-1}(A) \subset f^{-1}(I(G(A))) \subset I(G(f^{-1}(G(A)))) \subset I(G(G(f^{-1}(A)))) \subset I(G(f^{-1}(A)))$. Hence by Theorem 4.7, f is p^*c-convex. □

Theorem 4.14. *Let* (X, σ) *and* (Y, μ) *be two preconvexity spaces and* $f : X \to Y$ *a function. Then the following things are equivalent:*

1. *f is p^*c-convex.*
2. *For each $U \in \mathcal{PC}(Y)$, $f^{-1}(U) \in \mathcal{PC}(X)$.*
3. *$f(pG(A)) \subset pG(f(A))$ for all $A \subset X$.*
4. *$pG(f^{-1}(B)) \subset f^{-1}(pG(B))$ for all $B \subset Y$.*
5. *$f^{-1}(pI(B)) \subset pI(f^{-1}(B))$ for all $B \subset Y$.*

Proof. Obvious. □

Definition 4.15. Let (X, σ) and (Y, μ) be two preconvexity spaces and $f : X \to Y$ a function. Then

1. f is said to be pI-convex if for each $U \in \mathcal{I}(X)$, $f(U) \in \mathcal{P}(Y)$.
2. f is is said to be p^*I-convex if for each $U \in \mathcal{P}(X)$, $f(U) \in \mathcal{P}(Y)$.

Example 4.16. In Example 4.6, the function f is pI-convex but not I-convex. And f is not p^*I-convex because $f(\{c\}) = \{a\}$ is not p-convex for a p-convex set $\{c\}$ in X.

$$I\text{-convex} \Rightarrow pI\text{-convex} \Leftarrow p^*I\text{-convex}$$

Theorem 4.17. *Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. Then f is pI-convex iff $f(I(A)) \subset I(G(f(A)))$ for all $A \subset X$.*

Proof. Let f be pI-convex; then since $I(A) \in \mathcal{I}(X)$, $f(I(A)) \subset I(G(f(I(A)))) \subset I(G(f(A)))$.

Suppose that $f(I(A)) \subset I(G(f(A)))$ for all $A \subset X$. Since $U \in \mathcal{I}(X)$ iff $I(U) = U$, we have f is pI-convex. □

Theorem 4.18. *Let (X, σ) and (Y, μ) be two preconvexity spaces and $f : X \to Y$ a function. Then f is c-concave iff $f(I(U)) \subset I(f(U))$ for all $U \subset X$.*

Proof. Let U be a subset of X; then from Theorem 4.8, it follows

$$I(U) \subset I(f^{-1}(f(U)) \subset f^{-1}(I(f(U))).$$

Hence $f(I(U)) \subset I(f(U))$.

Similarly, we have the converse. □

Theorem 4.19. *Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. Then if f is c-convex and c-concave, then f is p^*I-convex.*

Proof. Let $U \in \mathcal{P}(X)$; then $U \subset I(G(U))$. From Theorem 4.9 (2) and Theorem 4.18, we have

$$f(U) \subset f(I(G(U))) \subset I(f(G(U))) \subset I(G(f(U))).$$

Hence $f(U) \in \mathcal{P}(Y)$. □

Corollary 4.20. Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. Then if f is c-convex and c-concave, then f is pI-convex.

Proof. Since every p^*I-convex function is pI-convex, by Theorem 4.19, f is pI-convex. □

Theorem 4.21. *Let $f : (X, \sigma) \to (Y, \mu)$ be a function on two preconvexity spaces. Then f is pI-convex and c-convex, then f is p^*I-convex.*

Proof. Let $U \in \mathcal{P}(X)$; then since $U \subset I(G(U))$, from Theorem 4.17 and c-convexity, we have

$$f(U) \subset f(I(G(U))) \subset I(G(f(G(U)))) \subset I(G(G(f(U)))) \subset I(G(f(U))).$$

Hence $f(U) \in \mathcal{P}(Y)$. □

References

[1] M. D. Guay, *An introduction to preconvexity spaces*, Acta Math. Hungar., vol. 105 (2004), no 3, pp. 241–248.

[2] W. K. Min, *Some results on preconvexity spaces*, Bull. Korean Math. Soc., vol. 45 (2008), no 1, pp. 39–44.

[3] W. K. Min, *A note on preconvexity spaces*, Honam Math. J., vol. 29(2007), no 4, 589-595.

[4] W. K. Min, *semi-preconvex sets on preconvexity spaces*, Commun. Korean Math. Soc., vol.23(2008), no, 2, pp. 251-256.

[5] W. K. Min, *β-preconvex sets on preconvexity spaces*, submitted.

[6] S. A. Naimpally and B. D. Warrack, *Proximity spaces*, Cambridge University Press (1970).

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