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CONVERGENCE PROPERTIES OF PREDATOR-PREY SYSTEMS WITH FUNCTIONAL RESPONSE

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Abstract. In the field of population dynamics and chemical reaction the possibility of the existence of spatially and temporally nonhomogeneous solutions is a very important problem. For last 50 years or so there have been many results on the pattern formation of chemical reaction systems studying reaction systems with or without diffusions to explain instabilities and nonhomogeneous states arising in biological situations. In this paper we study time-dependent properties of a predator-prey system with functional response and give sufficient conditions that guarantee the existence of stable limit cycles.

1. Introduction

One of the most interesting problems in the field of population dynamics and chemical reaction is the possibility of the existence of spatially and temporally nonhomogeneous solutions. In 1952 Turing had shown that a systems of equations with diffusions could have nonhomogeneous solutions describing spatially nonhomogeneous distribution of reacting chemicals. Since then, many papers on the pattern formation of chemical reaction systems studied reaction systems with diffusion as the most probable mechanism to explain instabilities and nonhomogeneous states arising in biological situations. Among many some the works on instabilities and nonhomogeneous states arising in biological situations are found in Gierer and Meinhardt [2], [3], Murray [6], [7], Levin and Segel [4].

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In the study of prey-predator models in population dynamics, the classical models use monotone increasing reaction functions to describe that the more prey, the better off the predator. However there have been some observations that indicate it is not always the case. As an effort to investigate such cases like of group defense, Rosenzweig [10], [11], [12], May [5] and Riebesell [9] have studied prey-predator systems with functional responses of various types. Bhattacharyya, et al. [1] have shown diffusion-driven instability phenomenon of a prey-predator systems with Holling type IV functional response as in the following :

$$\begin{cases} \frac{du}{d\tau} &= D_1 u_{xx} + u(1-\frac{u}{\gamma}) - \frac{uv}{(u^2/\alpha)+u+1} \\ \frac{dv}{d\tau} &= D_2 v_{xx} + \frac{\beta\delta uv}{(u^2/\alpha)+u+1} - \delta v. \end{cases}$$

Their result says that diffusive instability to small perturbations will take place, and it depends on the ratio $\frac{D_1}{D_2}$ of the diffusion coefficients.

In [13] the convergence properties have been studied for the following prey-predator system with functional response and diffusions :
(1.1)

$$\begin{cases} u_t = (du + \alpha_{12}uv)_{xx} + u(a_1 - b_1 u - \frac{c_1 v}{1+qu}) & \text{in } [0,1]\times(0,\infty), \\ v_t = (dv + \alpha_{21}uv)_{xx} + v(a_2 + \frac{b_2 u}{1+qu} - c_2 v) & \text{in } [0,1]\times(0,\infty), \\ u_x(x,t) = v_x(x,t) = 0 & \text{at } x = 0, 1, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } [0,1], \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. The coefficients d_1, d_2, b_i, c_i $(i = 1,2)$, q, and a_1 are positive constants. Only a_2 may be nonpositive. The initial functions $u_0(x)$, $v_0(x)$ are assumed not to be identically zero. The coefficients d_1 and d_2 are the diffusion rates of the two species, respectively. The positive constant a_1 means that the prey is assumed to be sharing limited resource so that its population can increase a bit in the absence of predator. If $a_2 > 0$ the predator is assumed to have another source of food supply than the prey, sufficient to increase the predator population somewhat in the absence of prey. If $a_2 \leq 0$ the predator population will be decreasing in the absence of prey. The coefficients b_1 and c_2 account for the competitions within the prey species and predator species, respectively. c_1 represents the death rate of the prey due to the encounter with predator. And, b_2 is the growth rate of the predator due to their prey consumption.

In system (1.1) u and v are nonnegative functions which represent the population densities of the prey and predator species, respectively, which are interacting and migrating in the same habitat Ω. By using the strong maximum principle and the Hopf boundary lemma for parabolic equations, it is shown as in [13] that

$$u(x,t) > 0 \text{ and } v(t,x) > 0 \text{ in } [0,1] \times (0,\infty).$$

For details in the biological background, we refer the reader to the monograph of Okubo and Levin [8]. The results on the asymptotic behaviors of the solution to system (1.1) in [13] are as given below.

Theorem 1 (Theorem 2 in [13]). *Assume that the initial functions* u_0, v_0 *are in* $W_2^2([0,1])$, *and let* $(u(x,t), v(x,t))$ *be the maximal solution of system* (1.1). *Then there exist positive constants* t_0 *and* $M = M(d, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1,2)$ *such that*

$$\max\{u(x,t),\ v(x,t) : (x,t) \in [0,1] \times (t_0, T)\} \leq M,$$

and $T = +\infty$. *In the case* $d \geq 1$, *the constant* M *is independent of* $d \geq 1$, *that is,* $M = M(\alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1,2)$.

Theorem 2 (Theorem 4 in [13]). *Suppose that* $-\frac{a_1 b_2}{b_1 c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$, *and* $0 < b_2 < c_1 + 2\min\{b_1, c_2\}$ *for the system* (1.1). *Let* u_0, v_0 *be in* $W_2^2([0,1])$. *If* $d \geq 1$ *satisfies that*

$$(1.2) \qquad (b_2^2 \alpha_{12}^2 \overline{u}^2 + c_1^2 \alpha_{21}^2 \overline{v}^2) M^2 < 4 b_2 c_1 \overline{u}\, \overline{v} d^2,$$

where M *is the positive constant in* Theorem 1 (*independent of* $d \geq 1$), *then the solution* $(u(x,t), v(x,t))$ *converges to* $(\overline{u}, \overline{v})$ *uniformly in* $[0,1]$ *as* $t \to \infty$, *and* $(\overline{u}, \overline{v})$ *is globally asymptotically stable.*

Regarding the convergence of the solution to the cross-diffusion system (1.1) to the steady-state $(\overline{u}, \overline{v})$ the result in Theorem 2 implies that under some condition a cross-diffusion prey-predator system has the same asymptotic property as its kinetic system. More specifically Theorem 2 gives sufficient conditions for system (1.1) in order to have the global convergence property of the solution.

In order to investigate under which conditions (1.1) possesses a nonconstant stable solution we have to analyze its kinetic system as in the following :

$$(1.3) \qquad \begin{cases} u_t = u(a_1 - b_1 u - \frac{c_1 v}{1+qu}) & \text{for } t \in (0,\infty), \\ v_t = v(a_2 + \frac{b_2 u}{1+qu} - c_2 v) & \text{for } t \in (0,\infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \end{cases}$$

In the course to understand the asymptotic behaviors of the solution $(u(t), v(t))$ of (1.3), the kinetic system of with prey-predator type functional response we study the existence of stable limit cycle of system (1.3). Based on the results for the kinetic system we advance our study to investigation of the existence of a limit cycle of the cross-diffusion system (1.1) as a small perturbation of the stable limit cycle of system (1.3).

In this paper we study the existence of nonconstant solutions of system (1.3) and their convergence property around the unstable positive constant steady-states $(\overline{u},\overline{v})$. The following theorem is the main result of the present paper dealing with the case that the kinetic system (1.3) has periodic solutions around the steady-state $(\overline{u},\overline{v})$.

Theorem 3. *Assume either of the two conditions below holds :*

(i) $a_2 \geq 0$ *and* $\frac{a_1}{c_1} > \frac{a_2}{c_2}$,

(ii) $a_2 < 0$, $\frac{a_1}{b_1} > -\frac{a_2}{b_2}$ *and* $0 \leq q < \frac{a_1 b_2 + a_2 b_1}{-a_2}$

for system (1.3). *System* (1.3) *has a unique stable limit cycle if and only if both of the following conditions hold :*

$$c_2(b_1(1+q\overline{u})^2 - c_1 q\overline{v}) + b_2 c_1 > 0$$

$$c_1\overline{u}(-2b_1 q\overline{u} + a_1 q - b_1) - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2 > 0$$

This paper consists of three sections : Section 1. Introduction. In Section 2 the existence results of the positive unstable constant steady-state $(\overline{u},\overline{v})$ and the stable limit cycle are proved. Section 3 gives some examples of system (1.3) with specific values of the parameters a_i, b_i, c_i for $i = 1, 2$, and q that posses a unique stable limit cycle around $(\overline{u},\overline{v})$.

2. Periodic solutions of system (1.3) with functional response

Let us consider the kinetic system of the following type :

$$(2.1) \qquad \begin{cases} u_t = f(u,v) & \text{for } t \in (0,\infty), \\ v_t = g(u,v) & \text{for } t \in (0,\infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \end{cases}$$

System (1.3) is obtained from system (2.1) with functional responses $f(u,v)$ and $g(u,v)$ as the following :

$$(2.2) \quad \begin{cases} f(u,v) = u(a_1 - b_1 u - \frac{c_1 v}{1+qu}) = u(\frac{(a_1 - b_1 u)(1+qu) - c_1 v}{1+qu}) \\ g(u,v) = v(a_2 + \frac{b_2 u}{1+qu} - c_2 v) = v(a_2 + \frac{b_2}{q} - \frac{b_2}{q^2(\frac{1}{q}+u)} - c_2 v). \end{cases}$$

In order to guarantee the existence of the positive constant steady-state $(\overline{u},\overline{v})$ for system (1.3), it is necessary to put some conditions on the parameters a_1, b_i, c_i $(i = 1, 2)$, and q.

Theorem 4. *In each of the cases*

(i) $a_2 \geq 0$ and $\frac{a_1}{c_1} > \frac{a_2}{c_2}$,

(ii) $a_2 < 0$, $\frac{a_1}{b_1} > -\frac{a_2}{b_2}$ and $0 \leq q < \frac{a_1 b_2 + a_2 b_1}{-a_2}$

the kinetic system (1.3) *has a unique positive steady-state* $(\overline{u}, \overline{v})$ *in the first quadrant* $\{(u,v) : u > 0$ and $v > 0\}$, *where*

$$\overline{v} = \frac{1}{c_1}(1 + q\overline{u})(a_1 - b_1\overline{u}) = \frac{a_2}{c_2} + \frac{b_2\overline{u}}{c_2(1+q\overline{u})}. \tag{2.3}$$

And if both of conditions (i) *and* (ii) *fail then system* (1.3) *has no limit cycle. If* $q = 0$, *then* $(\overline{u}, \overline{v})$ *is given by*

$$(\overline{u}, \overline{v}) = \left(\frac{a_1c_2 - a_2c_1}{b_1c_2 + b_2c_1}, \frac{a_2b_1 + a_1b_2}{b_1c_2 + b_2c_1}\right).$$

When $q > 0$, $\overline{u}$ *is expressed as the unique positive root of a cubic polynomial.*

Proof. Since $q \geq 0$ and $u \geq 0$, the zero sets of $f(u,v)$ and $g(u,v)$ are determined by each of the following two equations, respectively :

$$(a_1 - b_1 u)(1 + qu) - c_1 v = 0, \tag{2.4}$$

$$a_2 + \frac{b_2 u}{1 + qu} - c_2 v = 0. \tag{2.5}$$

To determine the relative locations of the zero sets of $f(u,v) = 0$ and $g(u,v) = 0$ in the phase plane, let us observe the equation

$$\frac{1}{c_1}(a_1 - b_1 u)(1 + qu)^2 = \frac{a_2}{c_2}(1 + qu) + \frac{b_2}{c_2}u$$

or equivalently,

$$(1 + qu)\left(\frac{1}{c_1}(a_1 - b_1 u)(1 + qu) - \frac{a_2}{c_2}\right) = \frac{b_2}{c_2}u.$$

This equation is reduced to

$$(1 + qu)\left(\frac{a_1}{c_1} - \frac{a_2}{c_2} + \frac{1}{c_1}(a_1 q - b_1)u - \frac{b_1 q}{c_1}u^2\right) = \frac{b_2}{c_2}u. \tag{2.6}$$

In order to analyze equation (2.6) let us define functions of U for the left and right hand side of (2.6) as follows :

$$\begin{aligned} h_1(u) &= (1 + qu)\left(\frac{a_1}{c_1} - \frac{a_2}{c_2} + \frac{1}{c_1}(a_1 q - b_1)u - \frac{b_1 q}{c_1}u^2\right) \\ h_2(u) &= \frac{b_2}{c_2}u \end{aligned}$$

In the case that $\frac{a_1}{c_1} > \frac{a_2}{c_2}$, the graphs of $h_1(u)$ and $h_2(u)$ are as sketched in Figure 1.

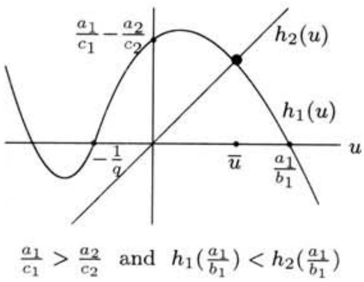


FIGURE 1. The cases in which $\overline{u} > 0$ exists for system (1.3).

We note from equation (2.4) that $u < \frac{a_1}{b_1}$ for $v > 0$. Thus it must hold that

$$h_1(\tfrac{a_1}{b_1}) < h_2(\tfrac{a_1}{b_1}). \tag{2.7}$$

By evaluating $h_1(\frac{a_1}{b_1})$ as follows

$$\begin{aligned} h_1(\tfrac{a_1}{b_1}) &= (1+q\tfrac{a_1}{b_1})\left(\tfrac{a_1}{c_1}-\tfrac{a_2}{c_2}+\tfrac{1}{c_1}(a_1q-b_1)\tfrac{a_1}{b_1}-\tfrac{b_1q}{c_1}(\tfrac{a_1}{b_1})^2\right) \\ &= (1+\tfrac{a_1q}{b_1})\left(\tfrac{a_1}{c_1}-\tfrac{a_2}{c_2}+\tfrac{a_1^2q}{b_1c_1}-\tfrac{a_1}{c_1}-\tfrac{a_1^2q}{b_1c_1}\right) \\ &= -\tfrac{a_2}{c_2}(1+\tfrac{a_1q}{b_1}) \end{aligned}$$

condition (2.7) is reduced to

$$-\tfrac{a_2}{c_2}(1+\tfrac{a_1q}{b_1}) < \tfrac{a_1b_2}{b_1c_2}. \tag{2.8}$$

Condition (2.8) holds if $a_2 \geq 0$. In the case that $a_2 < 0$ condition (2.8) is rewritten as

$$q < -(\tfrac{b_1}{a_1}+\tfrac{b_2}{a_2}). \tag{2.9}$$

Hence we conclude that system (1.3) has a unique positive steady-state $(\overline{u},\overline{v})$ in either of the following cases :

(i) $a_2 \geq 0$ and $\frac{a_1}{c_1} > \frac{a_2}{c_2}$

(ii) $a_2 < 0$, and $0 \leq q < -(\frac{b_1}{a_1}+\frac{b_2}{a_2})$.

Finally a positive value of $\overline{v}$ is obtained if $\overline{u}$ is positive from (2.3). □

The asymptotic behaviors of the solutions of system (1.3) are classified into the four cases in terms of the constants q, a_i, b_i, and c_i, $i=1,2$, and the directions of the flow for the system (1.3) are as illustrated in Figure 2 in the each case :

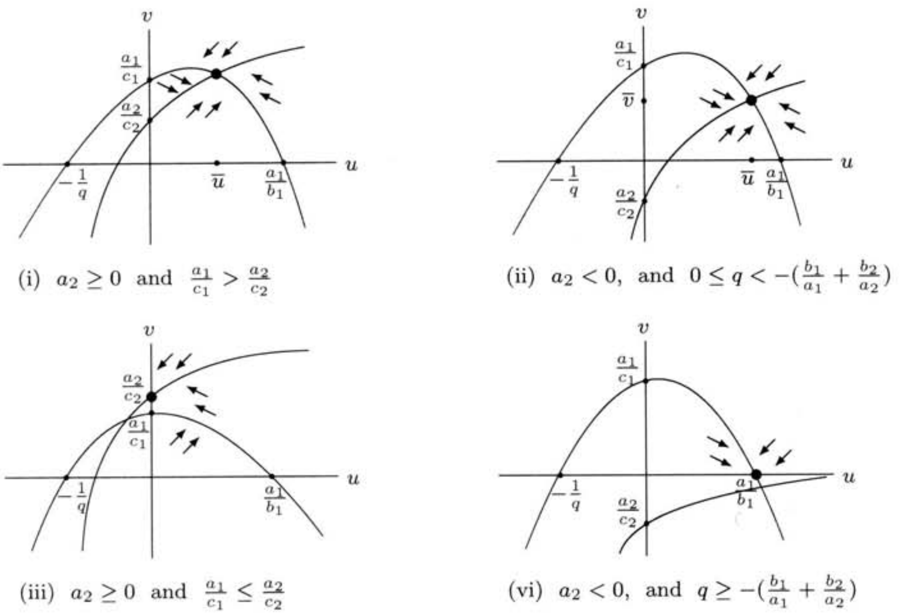


FIGURE 2. The flow directions for the system (1.3) in the first quadrant of the phase plane

(i) $a_2 \geq 0$ and $\frac{a_1}{c_1} > \frac{a_2}{c_2}$,
(ii) $a_2 < 0$, and $0 \leq q < -(\frac{b_1}{a_1} + \frac{b_2}{a_2})$
(iii) $a_2 \geq 0$ and $\frac{a_1}{c_1} \leq \frac{a_2}{c_2}$,
(vi) $a_2 < 0$, and $q \geq -(\frac{b_1}{a_1} + \frac{b_2}{a_2})$

In each case (i) and (ii) it is shown through an elementary but tedious way of computations that there exists a finite closed box in the phase plane which is an invariant set for the solutions of system (1.3) (see, for example, section 3.4 in [7]).

Theorem 5. *Assume either condition* (i) *or* (ii) *for system* (1.3). *The system* (1.3) *has a unique stable limit cycle if and only if both of conditions* (2.10) *and* (2.11) *below hold.*

$$c_2(b_1(1+q\overline{u})^2 - c_1 q\overline{v}) + b_2 c_1 > 0 \tag{2.10}$$

$$c_1\overline{u}(-2b_1 q\overline{u} + a_1 q - b_1) - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2 > 0 \tag{2.11}$$

Proof. For system (1.3) we note that

$$f(u,v) = u(a_1 - b_1 u - \frac{c_1 v}{1+qu}) = u\left(\frac{(a_1 - b_1 u)(1+qu) - c_1 v}{1+qu}\right)$$

and

$$g(u,v) = v(a_2 + \frac{b_2 u}{1+qu} - c_2 v) = u\left(a_2 + \frac{b_2}{q} - \frac{b_2}{q^2(\frac{1}{q}+u)} - c_2 v\right).$$

From Theorem 4 we have that $\overline{u} > 0, \overline{v} > 0$ and $\overline{v} = \frac{1}{c_1}(1+bu)(a_1 - b_b u) = \frac{a_2}{c_2} + \frac{b_2\overline{u}}{c_2(1+q\overline{u})}$.
For the linear analysis write

$$x(t) = u(t) - \overline{u}, \quad y(t) = v(t) - \overline{v}$$

which on substituting into system (1.3), linearizing with small $|x|$ and $|y|$ gives

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$

$$A = \begin{pmatrix} \frac{df}{du} & \frac{df}{dv} \\ \frac{dg}{du} & \frac{dg}{dv} \end{pmatrix}_{(\overline{u},\overline{v})} = \begin{pmatrix} a_1 - 2b_1\overline{u} - \frac{c_1\overline{v}(1+q\overline{u}) - c_1 q\overline{u}\overline{v}}{(1+q\overline{u})^2} & -\frac{c_1\overline{u}}{1+q\overline{u}} \\ \frac{b_2\overline{v}(1_q\overline{u}) - b_2 q\overline{u}\overline{v}}{(1+q\overline{u})^2} & a_2 - 2c_2\overline{v} + \frac{b_2\overline{u}}{1+q\overline{u}} \end{pmatrix}.$$

From the facts that

$$\begin{aligned} f(\overline{u},\overline{v}) = 0 \quad &\Leftrightarrow \quad a_1 - b_1\overline{u} - \frac{c_1\overline{v}}{1+q\overline{u}} = 0 \\ g(\overline{u},\overline{v}) = 0 \quad &\Leftrightarrow \quad a_2 + \frac{b_2\overline{u}}{1+q\overline{u}} - c_2\overline{v} = 0, \end{aligned}$$

it is possible to simplify the components of the community matrix A as

$$\begin{aligned} A &= \begin{pmatrix} a_1 - 2_1\overline{u} - \frac{c_1\overline{v}}{(1+q\overline{u})^2} & -\frac{c_1\overline{u}}{1+\overline{u}} \\ \frac{b_2\overline{v}}{(1+q\overline{u})^2} & -c_2\overline{v} \end{pmatrix} \\ &= \begin{pmatrix} -b_1\overline{u} + \frac{c_1\overline{v}}{1+q\overline{u}} - \frac{c_1\overline{v}}{(1+q\overline{u})^2} & -\frac{c_1\overline{u}}{1+\overline{u}} \\ \frac{b_2\overline{v}}{(1+q\overline{u})^2} & -c_2\overline{v} \end{pmatrix} \\ &= \begin{pmatrix} -b_1\overline{u} + \frac{c_1 q\overline{u}\overline{v}}{(1+q\overline{u})^2} & -\frac{c_1\overline{u}}{1+\overline{u}} \\ \frac{b_2\overline{v}}{(1+q\overline{u})^2} & -c_2\overline{v} \end{pmatrix}. \end{aligned}$$

The eigenvalues λ of the community matrix A satisfy that

$$\det(A - \lambda I) = 0 \quad \Rightarrow \quad \lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0.$$

The steady-state $(\overline{u},\overline{v})$ is linearly stable if and only if $\operatorname{Re}\lambda < 0$ and so the necessary and sufficient conditions for linear stability are

$$\det A > 0 \quad \text{and} \quad \operatorname{tr} A < 0.$$

Through a course of simple computations it is shown that the solution $(u(t), v(t))$ of system (1.3) is enclosed in a bounded box in the (u, v) phase plane. Thus by using Poincare-Bendixon Theorem we conclude that the positive steady state $(\overline{u},\overline{v})$ is unstable, and system (1.3) has a unique stable limit cycle around $(\overline{u},\overline{v})$ if and only if

$$\det A > 0 \quad \text{and} \quad \operatorname{tr} A > 0.$$

We investigate sufficient conditions that determines the signs of $\det A$ and $\operatorname{tr} A$.

$$\begin{aligned}
\det A &= b_1c_2\overline{u}\overline{v} - \frac{c_1c_2\overline{v}^2}{1+q\overline{u}} + \frac{c_1c_2\overline{v}^2}{(1+q\overline{u})^2} + \frac{b_2c_1\overline{u}\overline{v}}{(1+q\overline{u})^3} \\
&= b_1c_2\overline{u}\overline{v} - \frac{c_1c_2q\overline{u}\overline{v}^2}{(1+q\overline{u})^2} + \frac{b_2c_1\overline{u}\overline{v}}{(1+q\overline{u})^3} \\
&= \frac{c_2\overline{u}\overline{v}(b_1(1+q\overline{u})^2 - c_1q\overline{v})}{(1+q\overline{u})^2} + \frac{b_2c_1\overline{u}\overline{v}}{(1+q\overline{u})^3} \\
&= \frac{\overline{u}\overline{v}(b_1c_2(1+q\overline{u})^3 - c_1c_2q\overline{v}(1+q\overline{u}) + b_2c_1)}{(1+q\overline{u})^3}.
\end{aligned}$$

$$\begin{aligned}
\operatorname{tr} A &= -b_1\overline{u} + \frac{c_1q\overline{u}\overline{v}}{(1+q\overline{u})^2} - c_2\overline{v} \\
&= -b_1\overline{u} + \frac{(c_1q\overline{u} - c_2(1+q\overline{u})^2)}{(1+q\overline{u})^2}\cdot\overline{v} \\
&= -b_1\overline{u} + \frac{(c_1q\overline{u} - c_2(1+q\overline{u})^2)}{(1+q\overline{u})^2}\cdot\frac{1}{c_1}(1+q\overline{u})(a_1 - b_1\overline{u}) \\
&= -b_1\overline{u} + \frac{(c_1q\overline{u} - c_2(1+q\overline{u})^2)}{1+q\overline{u}}\cdot\frac{1}{c_1}(a_1 - b_1\overline{u}) \\
&= \frac{-b_1c_1\overline{u}(1+q\overline{u}) + (a_1 - b_1\overline{u})(c_1q\overline{u} - c_2(1+q\overline{u})^2)}{c_1(1+q\overline{u})} \\
&= \frac{-2b_1c_1q\overline{u}^2 + c_1(a_1q - b_1)\overline{u} - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2}{c_1(1+q\overline{u})} \\
&= \frac{c_1\overline{u}(-2b_1q\overline{u} + a_1q - b_1) - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2}{c_1(1+q\overline{u})}.
\end{aligned}$$

Thus the unique positive steady state $(\overline{u},\overline{v})$ is stable if and only if

$$\begin{aligned}
&b_1c_2(1+q\overline{u})^3 - c_1c_2q\overline{v}(1+q\overline{u}) + b_2c_1 > 0 \quad \text{and} \\
&c_1\overline{u}(-2b_1q\overline{u} + a_1q - b_1) - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2 < 0
\end{aligned} \tag{2.12}$$

The positive steady state $(\overline{u},\overline{v})$ becomes unstable and system (1.3) has a unique stable limit cycle around $(\overline{u},\overline{v})$ if and only if

$$\begin{aligned}
&b_1c_2(1+q\overline{u})^3 - c_1c_2q\overline{v}(1+q\overline{u}) + b_2c_1 > 0 \quad \text{and} \\
&c_1\overline{u}(-2b_1q\overline{u} + a_1q - b_1) - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2 > 0
\end{aligned} \tag{2.13}$$

□

3. Examples of system (1.3) with limit cycles

In this section we provide example sets of parameters a_1, b_1, c_1, a_2, b_2, c_2, and q for system (1.3) to have a unique stable limit cycle around the positive steady-state $(\overline{u}, \overline{v})$.

Example 1. [with $a_2 > 0$] Let $a_1 = 3.2$, $b_1 = 1$, $c_1 = 1$, $a_2 = 0.1$, $b_2 = 1$, $c_2 = 0.1$, and $q = 1$ in system (1.3).

$$(3.1) \qquad \begin{cases} u_t = u(3.2 - u - \frac{v}{1+u}) & \text{for } t \in (0, \infty), \\ v_t = v(0.1 + \frac{u}{1+u} - 0.1v) & \text{for } t \in (0, \infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \end{cases}$$

Solving the equation $(1+u)(3.2-u) = 1 + \frac{10u}{1+u}$, or equivalently, $5u^3 - 6u^2 + 28u - 11 = 0$, we obtain that

$$\overline{u} = \frac{2}{5} - \frac{128}{5}\sqrt[3]{\frac{2}{3(99+25\sqrt{40281})}} + \frac{\sqrt[3]{\frac{1}{2}(99+25\sqrt{40281})}}{\sqrt[3]{53^2}} \approx 0.4172,$$

and thus

$$\overline{v} = (1+\overline{u})(3.2 - \overline{u}) \approx 3.9438.$$

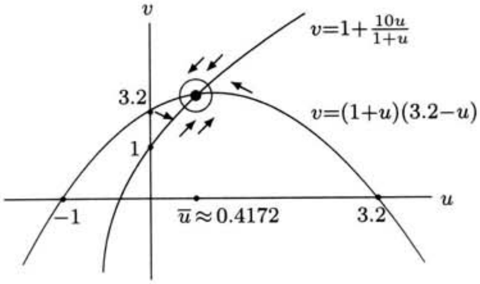
It is also verified that the coefficients of system (3.1) satisfy the required conditions as listed below that provide the existence of the unique stable limit cycle around the positive unstable steady-state $(\overline{u}, \overline{v})$:

(i) $a_2 \geq 0$ and $\frac{a_1}{c_1} > \frac{a_2}{c_2}$

(2.10) $c_2(b_1(1+q\overline{u})^2 - c_1 q\overline{v}) + b_2 c_1 > 0$

(2.11) $c_1\overline{u}(-2b_1 q\overline{u} + a_1 q - b_1) - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2 > 0$

Using the same values for the parameters b_1, c_1, a_2, b_2, c_2, and q as in Example 1 it is shown that system (1.3) still displays a stable limit cycle when a_1 is assumed to have the values in the following table by checking the necessary conditions (2.10) and (2.11).



(i) $a_2 = 0.1 \geq 0$ and $\frac{a_1}{c_1} = \frac{3.2}{1} > \frac{a_2}{c_2} = \frac{0.1}{0.1}$

FIGURE 3. An example of system (1.3) with $a_2 > 0$ which possesses a stable limit cycle.

a_1	(2.10)	(2.11)
3.2	0.8065	0.0108
3.4	0.7909	0.0461
3.6	0.0000	0.0000
3.8	0.7785	0.0139
3.82	0.7793	0.0026

Example 2. [with $a_2 < 0$] Let $a_1 = 0.4$, $b_1 = 0.2$, $c_1 = 1$, $a_2 = -0.2$, $b_2 = 1.2$, $c_2 = 0.1$, and $q = 2$ in system (1.3).

$$(3.2) \qquad \begin{cases} u_t = u(0.4 - 0.2u - \frac{v}{1+2u}) & \text{for } t \in (0, \infty), \\ v_t = v(-0.2 + \frac{1.2u}{1+2u} - 0.1v) & \text{for } t \in (0, \infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \end{cases}$$

Solving the equation $(1+2u)(0.4-0.2u) = -2 + \frac{12u}{1+2u}$, or equivalently, $4u^3 - 4u^2 + 33u - 12 = 0$, we obtain that

$$\overline{u} = \frac{1}{6}\left(2 - \sqrt[3]{\frac{195^2}{7+18\sqrt{106}}} + \sqrt[3]{5(7+18\sqrt{106})}\right) \approx 0.3743,$$

and thus

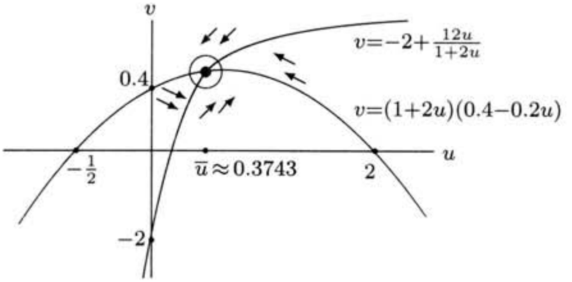
$$\overline{v} = (1+2\overline{u})(0.4-0.2\overline{u}) \approx 0.5685.$$

Then the coefficients of system (3.2) satisfy the required conditions as listed below that provide the existence of the unique stable limit cycle around the positive unstable steady-state $(\overline{u}, \overline{v})$:

(ii) $a_2 < 0$, $\frac{a_1}{b_1} > -\frac{a_2}{b_2}$ and $0 \leq q < \frac{a_1 b_2 + a_2 b_1}{-a_2}$

$$(2.10)\quad c_2(b_1(1+q\overline{u})^2 - c_1 q\overline{v}) + b_2 c_1 > 0$$

$$(2.11)\quad c_1\overline{u}(-2b_1 q\overline{u} + a_1 q - b_1) - c_2(a_1 - b_1\overline{u})(1+q\overline{u})^2 > 0$$



(ii) $a_2 = -0.2 < 0$, and $0 \le q = 2 < -(\frac{b_1}{a_1} + \frac{b_2}{a_2}) = -(\frac{0.2}{0.4} + \frac{1.2}{-0.2}) = 6.5$

FIGURE 4. An example of system (1.3) with $a_2 < 0$ which possesses a stable limit cycle.

Using the same values for the parameters b_1, c_1, a_2, b_2, c_2, and q as in Example 2 it is shown that system (1.3) still displays a stable limit cycle when a_1 is assumed to have the values in the following table by checking the necessary conditions (2.10) and (2.11).

a_1	(2.10)	(2.11)
0.4	1.1474	0.0131
0.6	1.08	0.01
0.8	1.001	0.1859
0.85	0.9835	0.1624

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