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# Equivalence Relations

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**Abstract**

We investigate the properties of fuzzy relations and $\odot$-equivalence relation on a stsc quantale lattice $L$ and a commutative cqm-lattice. In particular, fuzzy relations preserve $(*, \otimes)$-equivalence relations where $\otimes$ are compositions, $\Rightarrow$ and $\Leftarrow$.

**Key words** : stsc-quantales, commutative cqm-lattice, $\odot$-equivalence relations

## 1. Introduction and preliminaries

Quantales were introduced by Mulvey [11,12] as the non-commutative generalization of the lattice of open sets in topological spaces. Recently, quantales have arisen in an analysis of the semantics of linear logic systems developed by Girard [4], which supports part of foundation of theoretic computer science. Recently, Höhle [6-8,13] developed the algebraic structures and many valued topologies in a sense of quantales and cqm-lattices. Bělohlávek [1-3] investigate the properties of fuzzy relations and similarities on a residual lattice.

In this paper, we investigate the properties of fuzzy relations and $\odot$-equivalence relation on a stsc-quantale lattice and a commutative cqm-lattice. In particular, $L$-fuzzy relations preserve $(*, \otimes)$-equivalence relations where $\otimes$ are compositions, $\Rightarrow$ and $\Leftarrow$.

**Definition 1.1.** [6-8, 11-13] A triple $(L, \leq, \odot)$ is called a *strictly two-sided, commutative quantale* (stsc-quantale, for short) if it satisfies the following conditions:

(Q1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where $1$ is the universal upper bound and $0$ denotes the universal lower bound;

(Q2) $(L, \odot)$ is a commutative semigroup;

(Q3) $a = a \odot 1$, for each $a \in L$;

(Q4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

**Remark 1.2.** [6-8](1) A completely distributive lattice is a stsc-quantale. In particular, the unit interval $([0,1], \leq, \vee, \wedge, 0, 1)$ is a stsc-quantale.

(2) The unit interval with a left-continuous t-norm $t$, $([0,1], \leq, t)$, is a stsc-quantale.

(3) Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y \in L$, we define

$$x \rightarrow y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,
$(x \odot y) \leq z$ iff $x \leq (y \rightarrow z)$.

**Lemma 1.3.** [6-8,13] Let $(L, \leq, \odot)$ be a stsc-quantale. For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(5) $x \rightarrow (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \rightarrow y_i)$

(6) $(\bigwedge_{i\in\Gamma} x_i) \rightarrow y \geq \bigvee_{i\in\Gamma}(x_i \rightarrow y)$.

(7) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(8) $x \odot (x \rightarrow y) \leq y$ and $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

(9) $y \odot z \leq x \rightarrow (x \odot y \odot z)$ and $x \odot (x \odot y \rightarrow z) \leq y \rightarrow z$.

(10) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$.

(11) $x \rightarrow y = 1$ iff $x \leq y$.

**Definition 1.4.** [1-3], [6-8,13] A function $E : X \times X \rightarrow L$ is called an $\odot$-equivalence relation if it satisfies the following conditions:

(E1) $E(x,x) = 1$,

(E2) $E(x,y) = E(y,x)$,

(E3) $E(x,y) \odot E(y,z) \leq E(x,z)$.

An $\odot$-equivalence relation is called an $\odot$-equality if $E(x,y) = 1$ implies $x = y$.

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## 2. Commutative cqm-lattices and square roots

We define commutative cqm-lattice and square roots.

**Definition 2.1.** [6-8,13] A triple $(L, \leq, *)$ is called a *commutative cqm-lattice* if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound,

(C2) $a * (b * c) = (a * b) * c$ and $a * b = b * a$,

(C3) $a \leq a * 1$, for each $a \in L$,

(C4) If $a \leq b$ , then $a * c \leq b * c$

Let $(L, \odot)$ and $(L, *)$ be a stsc-quantale and a commutative cqm-lattice. An operation $*$ dominates $\odot$ if it satisfies

$$(a_1 * b_1) \odot (a_2 * b_2) \leq (a_1 \odot a_2) * (b_1 \odot b_2)$$

**Remark 2.2.** [6-8,13](1) A stsc-quantale is a commutative cqm-lattice.

(2) We define an operation $* : L \times L \to L$ as

$$a * b = \begin{cases} 1 & \text{if } a = 1 \text{ or } b = 1, \\ a \wedge b & \text{otherwise} \end{cases}$$

Then $(L, *)$ is a commutative cqm-lattice but not is a stsc-quantale because

$$a \neq 0, a < a * 1 = 1$$

and, for $a < b_i \neq 1$ with $\bigvee_{i\in I} b_i = 1$, we have

$$1 = a * (\bigvee_{i\in I} b_i) \neq \bigvee_{i\in I} (a * b_i) = a.$$

For a stsc-quantale $(L, \odot)$ with $a < b \neq 1$, $*$ does not dominate $\odot$ from

$$1 = (a * 1) \odot (1 * b) \not\leq (a \odot 1) * (1 \odot b) = a.$$

**Definition 2.3.** [6-8,13] A stsc-quantale $(L, \odot)$ has *square roots* if there exists a unary operator $S : L \to L$ satisfying the following conditions:

(S1) $S(a) \odot S(a) = a$

(S2) $b \odot b \leq a$ implies $b \leq S(a)$.

**Example 2.4.** (1) Let $x \odot y = 0 \vee (x + y - 1)$ be a t-norm. Then $S(a) = \frac{a+1}{2}$. But

$$0.5 = S(0.5 \odot 0.4) \neq S(0.4) \odot S(0.5) = 0.45.$$

(2) Let $x \odot y = x \wedge y$ be a t-norm. Then $S(a) = a$.

(3) Let $x \odot y = xy$ be a t-norm. Then $S(a) = \sqrt{a}$.

(4) Let $x \odot y = \frac{xy}{2-x-y+xy}$ be a t-norm. Then

$$S(a) = \begin{cases} \frac{-1+\sqrt{2a-a^2}}{1-a} & \text{if } a \neq 1, \\ 1 & \text{if } a = 1. \end{cases}$$

(5) Let a left-continuous t-norm $\odot$ defined as

$$x \odot y = \begin{cases} x \wedge y & \text{if } x + y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\odot$ has no square roots because there does not exists $S(a)$ such that $S(a) \odot S(a) = 0.5$.

**Theorem 2.5.** (1) Let $(L, \odot)$ be a stsc-quantale with square roots. Define

$$a * b = S(a) \odot S(b).$$

Then $(L, *)$ is a commutative cqm-lattice which $*$ dominates $\odot$.

(2) A t-norm $\wedge$ dominates every t-norm $\odot$.

*Proof.* (1) Since $a \odot a \leq a$ and $1 \odot 1 = 1$, we have

$$a = a \odot 1 \leq S(a) \odot S(1) = a * 1.$$

For each $x_1, x_2, y_1, y_2 \in L$,

$$\begin{aligned} &(x_1 \odot y_1) * (x_2 \odot y_2) \\ &= S(x_1 \odot y_1) \odot S(x_2 \odot y_2) \\ &\geq (S(x_1) \odot S(y_1)) \odot (S(x_2) \odot S(y_2)) \\ &= (x_1 * x_2) \odot (y_1 * y_2). \end{aligned}$$

Other cases are easily proved.

(2) For each $x_1, x_2, y_1, y_2 \in [0, 1]$,

$$(x_1 \odot y_1) \wedge (x_2 \odot y_2) \geq (x_1 \wedge x_2) \odot (y_1 \wedge y_2).$$

□

## 3. Equivalence Relations

**Theorem 3.1.** Let $(L, \odot)$ and $(L, *)$ be a stsc-quantale and a commutative cqm-lattice, respectively, which $*$ dominates $\odot$. We define $I : L^{X\times Y} \times L^{X\times Y} \to L$ as follows

$$I(u_1, u_2) = \bigwedge_{(x,y)\in X\times Y} \Big(u_1(x, y) \to u_2(x, y)\Big)$$

Then we have the following properties:

(1) $E_*(u_1, u_2) = I(u_1, u_2) * I(u_2, u_1)$ is an $\odot$-equivalence relation on $L^{X\times Y}$.

(2) If $a * b = 1$ implies $a = 1$ and $b = 1$, then $E_*$ is an $\odot$-equality on $L^{X\times Y}$.

(3) If $a * 1 = a$ for all $a \in L$, then $E_\odot \leq E_* \leq E_\wedge$.

*Proof.* (1) (E1) and (E2) are easy. We prove (E3) from the following statement

$$\begin{aligned} &E_*(u_1, u_2) \odot E_*(u_2, u_3) \\ &= (I(u_1, u_2) * I(u_2, u_1)) \odot (I(u_2, u_3) * I(u_3, u_2)) \\ &\leq (I(u_1, u_2) \odot I(u_2, u_3)) * (I(u_2, u_1) \odot I(u_3, u_2)) \\ &\leq I(u_1, u_3) * I(u_3, u_1) \\ &= E_*(u_1, u_3). \end{aligned}$$

(2) By Lemma 1.3(11), $E_*(u_1,u_2) = I(u_1,u_2) * I(u_2,u_1) = 1$ iff $I(u_1,u_2) = I(u_2,u_1) = 1$ iff $u_1 = u_2$.

(3) Since $a\odot b = (a*1)\odot(1*b) \leq (a\odot 1)*(1\odot b) = a*b$ and $a*b \leq a\wedge b$, it easily proved.

□

**Example 3.2.** Let $X = \{a,b\}, Y = \{c,d\}$ and $Z = \{e,f\}$ be sets and $L = [0,1]$ an unit interval. Define a binary operation $\otimes$ (called Łukasiewicz conjection) on $[0,1]$ by

$$x\odot y = \max\{0, x+y-1\},\ x\to y = \min\{1-x+y, 1\}.$$

Then $([0,1],\vee,\odot,0,1)$ is a stsc-quantale (ref.[6-8]). Let $u_1,u_2 \in [0,1]^{X\times Y}$ as follows:

$$u_1(a,c) = 0.7, u_1(a,d) = 0.5, u_1(b,c) = 0.8, u_1(b,d) = 0.1$$

$$u_2(a,c) = 0.4, u_2(a,d) = 0.6, u_2(b,c) = 0.5, u_2(b,d) = 0.7$$

We have

$$I(u_1,u_2) = \bigwedge_{(x,y)\in X\times Y}\Big(u_1(x,y)\to u_2(x,y)\Big) = 0.7,$$

$$I(u_2,u_1) = \bigwedge_{(x,y)\in X\times Y}\Big(u_2(x,y)\to u_1(x,y)\Big) = 0.4.$$

(1) We define $a*b = a\wedge b$. Then

$$E_*(u_1,u_2) = I(u_1,u_2)\wedge I(u_2,u_1) = 0.4$$

(2) We define $a*b = a\odot b$. Then

$$E_*(u_1,u_2) = I(u_1,u_2)\odot I(u_2,u_1) = 0.1$$

(3) We define $a*b = S(a)\odot S(b)$. Since $S(a) = \frac{a+1}{2}$, we have $a*b = S(a)\odot S(b) = \frac{a+b}{2}$. Hence

$$\begin{aligned} E_*(u_1,u_2) &= I(u_1,u_2) * I(u_2,u_1) \\ &= \tfrac{1}{2}(I(u_1,u_2) + I(u_2,u_1)) \\ &= 0.55 \end{aligned}$$

Since $a < a*1 = S(a)\odot S(1) = \frac{a+1}{2}$ for $a\in[0,1)$, we have

$$0.55 = E_*(u_1,u_2) \not\leq E_\wedge(u_1,u_2) = 0.4.$$

In following definitions and theorems, let $(L,\odot)$ and $(L,*)$ be a stsc-quantale and a commutative cqm-lattice, respectively, which $*$ dominates $\odot$.

**Definition 3.3.** For $u\in L^{X\times Y}$ and $v\in L^{Y\times Z}$, we define:

$$(u\circ v)(x,z) = \bigvee_{y\in Y}(u(x,y)\odot v(y,z))$$

$$(u\Rightarrow v)(x,z) = \bigwedge_{y\in Y}(u(x,y)\to v(y,z))$$

$$(u\Leftarrow v)(x,z) = \bigwedge_{y\in Y}(v(y,z)\to u(x,y))$$

$$(u\Leftrightarrow v)(x,z) = (u\Rightarrow v)(x,z) * (u\Leftarrow v)(x,z)$$

**Definition 3.4.** The relation $E_*$ preserves $(*,\otimes)$-equivalence relation if

$$E_*(u_1,u_2)\odot E_*(v_1,v_2) \leq E_*(u_1\otimes v_1, u_2\otimes v_2)$$

for every $u_i\in L^{X\times Y}$ and $v_i\in L^{Y\times Z}$

**Remark 3.5.** We regard $E_\wedge$ as the Bělohlávek's definition in [3].

**Theorem 3.6.** (1) The relation $I$ holds

$$I(u_1,u_2) \leq I(u_1\circ v, u_2\circ v)$$

$$I(v_1,v_2) \leq I(u\circ v_1, u\circ v_2)$$

for every $u,u_i\in L^{X\times Y}$ and $v,v_i\in L^{Y\times Z}$

(2) The relation $E_*$ holds

$$E_*(u_1,u_2) \leq E_*(u_1\circ v, u_2\circ v)$$

$$E_*(v_1,v_2) \leq E_*(u\circ v_1, u\circ v_2)$$

for every $u,u_i\in L^{X\times Y}$ and $v,v_i\in L^{Y\times Z}$

(3) The relation $E_*$ preserves $(*,\circ)$-equivalence relation.

*Proof.* (1) We show that $I(u_1,u_2)\leq I(u_1\circ v, u_2\circ v)$ from the following statements: for all $(x,z)\in(X,Z)$,

$$\begin{aligned} &I(u_1,u_2) \leq \textstyle\bigwedge_{(x,z)\in(X,Z)}(u_1\circ v(x,z)\to u_2\circ v(x,z)) \\ &\Leftrightarrow I(u_1,u_2) \leq (u_1\circ v(x,z)\to u_2\circ v(x,z)) \\ &\Leftrightarrow I(u_1,u_2)\odot(u_1\circ v)(x,z) \leq (u_2\circ v)(x,z) \\ &\Leftrightarrow I(u_1,u_2)\odot \textstyle\bigvee_y(u_1(x,y)\odot v(y,z)) \leq (u_2\circ v)(x,z) \\ &\Leftrightarrow \textstyle\bigvee_y I(u_1,u_2)\odot(u_1(x,y)\odot v(y,z)) \leq (u_2\circ v)(x,z) \end{aligned}$$

On the other hand,

$$\begin{aligned} &\textstyle\bigvee_y I(u_1,u_2)\odot(u_1(x,y)\odot v(y,z)) \\ &\leq \textstyle\bigvee_y\Big(\bigwedge_{(x_1,y_1)\in(X,Y)}(u_1(x_1,y_1)\to u_2(x_1,y_1))\Big) \\ &\quad\odot(u_1(x,y)\odot v(y,z)) \\ &\leq \textstyle\bigvee_y\Big(u_1(x,y)\to u_2(x,y)\Big)\odot(u_1(x,y)\odot v(y,z)) \\ &\leq \textstyle\bigvee_y(u_2(x,y)\odot v(y,z)) = (u_2\circ v)(x,z). \end{aligned}$$

By a similar method, we have $I(v_1,v_2)\leq I(u\circ v_1, u\circ v_2)$.

(2) Since $I(u_1,u_2)\leq I(u_1\circ v, u_2\circ v)$ and $I(u_2,u_1)\leq I(u_2\circ v, u_1\circ v)$, we have

$$\begin{aligned} &E_*(u_1,u_2) \\ &= I(u_1,u_2) * I(u_2,u_1) \\ &\leq I(u_1\circ v, u_2\circ v) * I(u_2\circ v, u_1\circ v) \\ &\leq E_*(u_1\circ v, u_2\circ v). \end{aligned}$$

Similarly, $E_*(v_1,v_2)\leq E_*(u\circ v_1, u\circ v_2)$.

(3) By (2) and (E3), we have

$$\begin{aligned} &E_*(u_1,u_2)\odot E_*(v_1,v_2) \\ &\leq E_*(u_1\circ v_1, u_2\circ v_1)\odot E_*(u_2\circ v_1, u_2\circ v_2) \\ &\leq E_*(u_1\circ v_1, u_2\circ v_2) \end{aligned}$$

□

**Example 3.7.** In Example 3.2, let $v_i \in L^{Y\times Z}$ as follows:

$$v_1(c,e)=0.6, v_1(c,f)=0.9, v_1(d,e)=0.7, v_1(d,f)=0.4.$$

$$v_2(c,e)=0.8, v_2(c,f)=0.4, v_2(d,e)=0.6, v_2(d,f)=0.7.$$

$$(u_1\circ v_1)(a,e)=0.3,\ (u_1\circ v_1)(a,f)=0.6,$$

$$(u_1\circ v_1)(b,e)=0.4,\ (u_1\circ v_1)(b,f)=0.7$$

$$(u_2\circ v_2)(a,e)=0.2,\ (u_2\circ v_2)(a,f)=0.3,$$

$$(u_2\circ v_2)(b,e)=0.3,\ (u_2\circ v_2)(b,f)=0.4$$

Hence

$$\begin{aligned} 0.15 &= E_*(u_1,u_2)\odot E_*(v_1,v_2)\\ &\le E_*(u_1\circ v_1, u_2\circ v_2)=0.7.\end{aligned}$$

**Theorem 3.8.** (1) The relation $I$ holds

$$I(u_2,u_1)\le I(u_1\Rightarrow v, u_2\Rightarrow v)$$

$$I(v_1,v_2)\le I(u\Rightarrow v_1, u\Rightarrow v_2)$$

for every $u,u_i\in L^{X\times Y}$ and $v,v_i\in L^{Y\times Z}$

(2) The relation $E_*$ holds

$$E_*(u_1,u_2)\le E_*(u_1\Rightarrow v, u_2\Rightarrow v)$$

$$E_*(v_1,v_2)\le E_*(u\Rightarrow v_1, u\Rightarrow v_2)$$

for every $u,u_i\in L^{X\times Y}$ and $v,v_i\in L^{Y\times Z}$

(3) The relation $E_*$ preserves $(*,\Rightarrow)$-equivalence relation.

*Proof.* (1) $I(u_2,u_1)\le I(u_1\Rightarrow v, u_2\Rightarrow v)$ from the following statements: for all $(x,z)\in(X,Z), y\in Y$,

$$\begin{aligned}
&I(u_2,u_1)\le \textstyle\bigwedge_{(x,z)\in(X,Z)}\big((u_1\Rightarrow v)(x,z)\to(u_2\Rightarrow v)(x,z)\big)\\
&\Leftrightarrow I(u_2,u_1)\le\big((u_1\Rightarrow v)(x,z)\to(u_2\Rightarrow v)(x,z)\big)\\
&\Leftrightarrow I(u_2,u_1)\odot(u_1\Rightarrow v)(x,z)\le(u_2\Rightarrow v)(x,z)\\
&\Leftrightarrow I(u_2,u_1)\odot(u_1\Rightarrow v)(x,z)\le\textstyle\bigwedge_y(u_2(x,y)\Rightarrow v(y,z))\\
&\Leftrightarrow u_2(x,y)\odot I(u_2,u_1)\odot(u_1\Rightarrow v)(x,z)\le v(y,z)
\end{aligned}$$

On the other hand, by Lemma 1.3(9),

$$\begin{aligned}
&u_2(x,y)\odot I(u_2,u_1)\odot(u_1\Rightarrow v)(x,z)\\
&=u_2(x,y)\odot\Big(\textstyle\bigwedge_{(x_1,y_1)\in(X,Y)}(u_2(x_1,y_1)\to u_1(x_1,y_1))\Big)\\
&\quad\odot\Big(\textstyle\bigwedge_{y_2\in Y}\Big(u_1(x,y_2)\to v(y_2,z)\Big)\\
&\le u_2(x,y)\odot\Big(u_2(x,y)\to u_1(x,y)\Big)\odot\Big(u_1(x,y)\to v(y,z)\Big)\\
&\le u_1(x,y)\odot\Big(u_1(x,y)\to v(y,z)\Big)\\
&\le v(y,z)
\end{aligned}$$

Similarly, $I(v_1,v_2)\le I(u\Rightarrow v_1, u\Rightarrow v_2)$.

(2) and (3) are similarly proved from Theorem 3.6.

□

**Theorem 3.9.** (1) The relation $I$ holds

$$I(u_1,u_2)\le I(u_1\Leftarrow v, u_2\Leftarrow v)$$

$$I(v_2,v_1)\le I(u\Leftarrow v_1, u\Leftarrow v_2)$$

for every $u,u_i\in L^{X\times Y}$ and $v,v_i\in L^{Y\times Z}$

(2) The relation $E_*$ holds

$$E_*(u_1,u_2)\le E_*(u_1\Leftarrow v, u_2\Leftarrow v)$$

$$E_*(v_1,v_2)\le E_*(u\Leftarrow v_1, u\Leftarrow v_2)$$

for every $u,u_i\in L^{X\times Y}$ and $v,v_i\in L^{Y\times Z}$

(3) The relation $E_*$ preserves $(*,\Leftarrow)$-equivalence relation.

*Proof.* It is similarly proved from Theorems 3.6 and 3.8.

□

**Theorem 3.10.** (1) The relation $I$ holds

$$I(u_2,u_1)\odot(u_1\Rightarrow v)\le u_2\Rightarrow v$$

$$I(u_1,u_2)\odot(u_1\Leftarrow v)\le u_2\Leftarrow v$$

for every $u_i\in L^{X\times Y}$ and $v\in L^{Y\times Z}$

(2) The relation $E_*$ holds

$$E_*(u_1,u_2)\le I(u_1\Leftrightarrow v, u_2\Leftrightarrow v)$$

$$E_*(v_1,v_2)\le I(u\Leftrightarrow v_1, u\Leftrightarrow v_2)$$

$$E_*(u_1,u_2)\odot E_*(v_1,v_2)\le I(u_1\Leftrightarrow v_1, u_2\Leftrightarrow v_2)$$

for every $u,u_i\in L^{X\times Y}$ and $v,v_i\in L^{Y\times Z}$

(3) The relation $E_\wedge$ preserves $(\wedge,\Leftrightarrow)$-equivalence relation.

*Proof.* (1) It is easy from Theorem 3.8 (1).

(2) $E_*(u_1,u_2)\le I(u_1\Leftrightarrow v, u_2\Leftrightarrow v)$ from:

$$\begin{aligned}
&E_*(u_1,u_2)\odot(u_1\Leftrightarrow v)(x,z)\\
&=\Big(I(u_2,u_1)*I(u_1,u_2)\Big)\odot\Big((u_1\Rightarrow v)(x,z)*(u_1\Leftarrow v)(x,z)\Big)\\
&\le\Big(I(u_2,u_1)\odot(u_1\Rightarrow v)(x,z)\Big)*\Big(I(u_1,u_2)\odot(u_1\Leftarrow v)(x,z)\Big)\\
&\le\Big((u_2\Rightarrow v)(x,z)\Big)*\Big((u_2\Leftarrow v)(x,z)\Big)\\
&=(u_2\Leftrightarrow v)(x,z)
\end{aligned}$$

Similarly, $E_*(v_1,v_2)\le I(u\Leftrightarrow v_1, u\Leftrightarrow v_2)$. It implies

$$\begin{aligned}
&E_*(u_1,u_2)\odot E_*(v_1,v_2)\\
&\le I(u_1\Leftrightarrow v_1,u_2\Leftrightarrow v_1)\odot I(u_2\Leftrightarrow v_1,u_2\Leftrightarrow v_2)\\
&\le I(u_1\Leftrightarrow v_1,u_2\Leftrightarrow v_2)
\end{aligned}$$

(3) By (2), we have

$$\begin{aligned}
&E_\wedge(u_1,u_2)\odot E_\wedge(v_1,v_2)\\
&\le I(u_1\Leftrightarrow v_1,u_2\Leftrightarrow v_2)\wedge I(u_2\Leftrightarrow v_2,u_1\Leftrightarrow v_1)
\end{aligned}$$

□

**Example 3.11.** In Examples 3.2 and 3.7, we have

$$(u_1 \Rightarrow v_1)(a,e) = 0.9, (u_1 \Rightarrow v_1)(a,f) = 0.9,$$

$$(u_1 \Rightarrow v_1)(b,e) = 0.8,\ (u_1 \Rightarrow v_1)(b,f) = 1$$

$$(u_1 \Leftarrow v_1)(a,e) = 0.8, (u_1 \Leftarrow v_1)(a,f) = 0.8,$$

$$(u_1 \Leftarrow v_1)(b,e) = 0.4,\ (u_1 \Leftarrow v_1)(b,f) = 0.7$$

We obtain

$$(u_1 \Leftrightarrow v_1)(a,e) = 0.85, (u_1 \Leftrightarrow v_1)(a,f) = 0.85,$$

$$(u_1 \Leftrightarrow v_1)(b,e) = 0.6,\ (u_1 \Leftrightarrow v_1)(b,f) = 0.85$$

Similarly,

$$(u_2 \Leftrightarrow v_2)(a,e) = 0.8, (u_2 \Leftrightarrow v_2)(a,f) = 0.95,$$

$$(u_2 \Leftrightarrow v_2)(b,e) = 0.8,\ (u_2 \Leftrightarrow v_2)(b,f) = 0.95$$

$$I(u_1 \Leftrightarrow v_1, u_2 \Leftrightarrow v_2) = 0.95$$

Hence

$$\begin{aligned} 0.15 &= E_*(u_1,u_2) \odot E_*(v_1,v_2) \\ &\le I(u_1 \Leftrightarrow v_1, u_2 \Leftrightarrow v_2) = 0.95 \end{aligned}$$

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