The Korean Journal of Applied Statistics Vol. 21, No. 3, 2008, pp. 485–495

Pricing Floating-Strike Lookback Options with Flexible Monitoring Periods

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Abstract

A floating-strike lookback call option gives the holder the right to buy at the lowest price of the underlying asset. Similarly, a floating-strike lookback put option gives the holder the right to sell at the highest price. This paper will present explicit pricing formulas for these floating-strike lookback options with flexible monitoring periods. The monitoring periods of these options start at an arbitrary date and end at another arbitrary date before maturity. Sections 3 and 4 assume that the underlying assets pay no dividends. In contrast, Section 5 will derive explicit pricing formulas for these options when their underlying asset pays dividends continuously at a rate proportional to its price.

Keywords: Lookback option, floating strike, Brownian motion.

1. Introduction

Lookback options are path-dependent contingent claims whose payoffs depend on the maximum(or minimum) of the underlying asset price over a certain period. A floating-strike lookback call option gives the holder the right to buy at the lowest price of the underlying asset. Similarly, a floating-strike lookback put option gives the holder the right to sell at the highest price. Goldman *et al.* (1979) derived explicit pricing formulas for floating-strike lookback options where the highest(or lowest) price of the underlying asset is attained during the whole life of the options. Since the payoffs of these options are greater than or equal to those of the corresponding plain-vanilla options, these options are more expensive. This makes the floating-strike lookback options less attractive to investors. Conze and Viswanathan (1991) derived explicit pricing formulas for partial floating-strike lookback options that give the holder the right to buy(or sell) at some percentage times the lowest(or highest) price. These options are less expensive than the floating-strike lookback options discussed by Goldman, *et al.* (1979).

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Heynen and Kat (1994) suggest a way of reducing the price of these partial floating-strike lookback options while preserving some of their good qualities. The solution, they say, lies in partial floating-strike lookback options whose monitoring period is not the entire life of the options but ends at an arbitrary date before the expiration date. If some investors believe that the underlying asset will increase(or decrease) between the beginning of the contract and the arbitrary date, these options will be very attractive.

However, those who have a specific view of the asset movement in a certain interval of the option life may be more interested in partial floating-strike lookback options whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity. If investors choose the monitoring period during which the underlying asset is believed to increase or decrease, they may not worry about the market entry problem. This paper will present explicit pricing formulas for these generalized options with flexible monitoring periods.

This paper is organized as follows. Section 3 and 4 will present explicit pricing formulas for the floating-strike lookback put and call options, respectively. Section 3 will show a duality relationship between the two options. In addition, Section 5 will derive explicit pricing formulas for these options when their underlying asset pays dividends continuously at a rate proportional to its price. These pricing formulas are generalization of the pricing formulas in Sections 3 and 4.

2. Esscher Transforms and Some Probability Distributions

This section describes the method of Esscher transforms developed by Gerber and Shiu (1994, 1996). Let $S(t)$ denote the time-t price of an equity. Assume that the equity is constructed with all dividends reinvested. Assume that for $t \geq 0$,

$$S(t) = S(0)e^{X(t)},$$

where $\{X(t)\}$ is a Brownian motion with drift μ and diffusion coefficient σ and $X(0) = 0$. Thus the Brownian motion is a stochastic process with independent and stationary increments and $X(t)$ has a normal distribution with mean μt and variance $\sigma^2 t$.

For a nonzero real number h, the moment generating function of $X(t)$, $\mathrm{E}[e^{hX(t)}]$, exists for all $t \geq 0$, because $\{X(t)\}$ is the Brownian motion as described above. The stochastic process

$$\{e^{hX(t)}E[e^{hX(1)}]^{-t}\}$$

is a positive martingale which can be used to define a new probability measure Q. We call Q the Esscher measure of parameter h.

For a random variable Y that is a real-valued function of $\{X(t),\ 0 \leq t \leq T\}$, the expectation of Y under the new probability measure Q is calculated as

$$E\left[Y\cdot\frac{e^{hX(T)}}{E\left[e^{hX(1)}\right]^T}\right], \tag{2.1}$$

which will be denoted by E[Y; h]. The risk-neutral Esscher measure is the Esscher measure of parameter $h = h^*$ under which the process $\{e^{-rt}S(t)\}$ is a martingale. Thus

$$E\left[e^{-rt}S(t);h^*\right] = S(0). \tag{2.2}$$

Therefore, h^* is the solution of

$$\mu + h^*\sigma^2 = r - \frac{\sigma^2}{2}. \tag{2.3}$$

For $t \geq 0$, the moment generating function of $X(t)$ under the Esscher measure of parameter h is

$$E\left[e^{zX(t)};h\right] = \exp\left\{(\mu + h\sigma^2)tz + \frac{\sigma^2 tz^2}{2}\right\}, \tag{2.4}$$

which implies that $X(t)$ has a normal distribution with mean $(\mu + h\sigma^2)t$ and variance $\sigma^2 t$ under the Esscher measure. It can be shown that the process $\{X(t)\}$ under the Esscher measure has independent and stationary increments. Thus, the process is a Brownian motion with drift $\mu + h\sigma^2$ and diffusion coefficient σ under the Esscher measure of parameter h.

Now, let us consider a special case of the factorization formula (Gerber and Shiu, 1994, p.177; 1996, p.188). For a random variable Y that is a real-valued function of $\{X(t),\ 0 \leq t \leq T\}$,

$$E\left[e^{cX(T)}Y;h\right] = E\left[e^{cX(T)};h\right]E\left[Y;h+c\right]. \tag{2.5}$$

In particular, for an event B whose condition is determined by $\{X(t),\ 0 \leq t \leq T\}$, the formula (2.5) can be expressed as follows:

$$E\left[e^{cX(T)}I(B);h\right] = E\left[e^{cX(T)};h\right]\Pr(B;h+c), \tag{2.6}$$

where $I(\cdot)$ denotes the indicator function and $\Pr(B;\ h)$ denotes the probability of the event B under the Esscher measure of parameter h.

Now, let us discuss distributions and calculate some expectations to derive the joint distribution function of random variables $X(T)$ and $M(s,\ t)$. For $0 \leq s \leq t$, let

$$M(s,t) = \max\{X(\tau),\ s \leq \tau \leq t\} \tag{2.7a}$$

be the maximum of the Brownian motion between time s and time t. For simplicity, let $M(t) = M(0,\ t)$. In addition, let

$$m(s,t) = \min\{X(\tau),\ s \leq \tau \leq t\}. \tag{2.7b}$$

For simplicity, let $m(t) = m(0, t)$.

Now, let $\boldsymbol{Z} = (Z_1, Z_2, Z_3)$ have a standard trivariate normal distribution with correlation coefficients $\mathrm{Corr}(Z_i, Z_j) = \rho_{ij}$ $(i, j = 1, 2, 3)$. The distribution function of the random vector $\boldsymbol{Z}$ is

$$\Phi_3(a, b, c; \rho_{12}, \rho_{13}, \rho_{23}) = \Pr(Z_1 \le a, Z_2 \le b, Z_3 \le c) \tag{2.8a}$$

and

$$\Phi_2(a, b; \rho_{12}) = \Pr(Z_1 \le a, Z_2 \le b). \tag{2.8b}$$

Let us calculate the expectations necessary for deriving the proposed lookback options. Let random variable X be normal with mean μ and variance σ^2. We assume that a, b, c, θ and θ_* are real numbers, $\sigma_1 > 0$ and $\sigma_2 > 0$. Then,

$$\begin{aligned} &E\left[e^{hX} I(X < a)\Phi_2\left(\frac{\theta X + b}{\sigma_1}, \frac{\theta_* X + b}{\sigma_2}; \rho\right)\right] \\ &= e^{h\mu + \frac{1}{2}h^2\sigma^2}\Phi_3\left(\frac{a - \mu_h}{\sigma}, \frac{b + \theta\mu_h}{\kappa}, \frac{c + \theta_*\mu_h}{\kappa}; -\theta\frac{\sigma}{\kappa}, -\theta_*\frac{\sigma}{\kappa_*}, \rho_*\right), \end{aligned} \tag{2.9}$$

where κ denotes $\sqrt{\theta^2\sigma^2 + \sigma_1^2}$, κ_* is $\sqrt{\theta_*^2\sigma^2 + \sigma_2^2}$, ρ_* is $(\rho\sigma_1\sigma_2 + \theta\theta_*\sigma^2)/(\kappa\kappa_*)$ and μ_h denotes $\mu + h\sigma^2$. In addition, we obtain another expectation,

$$\begin{aligned} &E\left[e^{hX} I(X > a)\Phi_2\left(\frac{\theta X + b}{\sigma_1}, \frac{\theta_* X + c}{\sigma_2}; \rho\right)\right] \\ &= e^{h\mu + \frac{1}{2}h^2\sigma^2}\Phi_3\left(-\frac{a - \mu_h}{\sigma}, \frac{b + \theta\mu_h}{\kappa}, \frac{c + \theta_*\mu_h}{\kappa}; \theta\frac{\sigma}{\kappa}, \theta_*\frac{\sigma}{\kappa_*}, \rho_*\right). \end{aligned} \tag{2.10}$$

In the particular case that c approaches infinity, it follows from (2.9) and (2.10) that

$$E\left[e^{hX} I(X < a)\Phi_2\left(\frac{\theta X + b}{\sigma_1}\right)\right] = e^{h\mu + \frac{1}{2}h^2\sigma^2}\Phi_2\left(\frac{a - \mu_h}{\sigma}, \frac{b + \theta\mu_h}{\kappa}; -\theta\frac{\sigma}{\kappa}\right), \tag{2.11}$$

$$E\left[e^{hX} I(X > a)\Phi\left(\frac{\theta X + b}{\sigma_1}\right)\right] = e^{h\mu + \frac{1}{2}h^2\sigma^2}\Phi_2\left(-\frac{a - \mu_h}{\sigma}, \frac{b + \theta\mu_h}{\kappa}; \theta\frac{\sigma}{\kappa}\right). \tag{2.12}$$

3. Floating-Strike Lookback Put Option

The proposed floating-strike lookback put option gives the holder the right to sell at some percentage of the highest price of the underlying asset attained in a certain interval of the option life. The floating-strike price is the greater of either some percentage times the maximum asset price or a minimum guaranteed strike price. The minimum guaranteed strike price might be interpreted as the maximum of the underlying asset price attained in the past.

Let us take a close look at the payoff of the floating-strike lookback put option. Assume that λ is the percentage over the highest price and L is used for the minimum guaranteed strike price. The payoff of this option is as follows:

$$S(0)\left(\lambda e^{\max(M(s,t),\,L)} - e^{X(T)}\right)_+. \tag{3.1}$$

Heynen and Kat (1994) assume that the percentage λ is less than or equal to one and greater than zero and that L is nonnegative. But the payoff (3.1) does not assume the magnitude of λ and the sign of L.

To simplify writing, we define all expectations in this and next sections as taken with respect to the risk-neutral measure. In other words, under this measure, the underlying stochastic process $\{X(\tau),\ \tau \geq 0\}$ is a Brownian motion with drift $r-\ \sigma^2/2$ and diffusion coefficient σ. By the fundamental theorem of asset pricing, the time-0 value of the payoff (3.1) is

$$S(0)e^{-rT}E\left[\left(\lambda e^{\max(M(s,t),\,L)} - e^{X(T)}\right)_+\right], \tag{3.2}$$

whose discounted expectation is a generalization of the partial floating-strike lookback put option (Heynen and Kat, 1997). Calculating this discounted expectation (3.2) seems to require much complicated and tedious integration, but conditional expectations, if obtained easily, can simplify and reduce many calculations. Lee (2003) calculated this discounted expectation (3.2). For a proof, see Lee (2003).

Therefore, the time-0 value of the floating-strike lookback put option with monitoring period from time s to time t,

$$\begin{aligned} S(0)e^{-rT}E\left[\left(\lambda e^{\max(M(s,t),\,L)} - e^{X(T)}\right)_+\right] &= S(0)\{\Phi(g_1)A + B\} \\ &=: V^{put}_{float}(S(0), \lambda, L, r, \sigma), \end{aligned} \tag{3.3}$$

where for $i = 1$ and 2, g_i denotes $[-L + \{r + (-1)^{i-1}1/2\sigma^2\}s]/(\sigma\sqrt{s})$. Here,

$$\begin{aligned} A := &- \Phi(k_1)\Phi\left(-e_1 + \frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \\ &+ \lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Phi_2\left(h_1 + \frac{\log\lambda}{\sigma\sqrt{T-s}}, -e_1 - \frac{\log\lambda}{\sigma\sqrt{T-t}}; -\sqrt{\frac{T-t}{T-s}}\right) \\ &- \lambda\frac{\sigma^2}{2r}e^{-r(T-s)}\Phi_2\left(k_1 - \frac{2r\sqrt{t-s}}{\sigma}, h_1 - \frac{2r\sqrt{T-s}}{\sigma} + \frac{\log\lambda}{\sigma\sqrt{T-s}}; \sqrt{\frac{t-s}{T-s}}\right) \\ &+ \lambda\left(1 + \frac{\sigma^2}{2r}\right)e^{-r(T-t)}\Phi(k_1)\Phi\left(-e_2 + \frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

$$
\begin{aligned}
&+ \lambda e^{-r(T-s)} \Phi_2 \left(-h_2 + \frac{\log \lambda}{\sigma\sqrt{T-s}}, -k_2; \sqrt{\frac{t-s}{T-s}} \right) \\
&- \Phi_2 \left(-h_1 + \frac{\log \lambda}{\sigma\sqrt{T-s}}, -k_1; \sqrt{\frac{t-s}{T-s}} \right), \qquad (3.4)
\end{aligned}
$$

where for $i = 1$ and 2, h_i is $[\{r + (-1)^{i-1} 1/2\sigma^2\}(T-s)]/(\sigma\sqrt{T-s})$ and k_i is $[\{r + (-1)^{i-1} 1/2\sigma^2\}(t-s)]/(\sigma\sqrt{t-s})$. In addition,

$$
\begin{aligned}
B :=& - \Phi_2 \left(-g_1, f_1; -\sqrt{\frac{s}{t}} \right) \Phi \left(-e_1 + \frac{\log \lambda}{\sigma\sqrt{T-t}} \right) \\
&+ \lambda^{2\frac{r}{\sigma^2}+1} \frac{\sigma^2}{2r} \Phi_3 \left(d_1 + \frac{\log \lambda}{\sigma\sqrt{T}}, -e_1 - \frac{\log \lambda}{\sigma\sqrt{T-t}}, -g_1; -\sqrt{1-\frac{t}{T}}, -\sqrt{\frac{s}{T}}, 0 \right) \\
&- \lambda \frac{\sigma^2}{2r} e^{-rT} e^{2\frac{r}{\sigma^2}L} \Phi_3 \left(f_1 - \frac{2r\sqrt{t}}{\sigma}, d_1 - \frac{2r\sqrt{T}}{\sigma} + \frac{\log \lambda}{\sigma\sqrt{T}}, -g_1 + \frac{2r\sqrt{s}}{\sigma}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{t}}, -\sqrt{\frac{s}{T}} \right) \\
&+ \lambda \left(1 + \frac{\sigma^2}{2r} \right) e^{-r(T-t+s)} \Phi_2 \left(-g_1, f_1; -\sqrt{\frac{s}{t}} \right) \Phi \left(-e_2 + \frac{\log \lambda}{\sigma\sqrt{T-t}} \right) \\
&+ \lambda e^{-rT} e^{L} \Phi_3 \left(-d_2 + \frac{\log \lambda}{\sigma\sqrt{T}}, -f_2, -g_2; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}} \right) \\
&- \Phi_3 \left(-d_1 + \frac{\log \lambda}{\sigma\sqrt{T}}, -f_1, -g_1; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}} \right), \qquad (3.5)
\end{aligned}
$$

where for $i = 1$ and 2, d_i denotes $[-L + \{r + (-1)^{i-1} 1/2\sigma^2\}T]/(\sigma\sqrt{T})$, e_i is $[\{r + (-1)^{i-1} 1/2\sigma^2\}(T-t)]/(\sigma\sqrt{T-t})$ and f_i is $[-L + \{r + (-1)^{i-1} 1/2\sigma^2\}t]/(\sigma\sqrt{t})$.

4. Floating-Strike Lookback Call Option

The proposed floating-strike lookback call option gives the holder the right to buy at some percentage of the lowest price of the underlying asset attained in a certain interval of the option life. The floating-strike price is the smaller of either some percentage times the minimum asset price and a maximum guaranteed strike price. The maximum guaranteed strike price might be interpreted as the minimum underlying asset price attained in the past. This section will derive an explicit pricing formula for the floating-strike lookback call option whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity.

Let us take a close look at the payoff of the floating-strike lookback call option. Assume that λ is some percentage and L is used for the maximum guaranteed strike price that can be interpreted as the highest asset price in a period of the past. The payoff of this option is

$$
S(0) \left(e^{X(T)} - \lambda e^{\min(m(s,t), L)} \right)_+ . \qquad (4.1)
$$

By the fundamental theorem of asset pricing, the time-0 value of the payoff is

$$S(0)e^{-rT}E\left[\left(e^{X(T)}-\lambda e^{\min(m(s,t),L)}\right)_+\right]. \tag{4.2}$$

Applying (2.11), (2.12) and (6.4) of Lee (2003), it can be shown that the time-0 value of the partial floating-strike lookback option call is

$$\begin{aligned}
&e^{-rT}E\left[\left(e^{X(T)}-\lambda e^{\min(m(t),L)}\right)_+\right]\\
&=-\Bigg\{-\Psi(f_1)\Psi\left(-e_1+\frac{\log\lambda}{\sigma\sqrt{T-t}}\right)\\
&\quad+\lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Psi_2\left(d_1+\frac{\log\lambda}{\sigma\sqrt{T}},-e_1-\frac{\log\lambda}{\sigma\sqrt{T-t}};-\sqrt{1-\frac{t}{T}}\right)\\
&\quad-\lambda\frac{\sigma^2}{2r}e^{-rT}e^{2\frac{r}{\sigma^2}L}\Psi_2\left(f_1-\frac{2r\sqrt{t}}{\sigma}.d_1-\frac{2r\sqrt{T}}{\sigma}+\frac{\log\lambda}{\sigma\sqrt{T}};\sqrt{\frac{t}{T}}\right)\\
&\quad+\lambda\left(1+\frac{\sigma^2}{2r}\right)e^{-r(T-t)}\Psi(f_1)\Psi\left(-e_2+\frac{\log\lambda}{\sigma\sqrt{T-t}}\right)\\
&\quad+\lambda e^{-rT}e^{L}\Psi_2\left(-d_2+\frac{\log\lambda}{\sigma\sqrt{T}},-f_2;\sqrt{\frac{t}{T}}\right)-\Psi_2\left(-d_1+\frac{\log\lambda}{\sigma\sqrt{T}},-f_1;\sqrt{\frac{t}{T}}\right)\Bigg\},
\end{aligned} \tag{4.3}$$

where $\Psi(x):=\Phi(-x)$ and $\Psi_2(x,\,y;\,\rho):=\Phi_2(-x,\,-y;\,\rho)$.

The time-0 price (4.2) can be expressed in the form of iterated expectations as follows:

$$e^{-rT}E\left[e^{X(s)}E\left[\left(e^{X(T)-X(s)}-\lambda e^{\min(m(s,t)-X(s),L-X(s))}\right)_+|X(s)\right]\right], \tag{4.4}$$

which can be decomposed into the sum of two terms,

$$\begin{aligned}
&e^{-rs}E\left[e^{X(s)}I\left(X(s)<L\right)e^{-r(T-s)}E\left[\left(e^{X(T)-X(s)}-\lambda e^{m(s,t)-X(s)}\right)_+|X(s)\right]\right]+e^{-rs}\\
&E\left[e^{X(s)}I\left(X(s)\geq L\right)e^{-r(T-s)}E\left[\left(e^{X(T)-X(s)}-\lambda e^{\min(m(s,t)-X(s),L-X(s))}\right)_+|X(s)\right]\right].
\end{aligned} \tag{4.5}$$

First, let us consider the first term of (4.5). Applying the fact that the random vector $(m(s,t)-X(s),\,X(T)-X(s))$ is independent of $X(s)$ and has the same distribution as the random vector $(m(t-s),\,X(T-s))$, we see that the first term of (4.5) can be the product of two discounted expectations

$$e^{-rs}E[e^{X(s)}I(X(s)<L)]e^{-r(T-s)}E[(e^{X(T-s)}-\lambda e^{m(t-s)})_+]. \tag{4.6}$$

Applying the factorization formula (2.6), we see that the first discounted expectation of (4.6) is

$$\begin{aligned}
e^{-rs}E[e^{X(s)}I(X(s)<L)]&=e^{-rs}E[e^{X(s)}]\Pr(X(s)<L;1)\\
&=\Psi(g_1).
\end{aligned} \tag{4.7}$$

From formula (4.5) with $L = 0$, $T = T - s$ and $t = t - s$, the second discounted expectation of (4.6) is

$$
\begin{aligned}
& e^{-r(T-s)}E\left[(e^{X(T-s)} - \lambda e^{m(t-s)})_+\right] \\
&= -\Bigg\{ - \Psi(k_1)\Psi\left(-e_1 + \frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \\
&\quad + \lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Psi_2\left(h_1 + \frac{\log\lambda}{\sigma\sqrt{T-t}}, -e_1 - \frac{\log\lambda}{\sigma\sqrt{T-t}}; -\sqrt{\frac{T-t}{T-s}}\right) \\
&\quad - \lambda\frac{\sigma^2}{2r}e^{-r(T-s)}\Psi_2\left(k_1 - \frac{2r\sqrt{t-s}}{\sigma}, h_1 - \frac{2r\sqrt{T-s}}{\sigma} + \frac{\log\lambda}{\sigma\sqrt{T-s}}; \sqrt{\frac{t-s}{T-s}}\right) \\
&\quad + \lambda\left(1 + \frac{\sigma^2}{2r}\right)e^{-r(T-t)}\Psi(k_1)\Psi\left(-e_2 + \frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \\
&\quad + \lambda e^{-r(T-s)}\Psi_2\left(-h_2 + \frac{\log\lambda}{\sigma\sqrt{T-s}}, -k_2; \sqrt{\frac{t-s}{T-s}}\right) \\
&\quad - \Psi_2\left(-h_1 + \frac{\log\lambda}{\sigma\sqrt{T-s}}, -k_1; \sqrt{\frac{t-s}{T-s}}\right)\Bigg\}. \qquad (4.8)
\end{aligned}
$$

Let us consider the second term of (4.5). It follows from the partial lookback option formula (4.5) with $L = L - X(s)$, $T = T - s$ and $t = t - s$ that the discounted conditional expectation on the second term of (4.5) is

$$
\begin{aligned}
& e^{-r(T-s)}E\left[\left(e^{X(T)-X(s)} - \lambda e^{\min(m(s,t)-X(s),L-X(s))}\right)_+ |X(s)\right] \\
&= -\Bigg\{ - \Psi(K_1)\Psi\left(-e_1 + \frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \\
&\quad + \lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Psi_2\left(D_1 + \frac{\log\lambda}{\sigma\sqrt{T-s}}, -e_1 - \frac{\log\lambda}{\sigma\sqrt{T-t}}; -\sqrt{\frac{T-t}{T-s}}\right) \\
&\quad - \lambda\frac{\sigma^2}{2r}e^{-r(T-s)}e^{2\frac{r}{\sigma^2}(L-X(s))}\Psi_2\left(F_1 - \frac{2r\sqrt{t-s}}{\sigma}, D_1 + \frac{\log\lambda - 2r(T-t)}{\sigma\sqrt{T-s}}; \sqrt{\frac{t-s}{T-s}}\right) \\
&\quad + \lambda\left(1 + \frac{\sigma^2}{2r}\right)e^{-r(T-t)}\Psi(F_1)\Psi\left(-e_2 + \frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \\
&\quad + \lambda e^{-r(T-s)}e^{L-X(s)}\Psi_2\left(-D_2 + \frac{\log\lambda}{\sigma\sqrt{T-t}}, -F_2; \sqrt{\frac{t-s}{T-s}}\right) \\
&\quad - \Psi_2\left(-D_1 + \frac{\log\lambda}{\sigma\sqrt{T-t}}, -F_1; \sqrt{\frac{t-s}{T-s}}\right)\Bigg\}, \qquad (4.9)
\end{aligned}
$$

where for $i = 1$ and 2, F_i denotes $[X(s) - L + \{r + (-1)^{i-1}1/2\sigma^2\}(t-s)]/(\sigma\sqrt{t-s})$ and D_i is $[X(s) - L + \{r + (-1)^{i-1}1/2\sigma^2\}(T-s)]/(\sigma\sqrt{T-s})$. Applying equations (2.10) and

(2.12), we can obtain the following formulas.

$$
\begin{aligned}
&E\left[e^{hX} I(X > a)\Psi_2\left(\frac{\theta X + b}{\sigma_1}, \frac{\theta_* X + c}{\sigma_2}; \rho\right)\right] \\
&= e^{h\mu + \frac{1}{2}h^2\sigma^2}\Psi_3\left(\frac{a - \mu_h}{\sigma}, \frac{b - \theta\mu_h}{\kappa}, \frac{c - \theta_*\mu_h}{\kappa_*}; -\theta\frac{\sigma}{\kappa}, -\theta_*\frac{\sigma}{\kappa_*}, \rho_*\right)
\end{aligned}
\tag{4.10}
$$

and

$$
E\left[e^{hX} I(X > a)\Psi\left(\frac{\theta X + b}{\sigma_1}\right)\right] = e^{h\mu + \frac{1}{2}h^2\sigma^2}\Psi_2\left(\frac{a - \mu_h}{\sigma}, \frac{b + \theta\mu_h}{\kappa}; -\theta\frac{\sigma}{\kappa}\right), \tag{4.11}
$$

where $\Psi_3(x, y, z; \rho_{12}, \rho_{13}, \rho_{23}) := \Phi_3(-x, -y, -z; \rho_{12}, \rho_{13}, \rho_{23})$. Thus, the second term of (4.5) is the discounted expectation of $e^{X(s)}I(X(s) > L)$ times (4.9). In other words, the expectation can be decomposed into the sum of six terms, which are expectations of $e^{X(s)}I(X(s) > L)$ times functions of the random variable $X(s)$ from each term of (4.9). Applying equation (4.10) or equation (4.11) to the six expectations, the second term of (4.5) becomes

$$
\begin{aligned}
&-\Bigg\{ -\Psi_2\left(-g_1, f_1; -\sqrt{\frac{s}{t}}\right)\Psi\left(-e_1 + -\frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \\
&+ \lambda^{2\frac{r}{\sigma^2}+1}\frac{\sigma^2}{2r}\Psi_3\left(d_1 + \frac{\log\lambda}{\sigma\sqrt{T}}, -e_1 - \frac{\log\lambda}{\sigma\sqrt{T-t}}, -g_1; -\sqrt{1 - \frac{t}{T}}, -\sqrt{\frac{s}{T}}, 0\right) \\
&- \lambda\frac{\sigma^2}{2r}e^{-rT}e^{2\frac{r}{\sigma^2}L}\Psi_3\left(f_1 - \frac{2r\sqrt{t}}{\sigma}, d_1 - \frac{2r\sqrt{T}}{\sigma} + \frac{\log\lambda}{\sigma\sqrt{T}}, -g_1 + \frac{2r\sqrt{s}}{\sigma}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{t}}, -\sqrt{\frac{s}{T}}\right) \\
&+ \lambda\left(1 + \frac{\sigma^2}{2r}\right)e^{-r(T-t+s)}\Psi_2\left(-g_1, f_1; -\sqrt{\frac{s}{t}}\right)\Psi\left(-e_2 + \frac{\log\lambda}{\sigma\sqrt{T-t}}\right) \\
&+ \lambda e^{-rT}e^{L}\Psi_3\left(-d_2 + \frac{\log\lambda}{\sigma\sqrt{T}}, -f_2, -g_2; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\
&- \Psi_3\left(-d_1 + \frac{\log\lambda}{\sigma\sqrt{T}}, -f_1, -g_1; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right)\Bigg\},
\end{aligned}
\tag{4.12}
$$

To calculate the time-0 value of the floating-strike lookback call option, the discounted expectation (4.2) is decomposed into the sum of the two discounted expectations of (4.5). The first discounted expectation is the product of (4.7) and (4.8) and the second discounted expectation is (4.12). Therefore, adding the two discounted expectations, we have the time-0 value of the floating-strike lookback call option with the monitoring period from time s to time t,

$$
\begin{aligned}
S(0)e^{-rT}E\left[\left(e^{X(T)} - \lambda e^{\min(m(s,t),\, L)}\right)_+\right] &= S(0)\{\Psi(g_1)(4.8) + (4.12)\} \\
&=: V^{call}_{float}(S(0), \lambda, L, r, \sigma).
\end{aligned}
\tag{4.13}
$$

Let us discuss a duality relationship between the floating-strike call and put options. If we take a close look at the two pricing formulas (3.3) and (4.13), we can observe that the call option formula (4.13) is -1 times the put option formula (3.3) with its components Φ, Φ_2 and Φ_3 replaced by Ψ, Ψ_2 and Ψ_3, respectively.

5. Continuous Constant-Yield Dividend

The previous sections have derived the explicit pricing formulas for the floating-strike lookback options whose underlying asset pays no dividends. The pricing formulas in Sections 3 and 4 can be extended to the case where the underlying asset pays dividends continuously at a rate proportional to its price. This section will derive explicit pricing formulas for this case.

Let $S(t)$ denote the time-t price of an underlying asset. Assume that δ is the constant nonnegative dividend yield rate such that the asset pays dividends $\delta S(t)dt$ between time t and time $t + dt$. The risk-neutral measure is the Esscher measure of parameter $h = h^{**}$ with respect to which the process $\{e^{-(r-\delta)t}S(t)\}$ is a martingale. Therefore, h^{**} is the solution of

$$\mu + h^{**}\sigma^2 = r - \delta - \frac{\sigma^2}{2}. \tag{5.1}$$

Note that the process $\{X(t)\}$ is a Brownian motion with drift $r-\delta-\sigma^2/2$ and diffusion coefficient σ under the risk-neutral measure. For further discussion, see Section 9 of Gerber and Shiu (1996).

By the fundamental theorem of asset pricing, the time-0 values of the payoffs (4.1) and (3.1) are

$$S(0)e^{-rT}E\left[\left(\lambda e^{\max(M(s,t),L)} - e^{X(T)}\right)_+; h^{**}\right] \tag{5.2}$$

and

$$S(0)e^{-rT}E\left[\left(e^{X(T)} - \lambda e^{\min(m(s,t),L)}\right)_+; h^{**}\right], \tag{5.3}$$

respectively, which are the same as the expectations (4.2) and (3.2) except that the underlying stochastic process is a Brownian motion with drift $r-\delta-\sigma^2/2$ and diffusion coefficient σ. The discounting factor e^{-rT} in the expectations (5.2) and (5.3) can be decomposed into the product of two terms $e^{-\delta T}$ and $e^{-(r-\delta)T}$. Thus we can see that the expectations of (5.2) and (5.3) times the discounting factor $e^{-(r-\delta)T}$ are the pricing formulas (4.2) and (3.2) with r equal to $r-\delta$. Therefore, the time-0 values of the floating-strike lookback call and put options are

$$e^{-\delta T}V^{call}_{float}(S(0), \lambda, L, r-\delta, \sigma) \tag{5.4}$$

and

$$e^{-\delta T} V_{float}^{put}(S(0), \lambda, L, r-\delta, \sigma) \tag{5.5}$$

respectively.

References

Conze, A. and Viswanathan (1991). Path dependent options: The case of lookback Options, *The Journal of Finance*, **46**, 1893–1907.

Gerber, H. U. and Shiu, E. S. W. (1994). Option pricing by esscher transforms, *Transactions of Society of Actuaries*, **46**, 99–140; Discussions 141–191.

Gerber, H. U. and Shiu, E. S. W. (1996). Actuarial bridges to dynamic hedging and option pricing, *Insurance: Mathematics and Economics*, **18**, 183–218.

Goldman, M. B., Sosin, H. B. and Gatto, M. A. (1979). Path dependent options: Buy at the low, sell at the high, *The Journal of Finance*, **34**, 1111–1127.

Heynen, R. C. and Kat, H. M. (1994). Partial barrier options, *The Journal of Financial Engineering*, **3**, 253–274.

Heynen, R. C. and Kat, H. M. (1997). *Look-Back Options Pricing and Applications*, In Exotic Options: The State of the Art, edited by Clewlow, C. and Strickland, C., International Thomson Besiness Press, London.

Lee, H. (2003). Pricing equity-indexed annuities with path-dependent options, *Insurance: Mathematics and Economics*, **33**, 677–690.

[2008 March Received , 2008 April Accepted]