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On the Flatness of Semi-Cubically Hyponormal Weighted Shifts

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Abstract. Let W_α be a weighted shift with positive weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. The semi-cubical hyponormality of W_α is introduced and some flatness properties of W_α are discussed in this note. In particular, it is proved that if $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $\alpha_{n+k} = \alpha_n$ for all $k \geq 1$.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is *weakly n-hyponormal* if $p(T)$ is hyponormal for any polynomial p with degree less than equal to n. And an operator T is *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial p. In particular, the weak 2-hyponormality (or weak 3-hyponormality) refered to as quadratical hyponormality (or cubical hyponormality, resp.), and has been considered in detail in [5], [6] and [8]. The flatness property makes an important role for detecting the bridges between subnormal and hyponormal operators. In [10] Stampfli proved that every subnormal weighted shift W_α with any two equal weights has a flatness property, i.e., it holds that if $\alpha_k = \alpha_{k+1}$ for some $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then $\alpha_1 = \alpha_2 = \cdots$. In [2], R. Curto proved that the 2-hyponormal weighted shift W_α with any two equal weights has flatness properties. And he obtained a quadratically hyponormal weighted shift W_α with first two equal weights but not the flatness property and gave a question: describe all quadratically hyponormal weighted shifts with first two equal weights, which can be applied to the detections of operator gaps (cf. [CuJ], [JuP1]). But it is still open whether there exists a cubically hyponormal weighted shift W_α with a weight sequence $\alpha : \alpha_0 = \alpha_1 < \alpha_2 < \cdots$. Also [1], Y. Choi proved that every polynomially

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hyponormal weighted shift W_α with any two equal weights has flatness properties.

In this note we introduce a semi-cubically hyponormal weighted shift (which will be defined below) and discuss their flatness properties. In particular, we proved that if W_α is a semi-cubically hyponormal weighted shift with $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $\alpha_{n+k} = \alpha_n$ for all $k \geq 1$. Hence this flatness property preserves in the case of cubical hyponormality. Some of the calculations in Section 2 were obtained through computer experiments using the software tool Mathematica [12].

2. Construction of formulas

For $A, B \in \mathcal{L}(\mathcal{H})$, we write $[A,B] := AB - BA$. Let W_α be a hyponormal weighted shift with a positive weight sequence $\alpha := \{\alpha_i\}_{i=0}^{\infty}$. Recall that a weighted shift W_α is *cubically hyponormal* if $W_\alpha + aW_\alpha^2 + bW_\alpha^3$ is hyponormal for any $a, b \in \mathbb{C}$ ([9]), i.e.,

$$D(a,b) := [(W_\alpha + aW_\alpha^2 + bW_\alpha^3)^*, W_\alpha + aW_\alpha^2 + bW_\alpha^3] \geq 0 \quad \text{for any } a, b \in \mathbb{C}.$$

Let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and let P_n be the orthogonal projection on $\vee_{i=0}^{n}\{e_i\}$. For $a, b \in \mathbb{C}$ we let

$$\begin{aligned} D_n(a,b) &= P_n[(W_\alpha + aW_\alpha^2 + bW_\alpha^3)^*, W_\alpha + aW_\alpha^2 + bW_\alpha^3]P_n \\ &= \begin{bmatrix} q_0 & r_0 & z_0 & 0 & & & & \\ \overline{r_0} & q_1 & r_1 & z_1 & 0 & & & \\ \overline{z_0} & \overline{r_1} & q_2 & r_2 & z_2 & 0 & & \\ 0 & \overline{z_1} & \overline{r_2} & q_3 & r_3 & z_3 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & z_{n-2} \\ & & & & \ddots & \ddots & \ddots & r_{n-1} \\ & & & & 0 & \overline{z_{n-2}} & \overline{r_{n-1}} & q_n \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} q_n &:= (\alpha_n^2 - \alpha_{n-1}^2) + (\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-2}^2\alpha_{n-1}^2)\,|a|^2 + (\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2)\,|b|^2, \\ r_n &:= \alpha_n(\alpha_{n+1}^2 - \alpha_{n-1}^2)\bar{a} + \alpha_n(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-1}^2\alpha_{n-2}^2)a\bar{b}, \\ z_n &:= \alpha_n\alpha_{n+1}(\alpha_{n+2}^2 - \alpha_{n-1}^2)\bar{b}. \end{aligned}$$

Then it is obvious that W_α is cubically hyponormal if and only if the pentadiagonal matrix $D_n(a,b) \geq 0$ for any $a, b \in \mathbb{C}$ and any $n \in \mathbb{N}$. To detect $D_n(a,b) \geq 0$ for any $a, b \in \mathbb{C}$ and any $n \in \mathbb{N}$, we consider $d_n(a,b) := \det D_n(a,b)$ and it follows from [11] that for $n \geq 5$,

$$\begin{aligned} d_n(a,b) &= \left(q_{n-1} - \frac{r_{n-3}r_{n-2}\overline{z_{n-3}}}{\left|r_{n-3}\right|^2}\right) d_{n-1} - \left(\left|r_{n-2}\right|^2 - \frac{q_{n-2}r_{n-3}r_{n-2}\overline{z_{n-3}}}{\left|r_{n-3}\right|^2}\right) d_{n-2} \\ &\quad - \left(\left|z_{n-3}\right|^2 q_{n-2} - r_{n-3}r_{n-2}\overline{z_{n-3}}\right) d_{n-3} \\ &\quad + \left|z_{n-4}\right|^2 \left(\left|z_{n-3}\right|^2 - \frac{q_{n-3}r_{n-3}r_{n-2}\overline{z_{n-3}}}{\left|r_{n-3}\right|^2}\right) d_{n-4} \\ &\quad + \frac{\left|z_{n-4}\right|^2 \left|z_{n-2}\right|^2 r_{n-3}r_{n-2}\overline{z_{n-3}}}{\left|r_{n-3}\right|^2} d_{n-5}. \end{aligned}$$

Hence if W_α is cubically hyponormal, then $d_n(a,b) \geq 0$ for any $a, b \in \mathbb{C}$.

Definition 2.1. (i) A weighted shift W_α is *semi-cubically hyponormal with type I* if $W_\alpha + sW_\alpha^3$ is hyponormal for any $s \in \mathbb{C}$.
(ii) A weighted shift W_α is *semi-cubically hyponormal with type II* if $W_\alpha^2 + sW_\alpha^3$ is hyponormal for any $s \in \mathbb{C}$.

2.1. Type I. Let $\{e_i\}_{i=0}^\infty$ be an orthonormal basis for $\mathcal{H}$ and let P_n be the orthogonal projection on $\vee_{i=0}^n \{e_i\}$. For $s \in \mathbb{C}$, we let

$$D_n^{[1]}(s) = P_n \left[\left(W_\alpha + sW_\alpha^3\right)^*, W_\alpha + sW_\alpha^3 \right] P_n.$$

It is obvious that a quadratically hyponormal weighted shift W_α is semi-cubically hyponormal with type I if and only if the matrix $D_n^{[1]}(s) \geq 0$ for any $s \in \mathbb{C}$ and $n \in \mathbb{N}$. Recall that for $s \in \mathbb{C}$,

$$\begin{aligned} D_n^{[1]}(s) &= P_n \left[\left(W_\alpha + sW_\alpha^3\right)^*, W_\alpha + sW_\alpha^3 \right] P_n \\ &= \begin{bmatrix} q_0 & 0 & z_0 & 0 & & & & \\ 0 & q_1 & 0 & z_1 & 0 & & & \\ \bar{z}_0 & 0 & q_2 & 0 & z_2 & 0 & & \\ 0 & \bar{z}_1 & 0 & q_3 & 0 & z_3 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & z_{n-2} \\ & & & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & & & 0 & \bar{z}_{n-2} & 0 & q_n \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} q_k &:= u_k + |s|^2 v_k, \quad |z_k|^2 := |s|^2 w_k, \\ u_k &:= \alpha_k^2 - \alpha_{k-1}^2, \quad v_k := \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 - \alpha_{k-3}^2\alpha_{k-2}^2\alpha_{k-1}^2, \\ w_k &:= \alpha_k^2\alpha_{k+1}^2(\alpha_{k+2}^2 - \alpha_{k-1}^2)^2, \end{aligned}$$

as usual, we set $\alpha_{-1} = \alpha_{-2} = \alpha_{-3} = 0$ for our convenience. To detect $D_n^{[1]}(s) \geq 0$ for any $s \in \mathbb{C}$ and any $n \in \mathbb{N}$, we consider $d_n^{[1]} := d_n^{[1]}(s) := \det D_n^{[1]}(s)$. Hence if W_α is semi-cubically hyponormal with type I, then $d_n^{[1]}(s) \geq 0$ for any $s \in \mathbb{C}$ and any $n \in \mathbb{N}$. Note that

$$d_n^{[1]}(t) = \sum_{i=0}^{n+1} c^{[1]}(n,i)t^i, \qquad \text{where} \quad t := |s|^2 .$$

For brevity, we will write $c(n,i)$ for $c^{[1]}(n,i)$ in this subsection without confusion.

Lemma 2.2 ([11]). *The following recursive relations hold.*

$$\begin{aligned} d_0^{[1]} &= u_0 + v_0 t, \\ d_1^{[1]} &= u_0u_1 + (v_0u_1 + u_0v_1)t + v_0v_1t^2, \\ d_n^{[1]} &= q_n d_{n-1}^{[1]} - |z_{n-2}|^2 q_{n-1} d_{n-3}^{[1]} + |z_{n-3}z_{n-2}|^2 d_{n-4}^{[1]} \quad (n \geq 2), \end{aligned}$$

where $d_{-1}^{[1]} = 1$ *and* $d_{-2}^{[1]} = 0$.

By direct computations, we obtain the following formulas.

Lemma 2.3. *The following formulas hold.*

(i) $c(0,1) = v_0,\ c(0,0) = u_0;\ c(1,2) = v_0v_1,\ c(1,1) = v_0u_1 + u_0v_1,\ c(1,0) = u_0u_1,$

(ii) $c(n,0) = u_0u_1 \cdots u_n,$

(iii) $c(n,1) = u_n c(n-1,1) + \frac{1}{u_{n-2}}(v_n u_{n-2} - w_{n-2})(u_0u_1 \cdots u_{n-2}u_{n-1}) \quad (n \geq 2),$

(iv) $c(n,n+1) = v_n v_{n-1} \cdots v_1 v_0,$

(v) $c(n,n) = u_n(v_{n-1} \cdots v_1v_0) + v_n c(n-1,n-1) - w_{n-2}v_{n-1}(v_{n-3} \cdots v_1v_0),$

(vi) $c(n,i) = u_n c(n-1,i) + v_n c(n-1,i-1)$
$\quad + w_{n-2}\left[w_{n-3}c(n-4,i-2) - v_{n-1}c(n-3,i-2) - u_{n-1}c(n-3,i-1)\right]$
$\quad (0 \leq i \leq n-1).$

Proof. Using formulas in Lemma 2.2 and comparing the coefficients of $d_n^{[1]} (n \geq 0)$, we can obtain formulas in this lemma. □

We now detect some flatness properties as following.

Theorem 2.4. *Let* W_α *be a semi-cubically hyponormal weighted shift with type I.*

Then we have the following assertions.

(i) *If* $\alpha_n = \alpha_{n+1}$ *for some* $n \geq 1$, *then* $\alpha_{n+k} = \alpha_n$ *for all* $k \geq 1$.

(ii) *If* $\alpha_n = \alpha_{n+1} = \alpha_{n+2} = \alpha_{n+3}$ *for some* $n \geq 2$, *then* $\alpha_1 = \alpha_2 = \alpha_3 \cdots$.

Proof. (i) We first claim that if $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \in \mathbb{N}_0$, then $\alpha_{n+k} = \alpha_n$ for all $k \geq 1$. For this proof, without loss of generality, we may assume that $n = 0$, i.e., $\alpha_0 = \alpha_1 = \alpha_2 = 1$. Then it is sufficient to show that $\alpha_3 = 1$. To do so, we detect the positivity of $D_n^{[1]}(t)$ for $t \geq 0$. Note that $c(n,0) = 0 = c(n,1)$. Since $d_n^{[1]}(t) \geq 0$ $(t \geq 0)$, the coefficient of t^2 which is the smallest order of $d_n^{[1]}(t)$ should be positive. By Lemma 2.3, we have

$$\begin{aligned} c(4,2) &= w_1w_2c(0,0) - w_2v_3c(1,0) + v_4c(3,1) - w_2u_3c(1,1) + u_4c(3,2) \\ &= -\alpha_4^2\left(\alpha_3^2 - 1\right)^3 \geq 0, \end{aligned}$$

which implies that $\alpha_3 = 1$. We now prove the assertion (i). Without loss of generality, we may assume $n = 1$, and $\alpha_1 = 1$, i.e., $\alpha_1 = \alpha_2 = 1$. And then it is sufficient to show that $\alpha_3 = 1$ or $\alpha_0 = 1$. Obviously $c(n,0) = 0$. The coefficient $c(n,1)$ of t which is the smallest order of $d_n(t)$ is positive, but since

$$\begin{aligned} c(4,1) &= u_4c\,(3,1) + \frac{1}{u_2}\left(v_4u_2 - w_2\right)\left(u_0u_1u_2u_3\right) \\ &= \alpha_4^2\alpha_0^2\left(1 - \alpha_3^2\right)^3\left(1 - \alpha_0^2\right), \end{aligned}$$

we have $\alpha_0 = 1$ or $\alpha_3 = 1$. Hence by the above claim we have this conclusion.

(ii) Without loss of generality, we may assume $n = 2$, i.e., $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1$. Then it is sufficient to show that $\alpha_1 = 1$. First note that $c(n,0) = 0$. By Lemma 2.3, we have $c(3,1) = \alpha_0^4\left(\alpha_1^2 - 1\right)^3 \geq 0$, and so $\alpha_1 \geq 1$. Hence $\alpha_1 = 1$. □

2.2. Type II. We use the same idea with type I. By direct calculation, we have that

$$D_n^{[2]} := D_n^{[2]}(s) = \left[\left(W_\alpha^2 + sW_\alpha^3\right)^*, \left(W_\alpha^2 + sW_\alpha^3\right)\right]$$

$$= \begin{bmatrix} \omega_0 & \bar{\phi}_0 & 0 & & & & & \\ \phi_0 & \omega_1 & \bar{\phi}_1 & 0 & & & & \\ 0 & \phi_1 & \omega_2 & \bar{\phi}_2 & 0 & & & \\ & 0 & \phi_2 & \omega_3 & \bar{\phi}_3 & 0 & & \\ & & 0 & \phi_3 & \omega_4 & \bar{\phi}_4 & \ddots & \\ & & & 0 & \phi_4 & \omega_5 & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where

$$\begin{aligned}\omega_k &:= \xi_k + |s|^2 \eta_k,\\ \phi_k &:= s\sqrt{\delta_k},\\ \xi_k &:= \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-2}^2\alpha_{k-1}^2,\\ \eta_k &:= \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 - \alpha_{k-3}^2\alpha_{k-2}^2\alpha_{k-1}^2,\\ \delta_k &:= \alpha_k^2\left(\alpha_{k+1}^2\alpha_{k+2}^2 - \alpha_{k-2}^2\alpha_{k-1}^2\right)^2 \qquad (k \geq 0),\end{aligned}$$

and $\alpha_{-1} = \alpha_{-2} = \alpha_{-3} := 0$. Clearly, $W_\alpha^2 + sW_\alpha^3$ is hyponormal if and only if $D_n^{[2]}(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$. Let $d_n^{[2]}(\cdot) := \det(D_n^{[2]}(\cdot))$. Then

$$\begin{aligned}d_0^{[2]} &= \omega_0,\\ d_1^{[2]} &= \omega_0\omega_1 - |\phi_0|^2,\\ d_{n+2}^{[2]} &= \omega_{n+2}d_{n+1}^{[2]} - |\phi_{n+1}|^2 d_n^{[2]} \quad (n \geq 0),\end{aligned}$$

and that $d_n^{[2]}$ is actually a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin expansion $d_n^{[2]}(t) := \sum_{i=0}^{n+1} c^{[2]}(n,i)\, t^i$. This gives at once the following lemma. For brevity, we will write $c(n,i)$ for $c^{[2]}(n,i)$ in Subsection 2.2 without confusion.

Lemma 2.5. *It holds that*

(i) $c(0,0) = \xi_0, \quad c(0,1) = \eta_0,$

(ii) $c(1,0) = \xi_1\xi_0, \quad c(1,1) = \xi_1\eta_0 + \xi_0\eta_1, \quad c(1,2) = \eta_1\eta_0,$

(iii) $c(n,0) = \xi_0\xi_1\cdots\xi_n > 0,$

(iv) $c(n,n+1) = \eta_0\eta_1\cdots\eta_n > 0,$

(v) $c(n+2,i) = \xi_{n+2}c(n+1,i) + \eta_{n+2}c(n+1,i-1) - \delta_{n+1}c(n,i-1)$
$(n \geq 0, \quad 0 \leq i \leq n+3).$

Proof. Repeat the methods which were used in Lemma 2.3. □

Theorem 2.6. *If W_α is semi-cubically hyponormal with type II such that $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then $\alpha_{n+k} = \alpha_n$ for all $k \geq 1$.*

Proof. We first prove that if $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some n, then $\alpha_{n+k} = \alpha_n$ for all $k \geq 1$. For this proof, without loss of generality, we assume that $n = 0$, i.e., $\alpha_0 = \alpha_1 = \alpha_2 = 1$. Then it is sufficient to show that $\alpha_3 = 1$. In fact,

$$c(6,0) = -\alpha_5^2\alpha_4^2\left(\alpha_3^2 - 1\right)\left(\alpha_3^2\alpha_4^2 - 1\right)\left(\alpha_4^2\alpha_5^2 - \alpha_3^2\right)\left(\alpha_5^2\alpha_6^2 - \alpha_3^2\alpha_4^2\right),$$

and $d_6^{[2]}(t) \geq 0$ for all $t \geq 0$, we must have $\alpha_3 = 1$. By using this claim, we prove this theorem. Without loss of generality, we may assume $n = 1$, and $\alpha_1 = 1$, i.e., $\alpha_1 = \alpha_2 = 1$. By the positivity of $c(6,0)$, we obtain $\alpha_3 = \alpha_0 = 1$. Hence the above assertion about three equal weights proves this theorem. □

Theorems 2.4 and 2.6 provide a question about the flatness of semi-cubically hyponormal weighted shifts as the following problem.

Problem 2.7. Let W_α be a semi-cubically hyponormal weighted shift with a weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ satisfying $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$. Does it hold that $\alpha_1 = \alpha_2 = \alpha_3 \cdots$?

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