KYUNGPOOK Math. J. 48(2008), 705-720

On the Hyers-Ulam-Rassias Stability of the Bi-Jensen Functional Equation

Kil-Woung Jun and Mi-Hyen Han
Department of Mathematics, Chungnam National University, Daejon 305-764, Korea
e-mail : kwjun@cnu.ac.kr *and* hmh1014@hotmail.com

Yang-Hi Lee
Department of Mathematics Education, Gongju National University of Education, Gongju 314-711, Korea
e-mail : yanghi2@hanmail.net

Abstract. In this paper, we obtain the Hyers-Ulam-Rassias stability of a bi-Jensen functional equation $4f(\frac{x+y}{2}, \frac{z+w}{2}) = f(x,z) + f(x,w) + f(y,z) + f(y,w)$.

1. Introduction

In 1940, S.M. Ulam [13] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D.H.Hyers [3] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th.M.Rassias [12] gave a generalization. Since then, the further generalization has been extensively investigated by a number of mathematicians [2], [6-10].

Throughout this paper, let X and Y be a normed space and a Banach space, respectively. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x) + g(y)$ (respectively $2g(\frac{x+y}{2}) = g(x) + g(y)$).

Received July 2, 2007.
2000 Mathematics Subject Classification: 39B52.
Key words and phrases: Hyers-Ulam-Rassias stability, bi-Jensen mapping, functional equation.

For a given mapping $f : X \times X \to Y$, we define

$$\begin{aligned} Jf(x,y,z,w) &:= 4f(\tfrac{x+y}{2}, \tfrac{z+w}{2}) - f(x,z) - f(x,w) - f(y,z) - f(y,w), \\ C_1 f(x,y,z) &:= f(x+y,z) - f(x,z) - f(y,z), \\ C_2 f(x,y,z) &:= f(x,y+z) - f(x,y) - f(x,z), \\ J_1 f(x,y,z) &:= 2f(\tfrac{x+y}{2}, z) - f(x,z) - f(y,z), \\ J_2 f(x,y,z) &:= 2f(x, \tfrac{y+z}{2}) - f(x,y) - f(x,z) \end{aligned}$$

for all $x, y, z, w \in X$. A mapping $f : X \times X \to Y$ is called a biadditive (Cauchy-Jensen, Jensen-Cauchy, bi-Jensen, respectively) mapping if f satisfies the functional equations $C_1 f = 0$ and $C_2 f = 0 (C_1 f = 0$ and $J_2 f = 0$, $C_2 f = 0$ and $J_1 f = 0$, $J_1 f = 0$ and $J_2 f = 0$, respectively).

In 2006, Park and Bae [1], [11] obtained the generalized Hyers-Ulam stability of Cauchy-Jensen mapping, bi-Jensen mapping and in 2007, Jun et al. [4], [5] improved the Park and Bae's results. It is easy to see that a mapping $f : X \times X \to Y$ is a bi-Jensen mapping if and only if the mapping f satisfies the functional equation $Jf(x,y,z,w) = 0$ for all $x, y, z, w \in X$.

In this paper, we investigate the Hyers-Ulam-Rassias stability of a bi-Jensen functional equation.

2. Stability of a bi-Jensen functional equation

We establish the basic properties of a bi-Jensen mapping in the following lemma.

Lemma 1. *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then*

$$\begin{aligned} f(x,y) &= \frac{1}{2^n} f(x, 2^n y) + (1 - \frac{1}{2^n}) f(x, 0), \\ f(x,y) &= \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n}(1 - \frac{1}{2^n}) f(0, 2^n y) + (1 - \frac{1}{2^n})^2 f(0,0) \\ &= \frac{1}{2^n} f(2^n x, y) + \frac{2^n - 1}{2^{2n+1}} (f(x, 2^n y) + f(-x, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0), \\ f(x,y) &= 2^n f(\frac{x}{2^n}, y) + 2^n (1 - 2^n) f(0, \frac{y}{2^n}) + (1 - 2^n)^2 f(0,0), \\ f(x,y) &= \frac{1}{2^n} f(2^n x, y) + (1 - \frac{1}{2^n}) 2^n f(0, \frac{y}{2^n}) + (1 - \frac{1}{2^n})(1 - 2^n) f(0,0) \end{aligned}$$

for all $x, y \in X$ and $n \in N$.

Proof. Since the equalities

$$\begin{aligned}
\frac{1}{2}J_1 f(2x,0,y) &= f(x,y) - \frac{1}{2}f(2x,y) - \frac{1}{2}f(0,y) = 0,\\
\frac{1}{2}J_2 f(x,2y,0) &= f(x,y) - \frac{1}{2}f(x,2y) - \frac{1}{2}f(x,0)) = 0,\\
J_1 f(x,0,y) &= 2f(\frac{x}{2},y) - f(x,y) - f(0,y) = 0,\\
J_2 f(x,y,0) &= 2f(x,\frac{y}{2}) - f(x,y) - \ f(x,0)) = 0
\end{aligned}$$

hold for all $x, y \in X$, we get

$$\begin{aligned}
f(x,y) - f(0,y) &= \frac{1}{2}(f(2x,y) - f(0,y)),\\
f(x,y) - f(x,0) &= \frac{1}{2}(f(x,2y) - f(x,0)),\\
f(x,y) - f(0,y) &= 2(f(\frac{x}{2},y) - f(0,y)),\\
f(x,y) - f(x,0) &= 2(f(x,\frac{y}{2}) - f(x,0))
\end{aligned}$$

for all $x, y \in X$. By induction on n, we have

$$\begin{aligned}
f(x,y) - f(0,y) &= \frac{1}{2^n}f(2^n x,y) - \frac{1}{2^n}f(0,y),\\
f(x,y) - f(x,0) &= \frac{1}{2^n}f(x,2^n y) - \frac{1}{2^n}f(x,0),\\
f(x,y) - f(0,y) &= 2^n f(\frac{x}{2^n},y) - 2^n f(0,y),\\
f(x,y) - f(x,0) &= 2^n f(x,\frac{y}{2^n}) - 2^n f(x,0)
\end{aligned}$$

for all $x, y \in X$. So the equalities

$$\begin{aligned}
f(x,y) &= \frac{1}{2^n}(f(2^n x,y) + (1 - \frac{1}{2^n})f(0,y)),\\
f(x,y) &= \frac{1}{2^n}f(x,2^n y) + (1 - \frac{1}{2^n})f(x,0),\\
f(0,y) &= \frac{1}{2^n}f(0,2^n y) + (1 - \frac{1}{2^n})f(0,0),\\
f(x,y) &= 2^n f(\frac{x}{2^n},y) + (1 - 2^n)f(0,y),\\
f(0,y) &= 2^n f(0,\frac{y}{2^n}) + (1 - 2^n)f(0,0)
\end{aligned}$$

hold for all $x, y \in X$ and $n \in N$.
By the above equalities, we easily obtain the desired results. □

Proposition 2. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_1, \delta_2, \delta_3, \delta_4$ be fixed positive real numbers with $p_1, p_2, p_3, q_1, q_2, q_3 < 1$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|J_1 f(x, y, z)\| \le (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)(\|z\|^{q_1} + \delta_2), \tag{1}$$

$$\|J_2 f(x, y, z)\| \le (\|x\|^{p_3} + \delta_3)(\|y\|^{q_2} + \|z\|^{q_3} + \delta_4), \tag{2}$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique Jensen-Cauchy mapping $F_2 : X \times X \to Y$ such that

$$\|f(x, y) - f(0, y) - F_1(x, y)\| \le (\frac{2^{p_1}}{2 - 2^{p_1}}\|x\|^{p_1} + \delta_1)(\|y\|^{q_1} + \delta_2), \tag{3}$$

$$\|f(x, y) - f(x, 0) - F_2(x, y)\| \le (\|x\|^{p_3} + \delta_3)(\frac{2^{q_2}}{2 - 2^{q_2}}\|y\|^{q_2} + \delta_4), \tag{4}$$

$$F_1(x, y) - F_1(x, 0) = F_2(x, y) - F_2(0, y) \tag{5}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x, y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y), \quad F_2(x, y) := \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y)$$

for all $x, y \in X$.

Proof. Letting $y = 0$ and replacing x, z by $2^{j+1}x, y$ in (1), we have

$$\begin{aligned} & \|\frac{1}{2^j}(f(2^j x, y) - f(0, y)) - \frac{1}{2^{j+1}}(f(2^{j+1}x, y) - f(0, y))\| \\ = \ & \frac{1}{2^{j+1}}\|J_1 f(2^{j+1}x, 0, y)\| \\ \le \ & ((\frac{2^{p_1}}{2})^{j+1}\|x\|^{p_1} + \frac{1}{2^{j+1}}\delta_1)(\|y\|^{q_1} + \delta_2) \end{aligned}$$

for all $x, y \in X$ and $j \in N$. For given integers l, m $(0 \le l < m)$,

$$\begin{aligned} & \|\frac{1}{2^l}(f(2^l x, y) - f(0, y)) - \frac{1}{2^m}(f(2^m x, y) - f(0, y))\| \\ \le \ & \sum_{j=l}^{m-1}((\frac{2^{p_1}}{2})^{j+1}\|x\|^{p_1} + \frac{1}{2^{j+1}}\delta_1)(\|y\|^{q_1} + \delta_2) \end{aligned}$$

for all $x, y \in X$. By $p_1 < 1$, the sequence $\{\frac{1}{2^j}(f(2^j x, y) - f(0, y))\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{2^j}(f(2^j x, y) - f(0, y))\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y) = \lim_{j\to\infty} \frac{1}{2^j}(f(2^j x, y) - f(0, y))$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (6), one can obtain the inequality (3).
By (1), (2) and the definition of F_1, we get

$$C_1F_1(x,y,z) = \lim_{j\to\infty} \frac{1}{2^j} J_1 f(2^j x, 2^j y, z) = 0,$$
$$J_2F_1(x,y,z) = \lim_{j\to\infty} \frac{1}{2^j} J_2 f(2^j x, y, z) = 0,$$
$$F_1(0,y) = 0$$

for all $x, y, z \in X$ and so F_1 is a Cauchy-Jensen mapping. Now, let $F_1' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (3). Then we have

$$\begin{aligned} \|F_1(x,y) - F_1'(x,y)\| &\le \frac{1}{2^n}\|f(2^n x, y) - f(0,y) - F_1(2^n x, y)\| \\ &+ \frac{1}{2^n}\|f(2^n x, y) - f(0,y) - F_1'(2^n x, y)\| \\ &\le (2(\frac{2^{p_1}}{2})^n \frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \frac{1}{2^n}\delta_1)(\|y\|^{q_1} + \delta_2) \end{aligned}$$

for all $x, y \in X$ and $n \in N$. As $n \to \infty$, we may conclude that $F_1(x,y) = F_1'(x,y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F_1 : X \times X \to Y$ is unique.

Next, by (2), one can obtain

$$\begin{aligned} (7)\qquad & \frac{1}{2^j}(f(x, 2^j y) - f(x,0)) - \frac{1}{2^{j+1}}\ (f(x, 2^{j+1}y) - f(x,0))\| \\ &= \|\frac{1}{2^{j+1}} J_2 f(x, 2^{j+1}y, 0)\| \\ &\le (\|x\|^{p_3} + \delta_3)((\frac{2^{q_2}}{2})^{j+1}\|y\|^{q_2} + \frac{1}{2^{j+1}}\delta_4) \end{aligned}$$

for all $x, y \in X$. By the same method as above, F_2 is a unique Jensen-Cauchy mapping which satisfies (4), where $F_2(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y)$ for all $x, y \in X$. Since F_1 is a Cauchy-Jensen and F_2 is a Jensen-Cauchy, the equalities

$$\begin{aligned} (8)\qquad F_1(x,y) - F_1(x,0) &= \frac{1}{2}(F_1(x,2y) - F_1(x,0)), \\ F_1(x,y) - F_1(x,0) &= \frac{1}{2^n}F_1(x, 2^n y) - \frac{1}{2^n}F_1(x,0), \\ F_2(x,y) &= \frac{1}{2^n}F_2(x, 2^n y) \end{aligned}$$

hold for all integer n and $x, y \in X$. Hence, the inequality

$$\begin{aligned}
& \|F_1(x,y) - F_1(x,0) - F_2(x,y) + F_2(0,y)\| \\
= \;& \frac{1}{2^n}\|F_1(x,2^n y) - F_1(x,0) - F_2(x,2^n y) + F_2(0,2^n y)\| \\
= \;& \frac{1}{2^n}[\|f(x,2^n y) - f(0,2^n y) - F_1(x,2^n y)\| + \|f(x,0) - f(0,0) - F_1(x,0)\| \\
+ \;& \|f(0,2^n y) - f(0,0) - F_2(0,2^n y)\| + \|f(x,2^n y) - f(x,0) - F_2(x,2^n y)\|] \\
\leq \;& (\frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \delta_1)((\frac{2^{q_1}}{2})^n\|y\|^{q_1} + \frac{2\delta_2}{2^n}) \\
+ \;& (\|x\|^{p_3} + 2\delta_3)((\frac{2^{q_2}}{2})^n\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \frac{\delta_4}{2^n})
\end{aligned}$$

holds for all $x, y \in X$ and $n \in N$. Taking $n \to \infty$ and using $q_1, q_2 < 1$, we have (5). □

Theorem 3. *Let $f, p_1, p_2, p_3,\ q_1, q_2, q_3, \delta_1, \delta_2, \delta_3, \delta_4$ be as in Proposition 2. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that*

$$\begin{aligned}
(9) \qquad & \|f(x,y) - F(x,y)\| \\
& \leq \min((\frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \delta_1)(\|y\|^{q_1} + \delta_2) + \delta_3(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4), \\
& (\frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \delta_1)\delta_2 + (\|x\|^{p_3} + \delta_3)(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4))
\end{aligned}$$

for all $x, y \in X$ with $f(0,0) = F(0,0)$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty}\frac{f(2^j x, y) + f(0, 2^j y)}{2^j} + f(0,0) = \lim_{j\to\infty}\frac{f(x, 2^j y) + f(2^j x, 0)}{2^j} + f(0,0)$$

for all $x, y \in X$.

proof. Let F_1, F_2 be as in Proposition 2. Then we can define the map F by

$$F(x,y) := F_1(x,y) + F_2(0,y) + f(0,0) = F_2(x,y) + F_1(x,0) + f(0,0)$$

and easily show that F is a bi-Jensen mapping. By (3) and (4), we get

$$\begin{aligned}
& \|f(x,y) - F(x,y)\| \\
& \quad = \|f(x,y) - F_2(0,y) - F_1(x,y) - f(0,0)\| \\
& \quad \leq \|f(x,y) - f(0,y) - F_1(x,y)\| + \|f(0,y) - f(0,0) - F_2(0,y)\| \\
& \quad \leq (\frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \delta_1)(\|y\|^{q_1} + \delta_2) + \delta_3(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4)
\end{aligned}$$

for all $x, y \in X$. Also, by the similar method, we get

$$\|f(x,y) - F(x,y)\| \leq (\frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \delta_1)\delta_2 + (\|x\|^{p_3} + \delta_3)(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4))$$

for all $x, y \in X$. By the above two inequalities, we get (9). Let F' be another bi-Jensen mapping satisfying (9) and the equality $F'(0,0) = f(0,0)$. By Lemma 1 and $F(0,0) = F'(0,0)$, we have

$$\begin{aligned}
&\|F(x,y) - F'(x,y)\| \\
&\quad \leq \|\frac{1}{2^n}F(2^n x, y) + \frac{1}{2^n}(1 - \frac{1}{2^n})F(0, 2^n y) - \frac{1}{2^n}F'(2^n x, y) - \frac{1}{2^n}(1 - \frac{1}{2^n})F'(0, 2^n y)\| \\
&\quad \leq \frac{1}{2^n}\|F(2^n x, y) - f(2^n x, y)\| + \frac{1}{2^n}\|f(2^n x, y) - F'(2^n x, y)\| \\
&\quad + \frac{1}{2^n}(\|F(0, 2^n y) - f(0, 2^n y)\| + \|f(0, 2^n y) - F'(0, 2^n y)\|) \\
&\quad \leq 2((\frac{2^{p_1}}{2})^n \frac{2^{p_1}}{2 - 2^{p_1}}\|x\|^{p_1} + \frac{\delta_1}{2^n})(\|y\|^{q_1} + \delta_2) + \frac{2\delta_3}{2^n}(\frac{2^{q_2}}{2 - 2^{q_2}}\|y\|^{q_2} + \delta_4) \\
&\quad + 2\delta_1((\frac{2^{q_1}}{2})^n\|y\|^{q_1} + \frac{\delta_2}{2^n}) + 2\delta_3((\frac{2^{q_2}}{2})^n \frac{2^{q_2}}{2 - 2^{q_2}}\|y\|^{q_2} + \frac{\delta_4}{2^n})
\end{aligned}$$

for all $x, y \in X$ and $n \in N$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Proposition 4. *Let p_1, p_2, p_3, q_1, q_2, q_3 be fixed positive real numbers with p_1, p_2, p_3, $q_1, q_2, q_3 > 1$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|J_1 f(x, y, z)\| \leq (\|x\|^{p_1} + \|y\|^{p_2})\|z\|^{q_1}, \tag{10}$$

$$\|J_2 f(x, y, z)\| \leq \|x\|^{p_3}(\|y\|^{q_2} + \|z\|^{q_3}) \tag{11}$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique Jensen-Cauchy mapping $F_2 : X \times X \to Y$ such that

$$\|f(x, y) - f(0, y) - F_1(x, y)\| \leq \frac{2^{p_1}}{2^{p_1} - 2}\|x\|^{p_1}\|y\|^{q_1}, \tag{12}$$

$$\|f(x, y) - f(x, 0) - F_2(x, y)\| \leq \frac{2^{q_2}}{2^{q_2} - 2}\|x\|^{p_3}\|y\|^{q_2}, \tag{13}$$

$$F_1(x, y) + f(0, y) = F_2(x, y) + f(x, 0) \tag{14}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x, y) := \lim_{j\to\infty} 2^j (f(\frac{x}{2^j}, y) - f(0, y)),$$

$$F_2(x, y) := \lim_{j\to\infty} 2^j (f(x, \frac{y}{2^j}) - f(x, 0))$$

for all $x, y \in X$.

Proof. Letting $y = 0$ and replacing x, z by $2^{j+1}x, y$ in (10), we have

$$\begin{aligned}
\|2^j (f(\frac{x}{2^j}, y) - f(0, y)) &- 2^{j+1}(f(\frac{x}{2^{j+1}}, y) - f(0, y))\| \\
&= 2^j \|J_1 f(\frac{x}{2^j}, 0, y)\| \leq (\frac{2}{2^{p_1}})^j \|x\|^{p_1}\|y\|^{q_1}
\end{aligned}$$

for all $x, y \in X$. For given integers l, m $(0 \le l < m)$,

$$\|2^l(f(\frac{x}{2^l}, y) - f(0, y)) - 2^m(f(\frac{x}{2^m}, y) - f(0, y))\| \le \sum_{j=l}^{m-1}(\frac{2}{2^{p_1}})^j \|x\|^{p_1}\|y\|^{q_1} \tag{15}$$

for all $x, y \in X$. By $p_1 > 1$, the sequence $\{2^j(f(\frac{x}{2^j}, y) - f(0, y))\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{2^j(f(\frac{x}{2^j}, y) - f(0, y))\}$ converges for all $x, y \in X$. Define $F_1 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j\to\infty} 2^j(f(\frac{x}{2^j}, y) - f(0, y))$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (15), one can obtain the inequality (12). By (10) and (11),

$$\begin{aligned} C_1F_1(x, y, z) &= \lim_{n\to\infty}(2^jJ_1f(\frac{x}{2^j}, \frac{y}{2^j}, z) - 2^jJ_1f(0, 0, z)) = 0,\\ J_2F_1(x, y, z) &= \lim_{n\to\infty}(2^jJ_2f(\frac{x}{2^j}, y, z) - 2^jJ_2f(0, y, z)) = 0,\\ F_1(0, y) &= 0 \end{aligned}$$

for all $x, y, z \in X$ and so F_1 is a Cauchy-Jensen mapping. Now, let $F_1' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (12). Then we have

$$\begin{aligned} \|F_1(x, y) - F_1'(x, y)\| &\le 2^n\|f(\frac{x}{2^n}, y) - f(0, y) - F_1(\frac{x}{2^n}, y)\|\\ &\quad + 2^n\|f(\frac{x}{2^n}, y) - f(0, y) - F_1'(\frac{x}{2^n}, y)\|\\ &\le (\frac{2}{2^{p_1}})^n \frac{2^{p_1}}{2^{p_1} - 2}\|x\|^{p_1}\|y\|^{q_1} \end{aligned}$$

for all $x, y \in X$ and $n \in N$. As $n \to \infty$, we may conclude that $F_1(x, y) = F_1'(x, y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F_1 : X \times X \to Y$ is unique.

Next, replacing z by 0 in (11), one can obtain

$$\|(f(x, y) - f(x, 0)) - 2(f(x, \frac{y}{2}) - f(x, 0))\| = \|J_2f(x, y, 0)\| \le \|x\|^{p_3}\|y\|^{q_2} \tag{16}$$

for all $x, y \in X$. By the same method for obtaining F_1, F_2 is a unique Jensen-Cauchy mapping which satisfies (13), where $F_2(x, y) := \lim_{j\to\infty} 2^j(f(x, \frac{y}{2^j}) - f(x, 0))$ for all $x, y \in X$. By (12) and (13), we have

$$\begin{aligned} f(x, 0) - f(0, 0) &= F_1(x, 0),\\ f(0, y) - f(0, 0) &= F_2(0, y) \end{aligned} \tag{17}$$

for all $x \in X$. By (8), (12), (13), and (17), the inequality

$$\begin{aligned}
&\|F_1(x,y) - F_1(x,0) - F_2(x,y) + F_2(0,y)\| \\
&\quad = \|2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x,0) - 2^n F_2(x, \frac{y}{2^n}) + 2^n F_2(0, \frac{y}{2^n})\| \\
&\quad = 2^n \|f(x, \frac{y}{2^n}) - f(0, \frac{y}{2^n}) - F_1(x, \frac{y}{2^n})\| \\
&\qquad + 2^n \|f(x, \frac{y}{2^n}) - f(x,0) - F_2(x, \frac{y}{2^n})\| \\
&\quad \le (\frac{2}{2^{q_1}})^n \frac{2^{p_1}}{2^{p_1} - 2} \|x\|^{p_1} \|y\|^{q_1} + (\frac{2}{2^{q_2}})^n \frac{2^{q_2}}{2^{q_2} - 2} \|x\|^{p_3} \|y\|^{q_2}
\end{aligned}$$

holds for all $x, y \in X$ and $n \in N$. Taking $n \to \infty$ and using $q_1, q_2 > 1$, we have (14). □

Theorem 5. *Let $f, p_1, p_2, p_3, q_1, q_2, q_3$ be as in Proposition 4. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that*

$$\begin{aligned}
(18) \qquad &\|f(x,y) - F(x,y)\| \\
&\quad \le \min(\frac{2^{p_1}}{2^{p_1} - 2} \|x\|^{p_1} \|y\|^{q_1}, \frac{2^{q_2}}{2^{q_2} - 2} \|x\|^{p_3} \|y\|^{q_2})
\end{aligned}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$\begin{aligned}
F(x,y) &:= \lim_{j \to \infty} [2^j f(\frac{x}{2^j}, y) - (2^j - 1) f(0,y)] \\
&= \lim_{j \to \infty} [2^j f(x, \frac{y}{2^j}) - (2^j - 1) f(x,0)]
\end{aligned}$$

for all $x, y \in X$.

Proof. Let F_1, F_2 be as in Proposition 4. Then we can define the map F by

$$F(x,y) := F_1(x,y) + f(0,y) = F_2(x,y) + f(x,0)$$

for all $x, y \in X$. From (17), we know that

$$F(x,y) = F_1(x,y) + F_2(0,y) + f(0,0)$$

and F is bi-Jensen mapping. By (12), (13) and the definition of F, we get (18).

If F' is another bi-Jensen mapping satisfying (18), then $F'(0,0) = f(0,0)$. By Lemma 1 and the equality $F(0,0) = F'(0,0)$, we have

$$\begin{aligned}
\|F(x,y) - F'(x,y)\| &\le \|2^n F(\frac{x}{2^n}, y) + 2^n(1 - 2^n) F(0, \frac{y}{2^n}) \\
&\quad - 2^n F'(\frac{x}{2^n}, y) - 2^n(1 - 2^n) F'(0, \frac{y}{2^n})\| \\
&\le 2^n \|F(\frac{x}{2^n}, y) - f(\frac{x}{2^n}, y)\| + 2^n \|f(\frac{x}{2^n}, y) - F'(\frac{x}{2^n}, y)\| \\
&\quad + 2^n \cdot 2^n (\|F(\frac{0}{2^n}, \frac{y}{2^n}) - f(\frac{0}{2^n}, \frac{y}{2^n})\| + \|f(\frac{0}{2^n}, \frac{y}{2^n}) - F'(\frac{0}{2^n}, \frac{y}{2^n})\|) \\
&\le 2(\frac{2}{2^{p_1}})^n \frac{2^{p_1}}{2^{p_1} - 2} \|x\|^{p_1} \|y\|^{q_1}
\end{aligned}$$

for all $x, y \in X$ and $n \in N$. Taking $n \to \infty$ and using $p_1 > 1$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Proposition 6. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_1, \delta_3$ be fixed positive real numbers with $p_1, p_2, p_3 < 1 < q_1, q_2, q_3$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|J_1 f(x,y,z)\| \le (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)\|z\|^{q_1},$$
$$\|J_2 f(x,y,z)\| \le (\|x\|^{p_3} + \delta_3)(\|y\|^{q_2} + \|z\|^{q_3})$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique Jensen-Cauchy mapping $F_2 : X \times X \to Y$ $F_2 : X \times X \to Y$ such that

$$\|f(x,y) - f(0,y) - F_1(x,y)\| \le (\frac{2^{p_1}}{2 - 2^{p_1}}\|x\|^{p_1} + \delta_1)\|y\|^{q_1}, \tag{19}$$

$$\|f(x,y) - f(x,0) - F_2(x,y)\| \le \frac{2^{q_2}}{2^{q_2} - 2}(\|x\|^{p_3} + \delta_3)\|y\|^{q_2}, \tag{20}$$

$$F_1(x,y) + F_2(0,y) + f(0,0) = F_2(x,y) + f(x,0) \tag{21}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x,y) = \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y), \tag{22}$$
$$F_2(x,y) = \lim_{j\to\infty} 2^j (f(x, \frac{y}{2^j}) - f(x,0))$$

for all $x, y \in X$.

proof. By the similar method in the proof of Proposition 2 and Proposition 4, there exist a unique Cauchy-Jensen mappings $F_1 : X \times X \to Y$ and a unique Jensen-Cauchy mapping $F_2 : X \times X \to Y$ satisfying (19) and (20) for all $x, y \in X$, where the mappings $F_1, F_2 : X \times X \to Y$ are given by (22). By (19),

$$f(x,0) - f(0,0) = F_1(x,0)$$

for all $x \in X$. By (8), (19), (20), and the above equality, the inequality

$$\begin{aligned}
&\|F_1(x,y) - F_1(x,0) - F_2(x,y) + F_2(0,y)\| \\
&\quad = \|2^n F_1(x, \frac{y}{2^n}) - 2^n F_1(x,0) - 2^n F_2(x, \frac{y}{2^n}) + 2^n F_2(0, \frac{y}{2^n})\| \\
&\quad = 2^n \|f(x, \frac{y}{2^n}) - f(0, \frac{y}{2^n}) - F_1(x, \frac{y}{2^n})\| + 2^n \|f(0, \frac{y}{2^n}) - f(0,0) - F_2(0, \frac{y}{2^n})\| \\
&\qquad + 2^n \|f(x, \frac{y}{2^n}) - f(x,0) - F_2(x, \frac{y}{2^n})\| + 2^n \|f(x,0) - f(0,0) - F_1(x,0)\| \\
&\quad \le (\frac{2}{2^{q_1}})^n (\frac{2^{p_1}}{2 - 2^{p_1}}\|x\|^{p_1} + \delta_1)\|y\|^{q_1} + (\frac{2}{2^{q_2}})^n \frac{2^{q_2}}{2^{q_2} - 2}(\|x\|^{p_3} + 2\delta_3)\|y\|^{q_2}
\end{aligned}$$

holds for all $x, y \in X$ and $n \in N$. Taking $n \to \infty$ and using $q_1, q_2 > 1$, we have (21). □

Theorem 7. *Let $f, p_1, p_2, p_3, q_1, q_2, q_3, \delta_1, \delta_3$ be as in Proposition 6. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \leq \min(\frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \delta_1)\|y\|^{q_1} + \frac{2^{q_2}}{2^{q_2}-2}\delta_3\|y\|^{q_2}, \frac{2^{q_2}}{2^{q_2}-2}(\|x\|^{p_3} + \delta_3)\|y\|^{q_2} \tag{23}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$\begin{aligned} F(x,y) &:= \lim_{j\to\infty}[\frac{1}{2^j}f(2^j x, y) + 2^j f(0, \frac{y}{2^j}) - (2^j - 1)f(0,0)] \\ &= \lim_{j\to\infty}[2^j f(x, \frac{y}{2^j}) - (2^j - 1)f(x,0)] \end{aligned}$$

for all $x, y \in X$.

Proof. Let F_1, F_2 be as in Proposition 6. Then we can define the map F by

$$F(x,y) := F_1(x,y) + F_2(0,y) + f(0,0) = F_2(x,y) + f(x,0)$$

and easily show that F is bi-Jensen mapping. By (19) and (20), we get

$$\begin{aligned} \|f(x,y) - F(x,y)\| &= \|f(x,y) - F_2(0,y) - F_1(x,y) - f(0,0)\| \\ &\leq \|f(x,y) - f(0,y) - F_1(x,y)\| + \|f(0,y) - f(0,0) - F_2(0,y)\| \\ &\leq (\frac{2^{p_1}}{2-2^{p_1}}\|x\|^{p_1} + \delta_1)\|y\|^{q_1} + \frac{2^{q_2}}{2^{q_2}-2}\delta_3\|y\|^{q_2} \end{aligned}$$

and

$$\begin{aligned} \|f(x,y) - F(x,y)\| &= \|f(x,y) - f(x,0) - F_2(x,y)\| \\ &\leq \frac{2^{q_2}}{2^{q_2}-2}(\|x\|^{p_3} + \delta_3)\|y\|^{q_2} \end{aligned}$$

for all $x, y \in X$. By the above inequalities, we get (23).

If F' is another bi-Jensen mapping satisfying (23), then $F'(0,0) = f(0,0)$ for all $x \in X$. By Lemma 1 and the equality $F(0,0) = F'(0,0)$, we have

$$\begin{aligned} \|F(x,y) - F'(x,y)\| &\leq \|\frac{1}{2^n}F(2^n x, y) + (1 - \frac{1}{2^n})2^n F(0, \frac{y}{2^n}) \\ &\quad - \frac{1}{2^n}F'(2^n x, y) - (1 - \frac{1}{2^n})2^n F'(0, \frac{y}{2^n})\| \\ &\leq \frac{1}{2^n}\|F(2^n x, y) - f(2^n x, y)\| + \frac{1}{2^n}\|f(2^n x, y) - F'(2^n x, y)\| \\ &\quad + 2^n\|F(0, \frac{y}{2^n}) - f(0, \frac{y}{2^n})\| + 2^n\|f(0, \frac{y}{2^n}) - F'(0, \frac{y}{2^n})\| \\ &\leq \frac{2\cdot 2^{q_2}}{2^{q_2}-2}((\frac{2^{p_3}}{2})^n\|x\|^{p_3} + \frac{\delta_3}{2^n})\|y\|^{q_2} + \frac{2\cdot 2^{q_2}\delta_3}{2^{q_2}-2}(\frac{2}{2^{q_2}})^n\|y\|^{q_2} \end{aligned}$$

for all $x, y \in X$ and $n \in N$. Taking $n \to \infty$ and using $p_3 < 1 < q_2$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Proposition 8. *Let $p_1, p_2, p_3, q_1, q_2, q_3, \delta_2, \delta_4$ be fixed positive real numbers with $p_1, p_2, p_3 > 1 > q_1, q_2, q_3$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|J_1 f(x, y, z)\| \le (\|x\|^{p_1} + \|y\|^{p_2})(\|z\|^{q_1} + \delta_2)$$
$$\|J_2 f(x, y, z)\| \le (\|x\|^{p_3})(\|y\|^{q_2} + \|z\|^{q_3} + \delta_4)$$

for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique Jensen-Cauchy mapping $F_2 : X \times X \to Y$ such that

$$\|f(x, y) - f(0, y) - F_1(x, y)\| \le \frac{2^{p_1}}{2^{p_1} - 2}\|x\|^{p_1}(\|y\|^{q_1} + \delta_2), \tag{24}$$

$$\|f(x, y) - f(x, 0) - F_2(x, y)\| \le \|x\|^{p_3}(\frac{2^{q_2}}{2 - 2^{q_2}}\|y\|^{q_2} + \delta_4), \tag{25}$$

$$F_1(x, y) + f(0, y) - f(0, 0) = F_2(x, y) + F_1(x, 0) \tag{26}$$

for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x, y) = \lim_{j\to\infty} 2^j (f(\frac{x}{2^j}, y) - f(0, y)), \tag{27}$$
$$F_2(x, y) = \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y)$$

for all $x, y \in X$.

proof. By the similar method in the proof of Proposition 2 and Proposition 4, there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique Jensen-Cauchy mapping $F_2 : X \times X \to Y$ satisfying (24) and (25) for all $x, y \in X$, where the mappings $F_1, F_2 : X \times X \to Y$ are given by (27). By (25),

$$f(0, y) - f(0, 0) = F_2(0, y)$$

for all $x \in X$. By (8), (24), and (25), the inequality

$$\begin{aligned}
&\|F_1(x, y) - F_1(x, 0) - F_2(x, y) + F_2(0, y)\| \\
&\quad = \|\frac{1}{2^n}F_1(x, 2^n y) - \frac{1}{2^n}F_1(x, 0) - \frac{1}{2^n}F_2(x, 2^n y) + \frac{1}{2^n}F_2(0, 2^n y)\| \\
&\quad = \frac{1}{2^n}\|f(x, 2^n y) - f(0, 2^n y) - F_1(x, 2^n y)\| + \frac{1}{2^n}\|f(0, 2^n y) - f(0, 0) - F_2(0, 2^n y)\| \\
&\qquad + \frac{1}{2^n}\|f(x, 2^n y) - f(x, 0) - F_2(x, 2^n y)\| + \frac{1}{2^n}\|f(x, 0) - f(0, 0) - F_1(x, 0)\| \\
&\quad \le \frac{2^{p_1}}{2^{p_1} - 2}\|x\|^{p_1}((\frac{2^{q_1}}{2})^n\|y\|^{q_1} + \frac{2\delta_2}{2^n}) + \|x\|^{p_3}((\frac{2^{q_2}}{2})^n\frac{2^{q_2}}{2 - 2^{q_2}}\|y\|^{q_2} + \frac{\delta_4}{2^n})
\end{aligned}$$

holds for all $x, y \in X$ and $n \in N$. Taking $n \to \infty$ and using $q_1, q_2 < 1$, we have (27). □

Theorem 9. *Let $f, p_1, p_2, p_3, q_1, q_2, q_3, \delta_2, \delta_4$ be as in Proposition 8. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that*

$$(28) \qquad \|f(x,y) - F(x,y)\| \le \min(\frac{2^{p_1}}{2^{p_1}-2}\|x\|^{p_1}(\|y\|^{q_1} + \delta_2), \frac{2^{p_1}}{2^{p_1}-2}\|x\|^{p_1}\delta_2 + \|x\|^{p_3}(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4))$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$\begin{aligned} F(x,y) &= \lim_{j\to\infty}[2^j f(\frac{x}{2^j}, y) - (2^j - 1)f(0,y) + f(0,0)] \\ &= \lim_{j\to\infty}[\frac{1}{2^j}f(x, 2^j y) + 2^j f(\frac{x}{2^j}, 0) - (2^j - 1)f(0,0)] \end{aligned}$$

for all $x, y \in X$.

proof. Let F_1, F_2 be as in Proposition 9. Then we can define the map F by

$$F(x,y) := F_1(x,y) + f(0,y) = F_2(x,y) + F_1(x,0) + f(0,0)$$

and easily show that F is bi-Jensen mapping. By (24) and (25), we get

$$\begin{aligned} \|f(x,y) - F(x,y)\| &= \|f(x,y) - f(0,y) - F_1(x,y)\| \\ &\le \frac{2^{p_1}}{2^{p_1}-2}\|x\|^{p_1}(\|y\|^{q_1} + \delta_2), \end{aligned}$$

and

$$\begin{aligned} \|f(x,y) - F(x,y)\| &= \|f(x,y) - F_1(x,0) - F_2(x,y) - f(0,0)\| \\ &\le \|f(x,y) - f(x,0) - F_2(x,y)\| + \|f(x,0) - f(0,0) - F_1(x,0)\| \\ &\le \frac{2^{p_1}}{2^{p_1}-2}\|x\|^{p_1}\delta_2 + \|x\|^{p_3}(\frac{2^{q_2}}{2-2^{q_2}}\|y\|^{q_2} + \delta_4), \end{aligned}$$

for all $x, y \in X$. By the above inequalities, we get (28). □

Let F' be another bi-Jensen mapping satisfying (28). Since $F - F'$ is a bi-Jensen mapping and $F(0,0) = F'(0,0)$, we can apply Lemma 1 and we get

$$\begin{aligned} \|(F-F')(x,0)\| &= \|(F-F')(x,0) - (F-F')(0,0)\| \\ &= 2^n\|(F-F')(\frac{x}{2^n}, 0) - (F-F')(0,0)\| \\ &= 2^n\|F(\frac{x}{2^n}, 0) - f(\frac{x}{2^n}, 0)\| + 2^n\|F'(\frac{x}{2^n}, 0) - f(\frac{x}{2^n}, 0)\| \\ &\le 2(\frac{2}{2^{p_1}})^n \frac{2^{p_1}}{2^{p_1}-2}\|x\|^{p_1}\delta_2 \end{aligned}$$

for all $x \in X$ and $n \in N$. Taking $n \to \infty$ and using (13),

$$(F - F')(x, 0) = 0$$

for all $x \in X$. By Lemma 1 and the above equality, we have

$$\begin{aligned}\|F(x,y) - F'(x,y)\| &\le \|\frac{1}{2^n}F(x,2^n y) + (1 - \frac{1}{2^n})F(x,0) \\ &\quad - \frac{1}{2^n}F'(x,2^n y) - (1 - \frac{1}{2^n})F'(x,0) \\ &\le \frac{1}{2^n}\|F(x,2^n y) - f(x,2^n y)\| + \frac{1}{2^n}\|f(x,2^n y) - F'(x,2^n y)\| \\ &\le (\frac{2}{2^{p_1}})^n \frac{2^{p_1}}{2^{p_1} - 2}\|x\|^{p_1}(\|y\|^{q_1} + \delta_2)\end{aligned}$$

for all $x, y \in X$ and $n \in N$. Taking $n \to \infty$,

$$F(x, y) - F'(x, y) = 0$$

for all $x, y \in X$ as desired. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. □

Corollary 10. *Let $p, q(\neq 1)$ be fixed positive real numbers and let $f : X \times X \to Y$ be a mapping such that*

$$\begin{aligned}\|J_1 f(x,y,z)\| &\le (\|x\|^p + \|y\|^p)\|z\|^q, \\ \|J_2 f(x,y,z)\| &\le \|x\|^p(\|y\|^q + \|z\|^q)\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \min(\frac{2^p}{|2 - 2^p|}, \frac{2^q}{|2 - 2^q|})\|x\|^p\|y\|^q$$

for all $x, y \in X$.

Corollary 11. *Let p_1, p_2 be positive real numbers satisfying one of the conditions $p_1, p_2 < 1$ or $1 < p_1, p_2$ and let p_3, p_4 be positive real numbers satisfying one of the conditions $p_3, p_4 < 1$ or $1 < p_3, p_4$. Let δ_1, δ_2 be nonnegative real numbers and let $f : X \times X \to Y$ be a mapping such that*

$$\|Jf(x,y,z,w)\| \le (\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)(\|z\|^{p_3} + \|w\|^{p_4} + \delta_2)$$

for all $x, y, z, w \in X$ where $\delta_1 = 0$ if $1 < p_1, p_2$ and $\delta_2 = 0$ if $1 < p_3, p_4$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\begin{aligned}&\|f(x,y) - F(x,y)\| \\ &\quad \le \min(\frac{1}{2}(\frac{2^{p_1}}{|2^{p_1} - 2|}\|x\|^{p_1} + \delta_1)(\|y\|^{p_3} + \|y\|^{p_4} + \delta_2) + \frac{1}{2}\delta_1(\frac{2^{p_3}}{|2^{p_3} - 2|}\|y\|^{p_3} + \delta_2), \\ &\qquad \frac{1}{2}(\|x\|^{p_1} + \|x\|^{p_2} + \delta_1)(\frac{2^{p_3}}{2^{p_3} - 2}\|y\|^{p_3} + \delta_2) + \frac{1}{2}(\frac{2^{p_1}}{|2 - 2^{p_1}|}\|x\|^{p_1} + \delta_1)\delta_2)\end{aligned}$$

for all $x, y \in X$ *with* $f(0,0) = F(0,0)$.

proof. From (31), we have

$$\|J_1 f(x,y,z)\| = \|\frac{1}{2} Jf(x,y,z,z)\| \leq \frac{1}{2}(\|x\|^{p_1} + \|y\|^{p_2} + \delta_1)(\|z\|^{p_3} + \|z\|^{p_4} + \delta_2),$$
$$\|J_2 f(x,y,z)\| = \|\frac{1}{2} Jf(x,x,y,z)\| \leq \frac{1}{2}(\|x\|^{p_1} + \|x\|^{p_2} + \delta_1)(\|y\|^{p_3} + \|z\|^{p_4} + \delta_2)$$

for all $x, y, z \in X$. For each case, we can apply the similar method used in the proof of Theorem 3, Theorem 5, Theorem 7 and Theorem 9 to get the results in this corollary. □

Corollary 12. *Let* ε *be a positive real number. Let* $f : X \times X \to Y$ *be a mapping such that*

$$\|Jf(x,y,z,w)\| \leq \varepsilon$$

for all $x, y, z, w \in X$. *Then there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$F(0,0) = f(0,0), \quad \|f(x,y) - F(x,y)\| \leq \varepsilon$$

for all $x, y \in X$.

Remark. Let $f, F, F' : X \times X \to Y$ be the bi-Jensen maps defined by

$$f(x,y) := 0, \quad F(x,y) := \varepsilon, \quad F'(x,y) := -\varepsilon$$

for all $x, y \in X$. Then f, F, F' satisfy the conditions in Corollary 12 but $F' \neq F$. Hence the condition $F(0,0) = f(0,0)$ is necessary to show that the map F is unique.
□

References

[1] J. H. Bae and W. G. Park, *On the solution of bi-Jensen functional equation and its stability*, Bull. Korean Math. Soc., **43**(2006), 499-507.

[2] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. and Appl.,**184**(1994), 431-436.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A., **27**(1941), 222-224.

[4] K. W. Jun, Y. H. Lee and Y. S. Cho, *On the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation, Abstract Appl. Anal.*, **ID 35151**(2007), 1-15.

[5] K. W. Jun, Y. H. Lee and J. H. Oh, *On the Rassias stability a bi-Jensen functional equation*, preprint.

[6] S. M. Jung, *Hyers-Ulam stability of linear differential equations of first order*, Jour Appl. Math. Lett., **17**(2004), 1135-1140.

[7] G. H. Kim. *On the stability of functional equations with square-symmetric operation*, Jour Math. Inequal. Appl., **4**(2001), 257-266.

[8] H. M. Kim, *On the stability problem for a mixed type of quartic and quadratic functional equation*, J. Math. Anal. Appl., **324**(2006), 358-372.

[9] Y. H. Lee and K. W. Jun, *On the stability of approximately additive mappings*, Jour Proc. Amer. Math. Soc., **128**(2000), 1361-1369.

[10] C. G. Park, *Linear functional equations in Banach modules over a C^*-algebra*, Acta Appl. Math., **77**(2003), 125-161.

[11] W. G. Park and J. H. Bae, *On a Cauchy-Jensen functional equation and its stability*, J. Math. Anal. Appl., **323**(2006), 634-643.

[12] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297-300.

[13] S. M. Ulam, A Collection of Mathematical Problems, Interscience, New York, 1968, p. 63.