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$MC2$ Rings

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Abstract. In this paper, we first study some characterizations of left $MC2$ rings. Next, by introducing left $nil-$injective modules, we discuss and generalize some well known results for a ring whose simple singular left modules are YJ-injective. Finally, as a byproduct of these results we are able to show that if R is a left $MC2$ left Goldie ring whose every simple singular left $R-$module is $nil-$injective and $GJcp-$injective, then R is a finite product of simple left Goldie rings.

Introduction

Throughout this paper R denotes an associative ring with identity, and $R-$modules are unital. For $a \in R$, $r(a)$ and $l(a)$ denote the right annihilator of a and the left annihilator of a, respectively. We write $J(R)$, $Z_l(R)$, $N(R)$, $N_1(R)$, $S_l(R)$, $P(R), K(R), B(R)$ and $BJ(R)$ for the Jacobson radical of R, the left singular ideal of R, the set of nilpotent elements of R, the set of non-nilpotent elements of R, the left socle of R, the prime radical, the Levitzki radical, antisimple radical and antisimple primitive radical, respectively. An element $k \in R$ is called left minimal if Rk is a minimal left ideal of R. An element $e \in R$ is called left minimal idempotent if e is a left minimal element and $e^2 = e$. An idempotent $e \in R$ is called left (right, resp) semicentral if, $ae = eae$ $(ea = eae)$ for all $a \in R$.

Recall that a ring R is left DS [1] if for every minimal element $k \in R$, Rk is a summand of $_RR$. These rings are also called left universally mininjective by W.K. Nicholson and M.F. Yousif in [3]. There, they proved that R is left DS ring iff $S_l(R) \cap J(R) = 0$. And, in [1], we give a lot of characterizations of left DS rings. For example, R is left DS ring iff R is left PS ring and left mininjective ring, where left mininjective rings are defined by W.K. Nicholson and M.F. Yousif in [3].

Call that a ring R is left $MC2$ if every left minimal idempotent element $e \in R$ is right minimal. In Theorem 1.1, we show that R is left $MC2$ ring if and only if for any left minimal elements $k, g^2 = g \in R$, $Rk \cong Rg$ as left $R-$module always implies

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$Rk = Re, e^2 = e \in R$. This characterization is equivalent to the definition appeared in [1]. In fact, in [1], we have shown that R is left $MC2$ ring iff $Z_l(R) \cap S_l(R) = J(R) \cap S_l(R)$. And, there, we have also shown that left DS rings are left $MC2$. Hence, we can easy see that R is left DS ring iff R is left PS ring and left $MC2$ ring. On the other hand, by [3, Proposition 1.11], we can see that left mininjective rings are left $MC2$.

In section 1, we first characterize left $MC2$ rings. Next, we generalize some results of left minsymmetric rings, left $C2$ rings to left $MC2$ rings. Finally, we show that every simple projective left $R-$module is mininjective if and only if R is left $MC2$ ring.

In section 2, left $MC2$ ring R whose simple singular left modules are $nil-$injective are studied. As a byproduct of these results we are able to show that if R is a left $MC2$ left Goldie ring whose every simple singular left $R-$module is $nil-$injective and left $GJcp-$injective, then R is a finite product of simple left Goldie rings.

1. Left $MC2$ rings

Theorem 1.1. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *For any left minimal elements $k, g^2 = g \in R$, $Rk \cong Rg$ as left $R-$module always implies $Rk = Re, e^2 = e \in R$;*
(3) *For any left minimal elements $k, g \in R$ with $k^2 = 0, g^2 = g$, $Rk \cong Rg$ as left $R-$module always implies $Rk = Re, e^2 = e \in R$.*

Proof. (1) $\Longrightarrow$ (2) Assume that R is left $MC2$ ring and $Rk \cong Rg$ for left minimal elements $k, g^2 = g \in R$. Evidently, there exists an idempotent element $h \in R$ such that $hk = k$ and $l(k) = l(h)$. So Rh is a minimal left ideal of R, by (1), hR is a minimal right ideal, thus $kR = hkR = hR$. Write $h = kc, c \in R$ and let $e = ck$. Then $Rk = Re, e^2 = e$.

(2) $\Longrightarrow$ (3) is evident.

(3) $\Longrightarrow$ (1) Let $e^2 = e \in R$ be a left minimal element. Let $a \in R$ such that $ea \neq 0$, then $Re \cong Rea$, so ea is a left minimal element. If $(ea)^2 \neq 0$, then clearly $Rea = Rg, g^2 = g \in R$. If $(ea)^2 = 0$, then by (3), $Rea = Rg, g^2 = g \in R$. Write $g = cea, c \in R$ and let $h = eac$, then $eaR = hR, h^2 = h$. Since $l(e) = l(ea) = l(h)$, $eR = rl(e) = rl(h) = hR = eacR = eaR$. This implies e is a right minimal element. Hence R is a left $MC2$ ring. □

Associated with each ring R is the monoid $(R, \circ)$, where $a \circ b = a + b - ab$, for each $a, b \in R$. We call $(R, \circ)$ the circle semigroup of R. An element $b \in R$ is quasi-regular if b is invertible in $(R, \circ)$; i.e., there exists $a \in R$ such that $b \circ a = a \circ b = 0$. The set of all quasi-regular elements in R is denoted by $q(R)$. Note that $N(R) \subseteq q(R)$ and Theorem 1.1, we have the following corollary.

Corollary 1.2. *R is left $MC2$ ring if and only if for any left minimal elements $k, g \in R$ with $k \in q(R), g^2 = g$, $Rk \cong Rg$ as left $R-$module always implies $Rk =$*

$Re, e^2 = e \in R$.

Recall that a ring R is idempotent reflexive [7] if $aRe = 0$ implies $eRa = 0$ for all $a, e^2 = e \in R$. Obviously any abelian rings and semiprime rings are idempotent reflexive rings.

With this idea, we characterize left $MC2$ rings as follows.

Theorem 1.3. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *For any $a \in R$ and left minimal idempotent $e \in R$ with $aRe = 0$ always implies $eRa = 0$;*
(3) *For any $a \in q(R)$ and left minimal idempotent $e \in R$ with $aRe = 0$ always implies $eRa = 0$;*
(4) *For any $a \in N(R)$ and left minimal idempotent $e \in R$ with $aRe = 0$ always implies $eRa = 0$;*
(5) *For any $a \in R$ and left minimal idempotent $e \in R$ with $a^2 = 0$ and $aRe = 0$ always implies $eRa = 0$;*
(6) *For any left minimal elements $k, e^2 = e \in R$ with $kRe = 0$ always implies $eRk = 0$;*
(7) *For any left minimal elements $k \in q(R), e^2 = e \in R$ with $kRe = 0$ always implies $eRk = 0$;*
(8) *For any left minimal elements $k, e^2 = e \in R$ with $k^2 = 0$ and $kRe = 0$ always implies $eRk = 0$;*

Proof. (1) $\Longrightarrow$ (2) Assume that $e^2 = e \in R$ is a left minimal element and $a \in R$ with $aRe = 0$. If $eRa \neq 0$, then there exists a $b \in R$ such that $eba \neq 0$. Clearly, $Re \cong Reba$. Hence, by Theorem 1.1, $Reba = Rg, g^2 = g \in R$. Thus $Reba = RebaReba = Reb(aRe)ba = 0$, which is a contradiction. Hence $eRa = 0$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (8) and (2) $\Longrightarrow$ (6) $\Longrightarrow$ (7) $\Longrightarrow$ (8) are trivial.

(8) $\Longrightarrow$ (1). Assume $k, e^2 = e \in R$ are left minimal elements with $Rk \cong Re$. Then there exists an idempotent element $h \in R$ such that $hk = k$ and $l(k) = l(h)$. If $(Rk)^2 = 0$, then $kRh = 0$ because $kR \subseteq l(k) = l(h)$. By hypothesis, $hRk = 0$, then $hRh = 0$ and so $h = 0$, which is a contradiction. Hence $(Rk)^2 \neq 0$, which implies that R is left $MC2$ ring. □

By Theorem 1.3, we can see that idempotent reflexive rings are left $MC2$ rings, so semiprime rings and abelian rings are all left $MC2$ rings.

Theorem 1.4. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *For any left minimal element k, if Rk is a summand in ${}_RR$, then k is right minimal element.*
(3) *For any left minimal element k, if ${}_RRk$ is projective, then k is right minimal element.*
(4) *For any left minimal element k, if ${}_RRk$ is nonsingular, then k is right minimal element.*

Proof. (1) $\Longrightarrow$ (2) Since Rk is a minimal left ideal of R, $_RRk$ is projective if and only if $_RRk$ is nonsingular. Hence, by Theorem 1.3, (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (2) $\Longrightarrow$ (1) are evident.

(1) $\Longrightarrow$ (3). Assume k is a left minimal element with $_RRk$ is projective, then there exists a left minimal idempotent $e \in R$ such that $ek = k$ and $l(e) = l(k)$. By (1), e is right minimal element, so $kR = eR$ is a minimal right ideal of R, hence k is right minimal element. □

Corollary 1.5. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *For any left minimal element k, if $k \notin q(R)$, then k is right minimal element;*
(3) *For any left minimal element k, if $k \notin J(R)$, then k is right minimal element;*
(4) *For any left minimal element k, if $k \notin Z_l(R)$, then k is right minimal element;*
(5) *For any left minimal element k, if $k \notin N(R)$, then k is right minimal element;*
(6) *For any left minimal element k, if $k \notin P(R)$, then k is right minimal element.*

Recall that a ring R is left minsymmetric [3] if every left minimal element is right minimal. From Theorem 1.4, we know that left minsymmetric ring is left $MC2$ ring. [3, Proposition 2.7] shows that R is left minsymmetric ring if and only if any left minimal element $k \in R$, $r(Rk \cap l(a)) = r(k) + aR$ for all $a \in R$. We can generalize the result as follows.

Theorem 1.6. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *If Rk is a projective minimal left ideal of R, then $r(Rk \cap l(a)) = r(k) + aR$ for all $a \in R$;*
(3) *For each left minimal idempotent $e \in R$, $r(Re \cap l(a)) = (1 - e)R + aR$ holds for all $a \in R$.*

Proof. (1) $\Longrightarrow$ (2). Assume that Rk is simple projective left ideal of R. By Theorem 1.4, k is a right minimal element. If $ka = 0$, then $Rk \subseteq l(a)$ and $aR \subseteq r(k)$, so $r(Rk \cap l(a)) = r(Rk) = r(k) = aR + r(k)$, and (2) follows. If $ka \neq 0$, then $Rk \cap l(a) = 0$ and $r(k) + aR = R$ because $r(k)$ is a maximal right ideal of R. Hence $r(Rk \cap l(a)) = r(0) = R = r(k) + aR$ and again (2) follows.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) Assume that $e^2 = e \in R$ is a left minimal element and $a \in R$ with $ea \neq 0$, then $Re \cap l(a) = 0$, so $R = r(0) = r(Re \cap l(a))) = (1 - e)R + aR$. Write $1 = x + ab, x \in (1 - e)R, b \in R$, then $e = ex + eab = eab$. Hence $eR = eabR = eaR$, which implies that eR is a minimal right ideal of R. □

Call a left $R-$module M mininjective [3] if for any left minimal element $k \in R$, every $R-$morphism $f : Rk \longrightarrow M$ extends to R. If $_RR$ is mininjective, we call R is left mininjective ring [3]. By [3, Theorem 1.14], left mininjective rings are left minsymmetric rings. Hence, left mininjective rings are left $MC2$ rings. [3, Theorem 1.14] shows that if R is left minsymmetric ring, then $S_l \subseteq S_r$. We do not know whether R is left $MC2$ ring, whence $S_l \subseteq S_r$.

Call a ring R left universally mininjective [3] if every left $R-$module is mininjective. In [1], we also call this ring left DS ring. Since left DS rings are left mininjective [3, Theorem 5.1], left DS rings are left $MC2$ rings.

Call a ring R left $C2$ [8] if every left ideal which isomorphic to a summand of $_RR$ is a summand. By Theorem 1.6, we see that left $C2$ rings are left $MC2$. Hence right Kasch rings are left $MC2$ because right Kasch rings are left $C2$ [8].

Recall that R is left $CM-$ring iff, for any maximal essential left ideal M of R, every complement left subideal is an ideal of M. [5, Proposition 3] shows that every simple projective module over left $CM-$ring is injective. Hence left $CM-$ring is left $MC2$. In fact, we have the following Theorem.

Theorem 1.7. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *Every nonsingular simple left $R-$module is mininjective;*
(3) *Every simple projective left $R-$module is mininjective;*

Proof. (1) $\Longrightarrow$ (2). Assume that R is left $MC2$ ring. Now let W be a nonsingular simple left $R-$module. Then $_RW$ is projective. Let Rk be any minimal left ideal of R and $f : Rk \longrightarrow W$ be any non-zero left $R-$morphism. Clearly, f is an isomorphism, so $Rk = Re, e^2 = e \in R$ by Theorem 1.1. Set $g : R \longrightarrow W$ defined by $g(x) = xf(e)$, Then g is a left $R-$morphism, and $g(k) = kf(e) = f(ke) = f(k)$ because $k = ke$. This shows $_RW$ is left mininjective.

(2) $\Longleftrightarrow$ (3) is trivial.

(3) $\Longleftrightarrow$ (1) Assume$Rk \cong Re$ where $Rk, Re, e^2 = e$ are minimal left ideals of R. Then by (3), $_RRk$ is mininjective, so the map $I : Rk \longrightarrow Rk$ via $xk \longmapsto xk, x \in R$ extends to $f : R \longrightarrow Rk$. Hence there exists a $c \in R$ such that $k = f(k) = kf(1) = kck$ where $f(1) = ck$. Write $ck = h$, then $h^2 = h$ and $Rk = Rh$, (1) follows. □

Let R be a left $MC2$ ring, $e^2 = e, g^2 = g \in R$ be any left minimal elements, if $Re \neq Rg$, then by [3], there exists a $h^2 = h \in R$ such that $Re \oplus Rg = Rh$.

Call a left ideal I of R is minimal finite generated if I is a direct sum of finite minimal left ideals of R.

Since direct sum of finite mininjective left $R-$modules is mininjective, we have the following corollary.

Corollary 1.8. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *Every nonsingular finite generated semisimple left $R-$module is mininjective;*
(3) *Every finite generated semisimple projective left $R-$module is mininjective;*
(4) *Every minimal finite generated projective left ideal of R is mininjective;*
(5) *Every nonsingular minimal finite generated left ideal of R is mininjective;*
(6) *Every minimal finite generated projective left ideal of R is a summand;*
(7) *Every nonsingular minimal finite generated left ideal of R is a summand.*

Call a left $R-$module M $nil-$injective if for each nilpotent element $k \in R$, there exists a positive integer n such that $k^n \neq 0$ and any left $R-$morphism $Rk^n \longrightarrow M$

extends to R. Example contain left $p-$injective modules and $YJ-$injective modules. R is called left $nil-$injective ring if ${}_RR$ is $nil-$injective. So left $YJ-$injective ring is left $nil-$injective ring. Obviously we have the following corollary.

Corollary 1.9. *Let R be a left $MC2$ ring and k be a left minimal element, then*
(1) *If ${}_RRk$ is injective, then k is right minimal element.*
(2) *If ${}_RRk$ is $p-$injective, then k is right minimal element.*
(3) *If ${}_RRk$ is mininjective, k is right minimal element.*
(4) *If ${}_RRk$ is $YJ-$injective, then k is right minimal element.*
(5) *If ${}_RRk$ is $nil-$injective, then k is right minimal element.*

It is well known that for any ring R, we have (1) $P(R) \subseteq K(R) \subseteq N(R) \cap J(R)$ and $N(R) \cup J(R) \subseteq q(R)$, (2)$B(R) \cup J(R) \subseteq BJ(R)$, (3) $Z_l(R) \cap S_l(R) \subseteq B(R) \cap P(R) \cap S_l(R)$. In (3), When does the equality hold? We have the following theorem.

Theorem 1.10. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) $Z_l(R) \cap S_l(R) = J(R) \cap S_l(R)$;
(3) $Z_l(R) \cap S_l(R) = P(R) \cap S_l(R)$;
(4) $Z_l(R) \cap S_l(R) = N(R) \cap S_l(R)$;
(5) $Z_l(R) \cap S_l(R) = K(R) \cap S_l(R)$,
(6) $Z_l(R) \cap S_l(R) = q(R) \cap S_l(R)$;
(7) $Z_l(R) \cap S_l(R) = B(R) \cap S_l(R)$
(8) $Z_l(R) \cap S_l(R) = BJ(R) \cap S_l(R)$.

Proof. It is obvious because $N(R), P(R), q(R), J(R), K(R), B(R), BJ(R)$ contain no nonzero idempotents. □

In [8], W.K.Nicholson shows that R is left $C2$ ring if and only if for any $a, e^2 = e \in R$, any $R-$isomorphism $Ra \longrightarrow Re$ extends to $R \longrightarrow R$ if and only if if $l(a) = l(e), a, e^2 = e \in R$, then $e \in aR$. Similar to the proof of the result, we can generalize the results as follows.

Theorem 1.11. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *Every $R-$morphism $f : Rk \longrightarrow Re$ with $k, e^2 = e \in R$ left minimal, extends to $R \longrightarrow R$;*
(3) *Every $R-$morphism $Rk \longrightarrow Re$ with $k, e^2 = e \in R$, left minimal and $k \in q(R)$, extends to $R \longrightarrow R$;*
(4) *Every $R-$morphism $Rk \longrightarrow Re$ with $k, e^2 = e \in R$, left minimal and $k^2 = 0$, extends to $R \longrightarrow R$.*

Proof. (1) $\Longrightarrow$ (2) Since Rk, Re are all minimal left ideals of R, f is an isomorphism. By (1) and Theorem 1.1, $Rk = Rg, g^2 = g \in R$. Set $\rho : R \longrightarrow R$ defined by $\rho(x) = xf(g), x \in R$, then ρ is a left $R-$homomorphism, and $\rho(k) = kf(g) = f(kg) = f(k)$ because $k = kg$. We are done.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are trivial.

(4) $\Longrightarrow$ (1) Assume that $\sigma : Rk \cong Re$, where $Rk, Re, e^2 = e$ are minimal left ideals of R. If $k^2 \neq 0$, then,certainly, $Rk = Rg$ for some $g^2 = g \in R$. If $k^2 = 0$, then by (4), there exists a left $R-$morphism $\rho : R \longrightarrow R$ such that $\rho|_{Rk} = \sigma$. Hence $e = \sigma(ak) = \rho(ak) = ak\rho(1) = akb, a \in R, b = \rho(1) \in R$. Since $Re = ReRe = Re\sigma(Rk) = Rakb\sigma(Rk) = \sigma(RakbRk) \subseteq \sigma(RkRk) \neq 0$, so $(Rk)^2 \neq 0$, which implies $Rk = Rg$ for some $g^2 = g \in R$, we again follow (1). □

Theorem 1.12. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *If $l(k) = l(e)$, $k, e^2 = e$ are left minimal elements, then $e \in kR$;*
(3) *If $l(k) = l(e)$, $k, e^2 = e$ are left minimal elements with $k \in q(R)$, then $e \in kR$;*
(4) *If $l(k) = l(e)$, $k, e^2 = e$ are left minimal elements with $k^2 = 0$, then $e \in kR$.*

Proof. (1) $\Longrightarrow$ (2) Since $Rk \cong R/l(k) = R/l(e) \cong Re$, By (1) and theorem 1.1, $Rk = Rg, g^2 = g$, so $kR = hR$ for some $h^2 = h \in R$. Hence $eR = rl(e) = rl(k) = rl(kR) = rl(hR) = rl(h) = hR = kR$, we are done.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1). Assume that $\sigma : Rk \cong Re$, where $Rk, Re, e^2 = e$ are minimal left ideals of R. Let $\sigma(ak) = e, \sigma(k) = be, a, b \in R$. Then $e = abe$. Set $g = bea$, then $g^2 = beabea = be(abe)a = beea = bea = g$, $gk = beak = be\sigma^{-1}(e) = \sigma^{-1}(bee) = k$ and clearly, $l(g) = l(k)$ because $l(g)$ is a maximal left ideal of R. We can assume that $k^2 = 0$, then by (4), $g \in kR$. This implies $(kR)^2 \neq 0$, so $(Rk)^2 \neq 0$, we follow (1). □

Recall that R is right minannihilator ring [3] if every minimal right ideal K of R is an annihilator, equivalently, $rl(K) = K$. Evidently, right DS rings [1] are right minannihilator rings. Certainly, left $p-$injective rings are also right minannihilator rings. With this idea, we give the following theorem.

Theorem 1.13. *The following conditions are equivalent for a ring R.*
(1) *R is left $MC2$ ring;*
(2) *For each projective minimal left ideal Rk of R, $rl(k) = kR$;*
(3) *For each projective minimal left ideal Rk of R with $k \in q(R)$, $rl(k) = kR$;*
(4) *For each projective minimal left ideal Rk of R with $k^2 = 0$, $rl(k) = kR$.*

Proof. (1) $\Longrightarrow$ (2) Since $_Rk$ is projective, then $Rk \cong Re, e^2 = e \in R$, so $Rk = Rh, h^2 = h \in R$ by (1) and Theorem 1.1. Thus we easy show that $kR = gR, g^2 = g \in R$. Consequently, $rl(k) = rl(kR) = rl(gR) = rl(g) = gR = kR$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are trivial.

(4) $\Longrightarrow$ (1) Let $Rk \cong Re, e^2 = e, k^2 = 0$ where Rk, Re be minimal left ideals of R. By (4), $rl(k) = kR$. Since there exists a $g^2 = g \in R$ such that $gk = k$ and $l(k) = l(g)$. Hence $kR = rl(g) = gR$, consequently, clearly, $(Rk)^2 \neq 0$, which implies that R is a left $MC2$ ring. □

2. Certain rings whose simple singular modules are $nil-$injective

Recall that a ring R is left $GQ-$injective [6] if, for any left ideal I isomorphic to a complement left ideal of R, every left $R-$homomorphism of I into R extends to an endomorphism of $_RR$. In [6], Roger Yue Chi Ming shows that if R is left $GQ-$injective ring, then $J(R) = Z_l(R), R/J(R)$ is regular.

Theorem 2.1. *Let R be a left $GQ-$injective ring whose simple singular left $R-$modules are nil$-$injective, then R is regular ring.*

Proof. Suppose that $Z_l(R) \neq 0$. Then there exists a $0 \neq a \in Z_l(R)$ such that $a^2 = 0$. If there exists a maximal essential left ideal M containing $Z_l(R) + l(a)$, then the $R-$morphism $f : Ra \longrightarrow R/M$ defined by $f(ra) = r + M, r \in R$ extends to R because R/M is simple singular left $R-$module, so is left $nil-$injective. Hence there exists a $c \in R$ such that $1 - ac \in M$. Since $ac \in Z_l(R) \subseteq M$, $1 \in M$, which is a contradiction. This implies that $Z_l(R) + l(a) = R$. Write $1 = x + y, x \in Z_l(R), y \in l(a)$. So $a = xa + ya = xa$, and then $(1 - x)a = 0$. Since $Z_l(R) = J(R)$, $1 - x$ is invertible. This shows that $a = 0$, which is a contradiction. Therefore $Z_l(R) = 0$ and so R is regular ring. □

Call a left $R-$module M $Gnp-$injective if for each non-nilpotent element $a \in R$, there exists a positive integer n such that any left $R-$morphism $Ra^n \longrightarrow M$ extends to $R \longrightarrow M$. Example contains left $YJ-$injective modules. R is called left $Gnp-$injective ring if $_RR$ is $Gnp-$injective. So left $YJ-$injective ring is left $Gnp-$injective ring. Obviously we have the following corollary.

Theorem 2.2. *Let R be a left $MC2$ ring. Then*
(1) *If $a \in R$ is not a left weakly regular element, then every maximal left ideal M of R containing $RaR + l(a)$ must be essential in $_RR$.*
(2) *If every simple singular left $R-$module is nil$-$injective, then for any non-zero nilpotent elementa $\in R$, there exists a positive integer n such that $a^n \neq 0$ and $RaR + l(a^n) = R$. Therefore $N(R) \cap J(R) = 0$. Consequently, R is NI ring if and only if R is reduced ring if and only if R is 2-prime ring.*
(3) *If every simple singular left $R-$module is $Gnp-$injective, then for any non-nilpotent elementa $\in R$, there exists a positive integer n such that $RaR + l(a^n) = R$. Therefore $N_1(R) \cap J(R) = 0$.*
(4) *If every simple singular left $R-$module is nil$-$injective, then for any $0 \neq a \in R$, $(Ra)^2 \neq 0$. Therefore R is semiprime ring.*

Proof. (1) Assume that $a \in R$ is not left weakly regular element. Then $RaR + l(a)$ is contained in some maximal left ideal M. If M is not essential, then $M = l(e), e^2 = e \in R$. Then $aRe = 0$. Since R is left $MC2$ ring and e is a left minimal idempotent, $eRa = 0$. Hence $e \in l(a) \subseteq M = l(e)$, which is a contradiction. This implies M is essential.

(2) Assume that $a^n \neq 0, a^{n+1} = 0$. if a^n is a left weakly regular element, then we are doen. Otherwise, by (1), there exists a maximal essential left ideal M containing $Ra^nR + l(a^n)$. Thus R/M is a simple singular left $R-$module, so is $nil-$injective. Hence the left $R-$morphism $f : Ra^n \longrightarrow R/M$ defined by $f(ra^n) = r + M$ extends to R, so there exists a $c \in R$ such that $1 - a^nc \in M$. Since $a^nc \in Ra^nR \subseteq M$,

$1 \in M$, which is a contradiction. Hence $R = Ra^nR + l(a^n) = RaR + l(a^n)$.

(3) Consider the chain $RaR+l(a) \subseteq RaR+l(a^2) \subseteq \cdots$. Let $\cup_{i=1}^{\infty}[RaR+l(a^i)] = I$. If $I \neq R$, then I is contained in a maximal essential left ideal M of R. Then R/M is left $Gnp-$injective. So there exists a positive integer n such that such that the left $R-$morphism $Ra^n \longrightarrow M$ defined by $ra^n \longmapsto r + M$ extends to R. By a similar way as in the previous process, we obtain a contradiction. Therefore $\cup_{i=1}^{\infty}[RaR + l(a^i)] = R$, then we can easy to show that $RaR + l(a^m) = R$ for some positive integer m.

(4) If $(Ra)^2 = 0$, then by (2), we have $RaR + l(a) = R$. Hence $a \in RaRa = 0$, which is a contradiction. Thus $(Ra)^2 \neq 0$. □

Corollary 2.3. *Let R be left $MC2$ ring whose simple singular left $R-$modules are nil$-$injective and $Gnp-$injective, then for any nonzero elementa $\in R$, there exists a positive integer n such that $a^n \neq 0$ and $RaR + l(a^n) = R$. Therefore $J(R) = 0$.*

A ring R is called ZI if $ab = 0$ implies $aRb = 0$ for all $a, b, \in R$. Evidently, ZI ring is abelian and 2-prime, so ZI ring is left $MC2$ and NI ring. N.K. Kim and J.Y. Kim [12, Theorem 4] shows that if R is a ZI ring whose every simple singular left $R-$module is $YJ-$injective, then R is reduced weakly regular ring. Then by [12, Proposition 8], we generalize above result as follows.

Theorem 2.4. *Let R be a left $MC2$ ring whose every simple singular left $R-$module is nil$-$injective, then the following conditions are equivalent.*
(1) R is reduced ring;
(2) R is ZI ring;
(3) R is 2-prime ring;
(4) R is NI ring.
In this case, R is weakly regular ring. And if R is also $MELT$ ring, then R is strongly regular ring.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obviously.

(4) $\Longrightarrow$ (1) Assume that $a \in R$ with $a^2 = 0$. If $a \neq 0$, then there exists a maximal left ideal M of R such that $l(a) \subseteq M$. If M is not essential in $_RR$, then $M = l(e)$ where $e^2 = e \in R$ is a left minimal element. Hence $ae = 0$ because $a \in l(a)$. If $aRe \neq 0$, then $RaRe = Re$. Since R is NI ring, then $N(R)$ is an ideal of R, so $RaRe \in N(R)$ because $a \in N(R)$. Thus $e \in N(R)$, which is a contradiction. This shows that $aRe = 0$. Hence $eRa = 0$ because R is left $MC2$ ring. Thus $e \in l(a) \subseteq l(e)$, which is also a contradiction. This implies that M is essential in $_RR$, then R/M is left *nil*$-$injective by hypothesis. Hence the left $R-$morphism $f : Ra \longrightarrow R/M$ defined by $f(xa) = x + M, x \in R$ extends to $R \longrightarrow R/M$, which implies that there exists a $c \in R$ such that $1 - ac \in M$. Since $ac \in N(R)$, $1 - ac$ is invertible, so $M = R$, which is a contradiction. This shows that $a = 0$ and so R is reduced. □

Call a left $R-$module M $GJcp-$injective if for each $a \notin Z_l(R)$, there exists a positive integer n such that $a^n \neq 0$ and any left $R-$morphism $Ra^n \longrightarrow M$

extends to $R \longrightarrow M$. Example contains left $YJ-$injective modules. R is called left $GJcp-$injective ring if ${}_RR$ is $GJcp-$injective. So left $YJ-$injective ring is left $GJcp-$injective ring.

Recall that a ring R is said to be left weakly $\pi-$regular if for every $x \in R$, there exists a positive integer n, depending on x, such that $x^n \in Rx^nRx^n$. Similar to the proof process of [14, Lemma 3.1], we have the following theorem.

Theorem 2.5. *Let R be a left $MC2$ left Goldie ring. If every simple singular left $R-$module is nil$-$injective and $GJcp-$injective, then R is a finite product of simple left Goldie rings.*

Proof. First we claim that R is semiprime ring. In fact, if $a \in R$ with $aRa = 0$, then $a = 0$. Otherwise, there exists a maximal left ideal M of R such that $RaR \subseteq l(a) \subseteq M$ because $l(a) \neq R$. If $M = l(e), e^2 = e \in R$, then $aRe = 0$ because $RaR \subseteq l(e)$. Since R is left $MC2$ ring and e is left minimal element, $eRa = 0$, which is a contradiction because $e \in l(a) \subseteq l(e)$. This shows that M is essential in ${}_RR$, so R/M is left $nil-$injective, and clearly, there exists a $c \in R$ such that $1 - ac \in M$. Since $ac \in RaR \subseteq M$, $1 \in M$, which is also a contradiction. Hence $a = 0$ and so R is semiprime ring.

Next note that for any nonzero element $0 \neq a \in R$, $RaR + l(Ra)$ is essential in ${}_RR$. Otherwise there exists a $0 \neq b \in R$ such that $(RaR + l(Ra)) \cap Rb = 0$. Hence $aRb \subseteq aR \cap Rb = 0$, so $bRa = 0$ because R is semiprime ring. Thus $b \in l(Ra) \cap Rb = 0$, which is a contradiction. Hence $RaR + l(Ra)$ is essential in ${}_RR$. Since R is a left Goldie ring, there exists a $c \in R$ with $l(c) = r(c) = 0$ and $c \in RaR + l(Ra)$. Clearly, $c \notin Z_l(R)$. Finally we show that $RcR = R$. If there exists a maximal left ideal M of R such that $RcR \subseteq M$, then M must be essential in ${}_RR$. In fact, if $M = l(e), e^2 = e \in R$, then $ce = 0$, so $e = 0$ because $r(c) = 0$, which is a contradiction. Thus M is essential, and so R/M is $GJcp-$injective. Hence there exists a positive integer n such that any $R-$morphism $Rc^n \longrightarrow R/M$ extends to $R \longrightarrow R/M$. Since $Rc^n \longrightarrow R/M$ via $xc^n \longmapsto x + M$ are well defined left $R-$morphism, so there exists a $d \in R$ such that $1 - c^nd \in M$, so $1 \in M$ because $c^nd \in RcR \subseteq M$, which is a contradiction. This implies $RcR = R$, so $RaR + l(Ra) = R$, further, $RaR + l(a) = R$. This implies R is left weakly regular ring. □

From the proof of Theorem 2.5, we can see that the following corollary holds

Corollary 2.6. *Let R be left $MC2$ ring whose each simple singular left $R-$module is nil$-$injective, then R is semiprime ring.*

Corollary 2.7. *Let R be a left Goldie ring with every minimal idempotent element of R be right semicentral. If every simple singular left $R-$module is nil$-$injective and $GJcp-$injective, then R is a finite product of simple left Goldie rings.*

Proof. By Theorem 2.5, we only show that R is left $MC2$ ring. In fact, if $aRe = 0$ where $a \in R$ and $e \in R$ is left minimal idempotent. By hypothesis, e is right semicentral in R. of R, thus $eRa = eRae = 0$. By Theorem 1.3, R is left $MC2$

ring. □

Recall that a ring R is said to be left weakly $\pi-$regular if for every $x \in R$, there exists a positive integer n, depending on x, such that $x^n \in Rx^nRx^n$. Similar to the proof of Theorem 2.5, we have the following theorem

Theorem 2.8. *Let R be a left $MC2$ left Goldie ring whose each simple singular left $R-$module is nil$-$injective. If R is left weakly $\pi-$regular, then R is a finite product of simple left Goldie rings.*

We do not know that whether $Z_l(R) = 0$, if R is left $MC2$ ring whose each simple singular left $R-$module is $nil-$injective? But we have the following theorem.

Theorem 2.9. *Let R be a left $MC2$ ring whose each simple singular left $R-$module is nil$-$injective, then $Z_r(R) = 0$.*

Proof. Suppose that $Z_r(R) \neq 0$, then there exists a $0 \neq a \in Z_r(R)$ such that $a^2 = 0$. We claim that $Z_r(R) + l(a) = R$. Otherwise, there exists a maximal left ideal M such that $Z_r(R) + l(a) \subseteq M$. If M is not essential, then $M = l(e), e^2 = e \in R$. Hence $ae = 0$ because $a \in l(a) \subseteq l(e)$. If $aRe \neq 0$, then $RaRe = Re$ because Re is a minimal left ideal of R. Since $a \in Z_r(R)$, $RaRe \subseteq Z_r(R)$, then $e \in Z_r(R)$, which is a contradiction. Hence $aRe = 0$. Since R is left $MC2$ ring, $eRa = 0$, $e \in l(a) \subseteq l(e)$, which is a contradiction. Hence M is essential in ${}_RR$. Thus R/M is $nil-$injective. Similarly to the proof of Theorem 2.5, there exists a $c \in R$ such that $1 - ac \in M$. Since $ac \in Z_r(R) \subseteq M$, $1 \in M$, which is a contradiction. Hence $Z_r(R) + l(a) = R$. Write $1 = x + y, x \in Z_r(R), y \in l(a)$, then $a = xa$. Since $x \in Z_r(R)$ and $r(x) \cap r(1-x) = 0$, $r(1-x) = 0$. Thus $a = 0$ because $a \in r(1-x)$, which is a contradiction. This implies that $Z_r(R) = 0$. □

We do not know wether the result hold if we obit the condition "R is left $MC2$ ring"? But we indeed have the following theorem.

Theorem 2.10. *Let R be a ring whose each simple singular left $R-$module is nil$-$injective, then $Z_l(R) \cap Z_r(R) = 0$.*

Proof. Suppose that $Z_l(R) \cap Z_r(R) \neq 0$, then there exists a $0 \neq a \in Z_l(R) \cap Z_r(R)$ such that $a^2 = 0$. We claim that $Z_r(R) + l(a) = R$. Otherwise, there exists a maximal essential left ideal M such that $Z_r(R) + l(a) \subseteq M$. Thus R/M is $nil-$injective. Similarly to the proof of theorem 2.5, there exists a $c \in R$ such that $1 - ac \in M$. Since $ac \in Z_r(R) \subseteq M$, $1 \in M$, which is a contradiction. Hence $Z_r(R) + l(a) = R$. Write $1 = x + y, x \in Z_r(R), y \in l(a)$, then $a = xa$. Since $x \in Z_r(R)$ and $r(x) \cap r(1-x) = 0$, $r(1-x) = 0$. Thus $a = 0$ because $a \in r(1-x)$, which is a contradiction. This implies that $Z_l(R) \cap Z_r(R) = 0$. □

With Theorem 2.10, we can obtain the following corollary. And from the following corollary, we can see that if R is right $GJcp-$injective, then $Z_r(R) \subseteq J(R)$.

Corollary 2.11. *Let R be a left $GQ-$injective, right $GJcp-$injective ring whose* *each simple singular left $R-$module is nil$-$injective, then R is regular ring.*

Proof. First for any $0 \neq a \in Z_r(R)$, $1-a \notin Z_r(R)$. Since R is right $GJcp-$injective, there exists a positive integer n such that $(1-a)^n \neq 0$ (in fact, $r(1-a)=0$) and any right $R-$morphism $(1-a)^nR \longrightarrow R$ extends to $R \longrightarrow R$. Hence the right $R-$morphism $(1-a)^nR \longrightarrow R$ defined by $(1-a)^n x \longmapsto x, x \in R$ extends to $R \longrightarrow R$, and so there exists a $c \in R$ such that $1 = c(1-a)^n$, which implies $Z_r(R) \subseteq J(R)$. Next R is left $GQ-$injective, then $Z_l(R) = J(R)$ and $R/J(R)$ is regular ring. Finally, since each simple singular left $R-$module is $nil-$injective, by Theorem 2.10, $Z_r(R) \cap Z_l(R) = 0$. Hence $Z_r(R) \subseteq J(R) = Z_l(R)$, and so $Z_r(R) = Z_r(R) \cap Z_l(R) = 0$. Consequently, R is right $YJ-$injective ring because R is right $GJcp-$injective. Thus $J(R) = Z_r(R)$, which implies that $J(R) = 0$, and so R is regular ring. $\square$

Corollary 2.12. *Let R be a right $GQ-$injective, left $GJcp-$injective ring whose each simple singular left $R-$module is nil$-$injective, then R is regular ring.*

Proof. First R is right $GQ-$injective, then $Z_r(R) = J(R)$ and $R/J(R)$ is regular ring. Next R is left $GJcp-$injective, then $Z_l(R) \subseteq J(R)$. Finally $Z_l(R) \cap Z_r(R) = 0$ by hypothesis and Theorem 2.10. Hence $Z_l(R) = 0$ because $Z_l(R) \subseteq Z_r(R)$. Consequently, R is left $YJ-$injective because R is left $GJcp-$injective. Hence $J(R) = Z_l(R) = 0$, which implies R is a regular ring. $\square$

References

[1] J. C. Wei, *The rings characterized by minimal left ideal*, Acta. Math. Sinica. English. Series, **21(3)**(2005), 473-482.

[2] W. K. Nicholson and J. F. Watters, *Rings with projective socle*, Proc. Amer. Math. Soc., **102**(1988), 443-450.

[3] W. K. Nicholson and M. F. Yousif, *Minijective rings*, J. Alg., **187**(1997), 548-578.

[4] T. Y. Lam and Alex S. Dugas, *Quas-duo rings and stable range descent*, Journal of pure and applied algebra, **195**(2005), 243-259.

[5] Roger Yue Chi Ming, *On Quasi-Frobeniusean and Artinian rings*, Publications De L'institut Math *é*matique, **33**(1983), 239-245.

[6] Roger Yue Chi Ming, *On quasi-injectivity and Von Neumann regularity*, Mounatshefte fr Math, **95**(1983), 25-32.

[7] Jin Yong Kim, *Certain rings whose simple singular modules are Gp$-$injective*, Proc. Japan Acad. Ser. A, **81**(2005), 125-128.

[8] W. K. Nicholson and M. F. Yousif, *Weakly continuous and C2-rings, Comm in Alg.*, **29(6)**(2001), 2429-2466.

[9] W. Xue, *A note on $YJ-$injectivity*, Riv. Mat. Univ. Parma, **6(1)**(1998), 31-37.

[10] J. Chen and N. Ding, *On generalizations of injectivity*, in International Symposium on Ring Theory (Kyongju, 1999), 85-94, Birkhauser, Boston, Boston, MA., 2001.

[11] Roger Yue Chi Ming, *On $p-$injectivity and generalizations*, Riv. Mat. Univ. Parma, **5(5)**(1996), 183-188.

[12] Roger Yue Chi Ming, N. K. Kim, S. B. Nam and J. Y. Kim, *On simple singular GP-injective modules*, Comm in Alg., **27(5)**(1999), 2087-2096.

[13] W. K. Nicholson and E. Sanchez Campos, *Rings with the dual of the isomorphism theorem*, Journal of Alg., **271**(2004), 391-406.

[14] G. O. Michler and O. E. Villamayor, *On rings whose simple modules are injective*, J. Alg., **25**(1973), 185-201.