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An Existence Result for Neumann Type Boundary Value Problems for Second Order Nonlinear Functional Differential Equation

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Abstract. New sufficient conditions for the existence of at least one solution of Neumann type boundary value problems for second order nonlinear differential equations

$$\begin{cases} (p(t)\phi(x'(t)))' = f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))), & t \in [0, T], \\ x'(0) = 0, \ \ x'(T) = 0, \end{cases}$$

are established.

1. Introduction

Recently, there have been many papers discussed the solvability of two-point, or multi-point boundary value problems for second order or higher order ordinary or functional differential equations, we refer the readers to the text books [1], [2] and papers [6]-[9], [13]-[15] and the references therein.

Xuan and Chen in [3] studied the solvability of singular one dimensional $p-$Laplacian-like equation with Neumann boundary conditions

$$(1) \qquad \begin{cases} (A(|u'|)u')' - g(u(t)) = h(t), \ \ 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases}$$

where $A(\xi)$ is positive for $\xi > 0$, $h(t) \in L^1[0, 1]$, g is a continuous function defined on $(-\infty, 0) \cup (0, +\infty)$ such that $g(\xi) \to 0$ as $|\xi| \to 0$, $g(\xi) \to +\infty$ as $\xi \to 0^+$, $g(\xi) \to -\infty$ as $\xi \to 0^-$, and $g(\xi)\xi > 0$ for $\xi \neq 0$. Suppose

(i) $H(\xi) = \xi A(|\xi|)$ is strictly increasing homeomorphism of $(0, +\infty)$ with $H(0) = 0$.

(ii) $\lim_{\xi\to 0^+} g(\xi) = +\infty$ and $\int_0^1 g(\xi)d\xi = +\infty$, $g(\xi) > 0$ and $\lim_{\xi\to+\infty} g(\xi) = 0$.

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It was proved that problem (1) has at least one solution for each $h \in L^1[0,1]$ if and only if $\int_0^1 h(t)dt < 0$.

Cabada, Habets and Lois in [5] considered the solvability of the following problem

$$\begin{cases} u'' = f(t,u,u'), & a < t < b, \\ u'(a) = u'(b) = 0, \end{cases} \tag{2}$$

where f is continuous. The existence and approximation of solutions of problem (2) were studied in the presence of lower and upper solutions in reverse order.

In [10], Boucherif and Al-Malki studied the following problem

$$y'' = f(t,y,y'), y'(0) = y'(1) = 0. \tag{3}$$

It was proved that if f is an L^1-Caratheodory function, and

(C_1) there exists $M_0 > 0$ such that $\left[\int_0^1 f(t,M_0,0)dt\right]\left[\int_0^1 f(t,-M_0,0)dt\right] < 0$;

(C_2) there exist $q \in L^1[0,1]$ and $\Phi \in [0,+\infty) \to (0,+\infty)$ nondecreasing with $1/\Phi$ integrable over bounded intervals, and

$$\int_{M_0}^{+\infty} \frac{d\sigma}{\Phi(\sigma)} > ||q||_{L^1}$$

such that $|f(t,y,z)| \le q(t)\Phi(|z|)$ for all $(t,y) \in I \times [-M_0,M_0]$ and all $z \in R$; Then problem (3) has at least one solution.

Atslaga in [11] studied the following problem

$$x'' = f(x), x'(0) = x'(1) = 0. \tag{4}$$

Under the assumptions that f is continuous, f has simple zeros at $p_1 < p_2 < p_3 < p_4 < p_5$, and $f(-\infty) = -\infty$ and $f(+\infty) = +\infty$, the multiplicity results for the problem (4) were proved.

In paper [12], Girg studied the following problem

$$\begin{cases} (\phi(u'(t)))' + g(u'(t)) + h(u(t)) = f(t), & 0 < t < T, \\ u'(0) = u'(T) = 0. \end{cases} \tag{5}$$

Let $f(t) = \widetilde{f} + \overline{f}$ with $\overline{f} = \frac{1}{T}\int_0^T f(t)dt$. Denote

$$\widetilde{C}[0,T] = \left\{u \in C[0,T] : \int_0^T u(t)dt = 0\right\}, \quad \widetilde{C_T} = C_T \cap \widetilde{C}[0,T].$$

Under the following assumptions:

(i) ϕ is an increasing homeomorphism of I_1 onto I_2, where $I_1, I_2 \subset R$ are open intervals containing zero and $\phi(0) = 0$.

(ii) g is continuous.

(iii) h is continuous, bounded real function having limits in $\pm\infty$ with

$$h(-\infty) := \lim_{\xi\to-\infty} h(\xi) < \lim_{\xi\to+\infty} h(\xi) =: h(+\infty).$$

(iv) ϕ is odd and there exist $c, \delta > 0$ and $p > 1$ such that for all $z \in (-\delta, \delta) \cap \mathrm{Dom}\phi : c|z|^{p-1} \le |\phi(z)|$.

It was proved that problem (5) has at least one solution if

$$\sqrt{\frac{3}{T}}b - \sqrt{T}\sup_{\xi\in R}|h(\xi)| > 0,$$

$$||\widetilde{f}||_{L^2} < \sqrt{\frac{3}{T}}b - \sqrt{T}\sup_{\xi\in R}|h(\xi)|,$$

and

$$s(\widetilde{f}) + h(-\infty) < \overline{f} < s(\widetilde{f}) + h(+\infty).$$

We note that the solvability of problem (5) is not addressed in paper [12] when h in (5) is unbounded, and there is no paper concerned with the solvability of Neumann boundary value problems for second order functional differential equations.

Mawhin′s continuation theorem of coincidence degree, see [15], is used to get periodic solutions of first or second order ordinary or functional differential equations, and get solutions of multi-point boundary value problems for second or three order differential equations. The known literature shows us that this theorem is an effective tool to get solutions of differential equations, however, there is no paper concerned with the existence of solutions of Neumann boundary value problems for second order differential equations with $p-$Laplacian-like operator.

Motivated by papers mentioned above, we study boundary value problems for second order nonlinear functional differential equation with $p-$Laplacian-like operator

$$\begin{cases} (p(t)\phi(x'(t)))' = f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))), t \in [0, T], \\ x'(0) = 0, \\ x'(T) = 0, \end{cases} \tag{6}$$

where $T > 0$, f is a continuous function, p continuous with $p(t) > 0$ for all $t \in [0, T]$, ϕ is continuous with $y\phi(y) > 0$ for $y \ne 0$, $\tau_i : [0, T] \to [0, T]$ are continuous differentiable functions with $\tau_i'(t) \ne 0$ for all $t \in [0, T]$.

The purpose of this paper is to establish new existence results for solutions of problem (6), by using Mawhin′s continuation theorem of coincidence degree, via to establish sufficient conditions for the existence of at least one solutions of BVP (6). It is interesting that we allow f to be sublinear, at most linear or superlinear.

This paper is organized as follows. In Section 2, we make preparations, and in Section 3, the main results are given, the examples will be presented in Section 4.

2. Preparations

Let C^0 be the set of all continuous functions on $[0,T]$ and $X = C^0 \times C^0, Y = X \times R^2$, the norm is defined by $||(x,y)|| = \max\{||x||_\infty, ||y||_\infty\}$ for $(x,y) \in X$ and $||(u,v,a,b)|| = \max\{||u||_\infty, ||v||_\infty, |a|, |b|\}$ for each $(u,v,a,b) \in Y$. Then X and Y are Banach spaces.

Let $D(L) = \{(x,y) \in X : x' \in C^0, (py)' \in C^0\}$. Define the linear operator $L : D(L) \cap X \to Y$ by

$$L\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ (p(t)y(t))' \\ y(0) \\ y(T) \end{pmatrix} \text{ for all } (x,y) \in D(L) \cap X.$$

Define the nonlinear operator $N : X \to Y$, for all $(x,y) \in X$, by

$$N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \phi^{-1}(y(t)) \\ f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))) \\ 0 \\ 0 \end{pmatrix}.$$

We omit the proofs of the following results since the proofs are simple and standard.

(i) $\mathrm{Ker}L = \{(a,0) : a \in R\}$;

(ii) $\mathrm{Im}L = \{(u,v,a,b) \in Y : \int_0^T v(t)dt = p(T)b - p(0)a\}$;

(iii) L is a Fredholm operator of index zero;

(iv) There are projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Ker}L = \mathrm{Im}P$ and $\mathrm{Ker}Q = \mathrm{Im}L$. There is an isomorphism $\wedge : \mathrm{Ker}L \to Y/\mathrm{Im}L$.

(v) Let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap D(L) \neq \emptyset$, then N is $L-$compact on $\overline{\Omega}$;

(vi) $(x,y) \in D(L)$ is a solution of the operator equation $L(x,y) = N(x,y)$ implies that x is a solution of problem (6).

Let $F(t) = f(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t)))$. In fact, we have, for $a, b \in R$, $(x,y) \in X$ and $(u,v,a,b) \in Y$, that

$$P\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0) \\ 0 \end{pmatrix},$$

$$Q\begin{pmatrix} u(t) \\ v(t) \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{T}\left(\int_0^T v(t)dt - p(T)b + p(0)a\right) \\ 0 \\ 0 \end{pmatrix},$$

$$K_p\begin{pmatrix} u(t) \\ v(t) \\ a \\ b \end{pmatrix} = \begin{pmatrix} \int_0^t u(s)ds \\ \frac{p(0)}{P(t)}a + \frac{1}{p(t)}\int_0^t v(s)ds \end{pmatrix} \text{ if } \int_0^T v(t)dt = p(T)b - p(0)a,$$

$$
\begin{aligned}
K_p(I-Q)N\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= K_p(I-Q)\begin{pmatrix} \phi^{-1}(y(t)) \\ f(t,x(t),\phi^{-1}(y(t))) \\ 0 \\ 0 \end{pmatrix} \\
&= K_p\begin{pmatrix} \phi^{-1}(y(t)) \\ F(t)-\frac{1}{T}\left(\int_0^T F(t)dt\right) \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \int_0^t \phi^{-1}(y(s))ds \\ \frac{1}{p(t)}\left(\int_0^t F(s)ds - \frac{t}{T}\left(\int_0^T F(t)dt\right)\right) \end{pmatrix}, \\
\wedge\begin{pmatrix} a \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

To get the existence results for solutions of BVP (6), we need a fixed point theorem. Let X and Y be Banach spaces, $L: D(L) \subset X \to Y$ be a Fredholm operator of index zero, $P: X \to X$, $Q: Y \to Y$ be projectors such that

$$\mathrm{Im}\ P = \mathrm{Ker}\ L,\ \mathrm{Ker}\ Q = \mathrm{Im}\ L,\ \ X = \mathrm{Ker}\ L \oplus \mathrm{Ker}\ P,\ \ Y = \mathrm{Im}\ L \oplus \mathrm{Im}\ Q.$$

It follows that

$$L|_{D(L)\cap \mathrm{Ker}\ P}: D(L)\cap \mathrm{Ker}\ P \to \mathrm{Im}\ L$$

is invertible, we denote the inverse of that map by K_p. If Ω is an open bounded subset of X, $D(L)\cap\overline{\Omega} \neq \emptyset$, the map $N: X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N: \overline{\Omega} \to X$ is compact.

Lemma 2.1[4]. *Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:*
(i) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(D(L)\setminus \mathrm{Ker}L)\cap\partial\Omega]\times(0,1)$;*
(ii) *$Nx \notin \mathrm{Im}L$ for every $x \in \mathrm{Ker}L\cap\partial\Omega$;*
(iii) *$\deg(\wedge QN\Big|_{\mathrm{Ker}L}, \Omega\cap\mathrm{Ker}L, 0) \neq 0$, where $\wedge: Y/\mathrm{Im}L \to \mathrm{Ker}L$ is the isomorphism. Then the equation $Lx = Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$.*

3. Main results

In this section, we prove the main results of this paper.

Lemma 3.1. *Suppose*
(A) *there exists constant $A > 0, p > 1$ such that*

$$|x|^p \leq A\phi(x)x \text{ for all } x \in R.$$

(C) there exist continuous functions $h : [0,T] \times R^{m+1} \to R$, $g_i : [0,T] \times R \to R$, and r such that
(i) *$f(t,x,x_1,\cdots,x_m) = h(t,x,x_1,\cdots,x_m) + g_0(t,x) + \sum_{i=1}^m g_i(t,x_i) + r(t)$ holds for all $(t,x,x_1,\cdots,x_m) \in [0,T] \times R^{m+1}$.*
(ii) *there exist constants $\theta \geq 1$ and $\beta > 0$ such that*

$$h(t,x,x_1,\cdots,x_m)x \geq \beta |x|^{\theta+1}$$

holds for all $(t,x,x_1,\cdots,x_m) \in [0,T] \times R^{m+1}$.
(iii) $\lim_{|x|\to+\infty} \sup_{t\in[0,T]} \frac{|g_i(t,x)|}{|x|^\theta} = r_i \in [0,+\infty)(i = 0,\cdots,m)$.
Let $\Omega_1 = \{(x,y) : L(x,y) = \lambda N(x,y),\ ((x,y),\lambda) \in [(D(L) \setminus \mathrm{Ker}L)] \times (0,1)\}$. Then Ω_1 is bounded if $r_0 + \sum_{i=1}^m \delta_i^{\frac{\theta}{\theta+1}} r_i < \beta$, where $\delta_i = \max_{t\in[0,T]} \frac{1}{|\tau_i'(t)|}$.

Proof. For $(x,y) \in \Omega_1$, we have $L \bullet (x,y) = \lambda N \bullet (x,y)$, $\lambda \in (0,1)$, i.e.

$$\begin{cases} x'(t) = \lambda \phi^{-1}(y(t)), \\ (p(t)y(t))' = \lambda f(t,x(t),,x(\tau_1(t)),\cdots,x(\tau_m(t))), \\ y(0) = 0, y(T) = 0. \end{cases} \tag{7}$$

It follows that

$$\begin{cases} [p(t)\phi(x'(t))]' = \phi(\lambda)\lambda f\left(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t))\right), \\ x'(0) = 0, x'(T) = 0. \end{cases}$$

Then

$$\begin{aligned} &-\int_0^T p(t)\phi(x'(t))x'(t)dt = \int_0^T [p(t)\phi(x'(t))]'x(t)dt \\ = \quad &\phi(\lambda)\lambda \int_0^T f\left(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t))\right)x(t)dt. \end{aligned}$$

Since $p(t) > 0$ for all $t \in [0,T]$, $\phi(x)x \geq 0$ for all $x \in R$, we get

$$\int_0^T f\left(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t))\right)x(t)dt \leq 0.$$

It follows from $(C)(i)$ that

$$\begin{aligned} &\int_0^T h\left(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t))\right)x(t)dt + \int_0^T g_0(t,x(t))x(t)dt \\ &+ \int_0^T r(t)x(t)dt + \sum_{i=1}^m \int_0^T g_i(t,x(\tau_i(t)))x(t)dt \leq 0. \end{aligned}$$

$(C)(ii)$ implies that

$$\beta \int_0^T |x(t)|^{\theta+1}dt \quad \leq \quad -\int_0^T g_0(t,x(t))x(t)dt - \int_0^T r(t)x(t)dt - \sum_{i=1}^m \int_0^T g_i(t,x(\tau_i(t)))x(t)dt.$$

Choose $\epsilon > 0$ such that

$$\beta > (r_0 + \epsilon) + \sum_{i=1}^{m} \delta_i^{\frac{\theta}{\theta+1}} (r_i + \epsilon). \tag{8}$$

Choose $\delta > 0$ such that

$$|g_i(t, x)| \leq |x|^\theta (r_i + \epsilon),\ |x| > \delta,\ t \in [0, T],\ i = 0, \cdots, m.$$

Then

$$\begin{aligned}
& \beta \int_0^T |x(t)|^{\theta+1} dt \\
\leq \;& \int_0^T |g_0(t, x(t))||x(t)|dt + \sum_{i=1}^{m} \int_0^T |g_i(t, x(\tau_i(t)))||x(t)dt + ||r|| \int_0^T |x(t)|dt \\
\leq \;& \int_{t\in[0,T],|x(t)|>\delta} |g_0(t, x(t))||x(t)|dt + \int_{t\in[0,T],|x(t)|\leq\delta} |g_0(t, x(t))||x(t)|dt \\
& + ||r|| \int_0^T |x(t)|dt + \sum_{i=1}^{m} \int_{t\in[0,T],|x(\tau_i(t))|>\delta} |g_i(t, x(\tau_i(t)))||x(t)|dt \\
& + \sum_{i=1}^{m} \int_{t\in[0,T],|x(\tau_i(t))|\leq\delta} |g_i(t, x(\tau_i(t)))||x(t)|dt \\
\leq \;& ||r|| \int_0^T |x(t)|dt + \delta T \max_{t\in[0,T],|x|\leq\delta} |g_0(t, x)| + (r_0 + \epsilon) \int_{t\in[0,T],|x(t)|>\delta} |x(t))|^{\theta+1} dt \\
& + \sum_{i=1}^{m} \max_{t\in[0,T],|x|\leq\delta} |g_i(t, x)| \int_{t\in[0,T],|x(\tau_i(t))|\leq\delta} |x(t)|dt \\
& + \sum_{i=1}^{m} (r_i + \epsilon) \int_{t\in[0,T],|x(\tau_i(t))|>\delta} |x(\tau_i(t))|^\theta |x(t)|dt \\
\leq \;& ||r|| \int_0^T |x(t)|dt + T\delta \max_{t\in[0,T],|x|\leq\delta} |g_0(t, x)| \\
& + (r_0 + \epsilon) \int_0^T |x(t))|^{\theta+1} dt + \sum_{i=1}^{m} (r_i + \epsilon) \int_0^T |x(\tau_i(t))|^\theta |x(t)|dt \\
& + \sum_{i=1}^{m} \max_{t\in[0,T],|x|\leq\delta} |g_i(t, x)| \int_{t\in[0,T],|x(\tau_i(t))|\leq\delta} |x(t)|dt
\end{aligned}$$

$$
\begin{aligned}
\leq\; & ||r||\int_0^T |x(t)|dt + T\delta \max_{t\in[0,T],|x|\leq\delta} |g_0(t,x)| + (r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt \\
& + \sum_{i=1}^m \max_{t\in[0,T],|x|\leq\delta} |g_i(t,x)| \int_{t\in[0,T],|x(\tau_i(t))|\leq\delta} |x(t)|dt \\
& + \sum_{i=1}^m (r_i+\epsilon)\left(\int_0^T |x(\tau_i(t))|^{\theta+1}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
=\; & ||r||\int_0^T |x(t)|dt + T\delta \max_{t\in[0,T],|x|\leq\delta} |g_0(t,x)| + (r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt \\
& + \sum_{i=1}^m \max_{t\in[0,T],|x|\leq\delta} |g_i(t,x)| \int_{t\in[0,T],|x(\tau_i(t))|\leq\delta} |x(t)|dt \\
& + \sum_{i=1}^m (r_i+\epsilon)\left(\int_{\tau_i(0)}^{\tau_i(T)} \frac{|x(s)|^{\theta+1}}{\tau_i'(t)}ds\right)^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
\leq\; & ||r||T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} + T\delta \max_{t\in[0,T],|x|\leq\delta} |g_0(t,x)| \\
& + (r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt + \sum_{i=1}^m \max_{t\in[0,T],|x|\leq\delta} |g_i(t,x)| \int_0^T |x(t)|dt \\
& + \sum_{i=1}^m (r_i+\epsilon)\delta_i^{\frac{\theta}{\theta+1}} \int_0^T |x(s)|^{\theta+1}ds \\
\leq\; & ||r||T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} + T\delta \max_{t\in[0,T],|x|\leq\delta} |g_0(t,x)| \\
& + (r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt + \sum_{i=1}^m \max_{t\in[0,T],|x|\leq\delta} |g_i(t,x)| T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
& + \sum_{i=1}^m (r_i+\epsilon)\delta_i^{\frac{\theta}{\theta+1}} \int_0^T |x(s)|^{\theta+1}ds.
\end{aligned}
$$

It follows from (8) that there is a constant $M_1 > 0$ such that $\int_0^T |x(t)|^\theta dt \leq M_1$. There exists $\mu \in [0,T]$ such that $|x(\mu)| \leq (M_1/T)^{\frac{1}{\theta}}$.

It is easy to see that there exists $\delta > 0$ such that $p(t) \geq \delta$ for all $t \in [0,T]$. Then

$$
\begin{aligned}
& \delta\int_0^T \phi(x'(t))x'(t)dt \leq \int_0^T p(t)\phi(x'(t))x'(t)dt = -\int_0^T [p(t)\phi(x'(t))]'x(t)dt \\
=\; & -\phi(\lambda)\lambda\int_0^T f\left(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))\right)x(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\phi(\lambda)\lambda\left[\int_0^T h\left(t, x(t), x(\tau_1(t)), \cdots, x(\tau_m(t))\right)x(t)dt\right.\\
&\quad \left.+\int_0^T g_0(t,x(t))x(t)dt+\int_0^T r(t)x(t)dt+\sum_{i=1}^m\int_0^T g_i(t,x(\tau_i(t)))x(t)dt\right]\\
&\le -\phi(\lambda)\lambda\left[\beta\int_0^T |x(t)|^{\theta+1}dt+\int_0^T g_0(t,x(t))x(t)dt\right.\\
&\quad \left.+\int_0^T r(t)x(t)dt+\sum_{i=1}^m\int_0^T g_i(t,x(\tau_i(t)))x(t)dt\right]\\
&\le \int_0^T |g_0(t,x(t))||x(t)|dt+\int_0^T |r(t)||x(t)|dt+\sum_{i=1}^m\int_0^T |g_i(t,x(\tau_i(t)))||x(t)|dt\\
&\le \int_{t\in[0,T],|x(t)|>\delta}|g_0(t,x(t))||x(t)|dt+\int_{t\in[0,T],|x(t)|\le\delta}|g_0(t,x(t))||x(t)|dt\\
&\quad +||r||\int_0^T |x(t)|dt+\sum_{i=1}^m\int_{t\in[0,T],|x(\tau_i(t))|>\delta}|g_i(t,x(\tau_i(t)))||x(t)|dt\\
&\quad +\sum_{i=1}^m\int_{t\in[0,T],|x(\tau_i(t))|\le\delta}|g_i(t,x(\tau_i(t)))||x(t)|dt\\
&\le ||r||\int_0^T |x(t)|dt+\delta T\max_{t\in[0,T],|x|\le\delta}|g_0(t,x)|+(r_0+\epsilon)\int_{t\in[0,T],|x(t)|>\delta}|x(t))|^{\theta+1}dt\\
&\quad +\sum_{i=1}^m\max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|\int_{t\in[0,T],|x(\tau_i(t))|\le\delta}|x(t)|dt\\
&\quad +\sum_{i=1}^m(r_i+\epsilon)\int_{t\in[0,T],|x(\tau_i(t))|>\delta}|x(\tau_i(t))|^{\theta}|x(t)|dt\\
&\le ||r||\int_0^T |x(t)|dt+T\delta\max_{t\in[0,T],|x|\le\delta}|g_0(t,x)|\\
&\quad +(r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt+\sum_{i=1}^m(r_i+\epsilon)\int_0^T |x(\tau_i(t))|^{\theta}|x(t)|dt\\
&\quad +\sum_{i=1}^m\max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|\int_{t\in[0,T],|x(\tau_i(t))|\le\delta}|x(t)|dt\\
&\le ||r||\int_0^T |x(t)|dt+T\delta\max_{t\in[0,T],|x|\le\delta}|g_0(t,x)|+(r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt\\
&\quad +\sum_{i=1}^m\max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|\int_{t\in[0,T],|x(\tau_i(t))|\le\delta}|x(t)|dt\\
&\quad +\sum_{i=1}^m(r_i+\epsilon)\left(\int_0^T |x(\tau_i(t))|^{\theta+1}dt\right)^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}}
\end{aligned}
$$

$$\begin{aligned}
&= \quad ||r||\int_0^T |x(t)|dt + T\delta \max_{t\in[0,T],|x|\le\delta}|g_0(t,x)| + (r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt \\
&\quad + \sum_{i=1}^m \max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|\int_{t\in[0,T],|x(\tau_i(t))|\le\delta}|x(t)|dt \\
&\quad + \sum_{i=1}^m (r_i+\epsilon)\left(\int_{\tau_i(0)}^{\tau_i(T)} \frac{|x(s)|^{\theta+1}}{\tau_i'(t)}ds\right)^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} \\
&\le \quad ||r||T^{\frac{\theta}{\theta+1}}\left(\int_0^T |x(t)|^{\theta+1}dt\right)^{\frac{1}{\theta+1}} + T\delta \max_{t\in[0,T],|x|\le\delta}|g_0(t,x)| \\
&\quad + (r_0+\epsilon)\int_0^T |x(t))|^{\theta+1}dt + \sum_{i=1}^m \max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|\int_{t\in[0,T],|x(\tau_i(t))|\le\delta}|x(t)|dt \\
&\quad + \sum_{i=1}^m (r_i+\epsilon)\delta_i^{\frac{\theta}{\theta+1}}\int_0^T |x(s)|^{\theta+1}ds \\
&\le \quad ||r||T^{\frac{\theta}{\theta+1}}M_1^{\frac{1}{\theta+1}} + T\delta \max_{t\in[0,T],|x|\le\delta}|g_0(t,x)| + (r_0+\epsilon)M_1 \\
&\quad + \sum_{i=1}^m \max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|T^{\frac{\theta}{\theta+1}}M_1^{\frac{1}{\theta+1}} + \sum_{i=1}^m (r_i+\epsilon)\delta_i^{\frac{\theta}{\theta+1}}M_1 \\
&=: \quad M_2.
\end{aligned}$$

It follows that $\int_0^T \phi(x'(t))x'(t)dt \le M_2/\delta$. Hence (A) implies that

$$\begin{aligned}
&|x(t)| \le \left|x(\mu) + \int_\mu^t x'(s)ds\right| \\
&\le \quad (M_1/T)^{\frac{1}{\theta}} + \int_0^T |x'(t)|dt \le (M_1/T)^{\frac{1}{\theta}} + T^{\frac{p-1}{p}}\left(\int_0^T |x'(t)|^p dt\right)^{\frac{1}{p}} \\
&\le \quad (M_1/T)^{\frac{1}{\theta}} + T^{\frac{p-1}{p}}A^{\frac{1}{p}}\left(\int_0^T \phi(x'(t))x'(t)dt\right)^{\frac{1}{p}} \\
&\le \quad (M_1/T)^{\frac{1}{\theta}} + T^{\frac{p-1}{p}}A^{\frac{1}{p}}M_2/\delta.
\end{aligned}$$

Then $||x|| \le (M_1/T)^{\frac{1}{\theta}} + T^{\frac{p-1}{p}}A^{\frac{1}{p}}(M_2/\delta)^{\frac{1}{p}}$. Hence

$$\begin{aligned}
&|p(t)y(t)| = \left|p(0)y(0) + \int_0^t [p(s)y(s)]'ds\right| \\
&\le \quad \int_0^T |f(t,x(t),x(\tau_1(t)),\cdots,x(\tau_m(t)))|dt \\
&\le \quad T \max_{t\in[0,T],|x_i|\le(M_1/T)^{\frac{1}{\theta}}+T^{\frac{p-1}{p}}A^{\frac{1}{p}}(M_2/\delta)^{\frac{1}{p}},i=0,\cdots,m}|f(t,x_0,x_1,\cdots,x_m)|.
\end{aligned}$$

It follows that

$$||y|| \le \frac{T\max_{t\in[0,T],|x_i|\le(M_1/T)^{\frac{1}{\delta}}+T^{\frac{p-1}{p}}A^{\frac{1}{p}}(M_2/\delta)^{\frac{1}{p}},i=0,\cdots,m}|f(t,x_0,x_1,\cdots,x_m)|}{\delta}.$$

Hence, for $(x,y)\in\Omega_1$, there is $H>0$ such that $||(x,y)||\le H$. Hence Ω_1 is bounded. □

Suppose
(B) There exists a constant $M_0>0$ such that

$$a\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt<0 \text{ for all } t\in[0,T],\ |a|>M_0, \tag{9}$$

or

$$a\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt>0 \text{ for all } t\in[0,T],\ |a|>M_0. \tag{10}$$

Lemma 3.2. *Suppose* (B) *holds. Then* $\Omega_2=\{(x,y)\in\text{Ker } L:\ N(x,y)\in\text{Im } L\}$ is bounded.

Proof. For $(a,0)\in\text{Ker}L$, we have $N(a,0)=(0,f(t,a,\overbrace{a,\cdots,a}^{m}),0,0)$. $Nx\in\text{Im}L$ implies that

$$\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt=0.$$

If $|a|>M_0$, then (B) implies that

$$\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt\ne 0,$$

a contradiction. Hence $|a|\le M_0$. Thus Ω_2 is bounded. □

Lemma 3.3. *Suppose* (B) *holds. Let*

$$\Omega_3=\{(x,y)\in\text{Ker}L:\ \lambda\wedge(x,y)-(1-\lambda)QN(x,y)=0,\ \lambda\in[0,1]\} \tag{11}$$

if (10) *holds, and*

$$\Omega_3=\{(x,y)\in\text{Ker}L:\ \lambda\wedge(x,y)+(1-\lambda)QN(x,y)=0,\ \lambda\in[0,1]\} \tag{12}$$

if (11) *holds. Then* Ω_3 *is bounded at either case, where* $\wedge:\text{Ker}L\to Y/\text{Im}L$ *defined by* $\wedge(a,0)=(0,a,0,0)$.

Proof. Consider the case when (9) holds. We note

$$\Omega_3 = \{(x,y) \in \mathrm{Ker}L : \ \lambda \wedge (x,y) - (1-\lambda)QN(x,y) = 0, \ \lambda \in [0,1]\}.$$

We will prove that Ω_3 is bounded. For $(a,0) \in \Omega_3$, and $\lambda \in [0,1]$, we have

$$\lambda(0,a,0,0) - (1-\lambda)\left(0, \frac{1}{T}\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt, 0, 0\right) = 0.$$

Then

$$\lambda a = (1-\lambda)\frac{1}{T}\left(\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt\right).$$

Then we have

$$\lambda a^2 = (1-\lambda)a\frac{1}{T}\left(\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt\right).$$

If $\lambda = 1$, then $a = 0$. If $\lambda \in [0,1)$ and $|a| > M_0$, from condition (B), we get that

$$0 \le \lambda a^2 = (1-\lambda)a\frac{1}{T}\left(\int_0^T f(t,a,\overbrace{a,\cdots,a}^{m})dt\right) < 0,$$

a contradiction. Hence $|a| \le M_0$. Thus Ω_3 is bounded. Similarly, we can prove that Ω_3 defined in (12) is bounded when (10) holds. □

Theorem L. *Suppose* (A), (C) *and* (B) *hold. Then equation* (6) *has at least one* $T-$*periodic solution if* $r_0 + \sum_{i=1}^m \delta_i^{\frac{\theta}{\theta+1}} r_i < \beta$.

Proof. We know that L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. Since (x,y) is a solution of $L(x,y) = N(x,y)$ implies that x is a solution of equation (5). It suffices to get a solution (x,y) of $L(x,y) = N(x,y)$. To do this, we construct an open bounded set Ω such that (i), (ii) and (iii) of Theorem GM hold.

Set Ω be a open bounded subset of X such that $\Omega \supset \cup_{i=1}^3 \overline{\Omega_i}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus, from Lemma 3.1 and Lemma 3.2, that $L(x,y) \ne \lambda N(x,y)$ for $(x,y) \in D(L) \setminus \mathrm{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $N(x,y) \notin \mathrm{Im}L$ for $(x,y) \in \mathrm{Ker}L \cap \partial\Omega$.

In fact, let $H((x,y),\lambda) = \pm\lambda\wedge(x,y) + (1-\lambda)QN(x,y)$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H((x,y),\lambda) \ne 0$ for $(x,y) \in \partial\Omega \cap \mathrm{Ker}L$, thus, from Lemma 3.3, by homotopy property of degree,

$$\begin{aligned} & \deg(QN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) = \deg(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0) \\ = \ & \deg(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0) = \deg(\pm\wedge, \Omega \cap \mathrm{Ker}L, 0) \ne 0. \end{aligned}$$

Thus by Theorem GM, $L(x,y) = N(x,y)$ has at least one solution in $D(L) \cap \overline{\Omega}$, then x is a $T-$solution of equation (6). The proof is completed. □

4. An example

In this section, we present an example, which can not be solved by known theorems in [3], [10], [11], to illustrate the main result given in Section 3.

Example 4.1. Consider the problem

$$(13)\quad \begin{cases} [(1+t^2)\phi(x'(t))]' = \frac{[x(t)]^5}{1+2[\sin x(t)]^8} + q_1(t)[x(t)]^5 + q_2(t)[x(t/3)]^5 + r(t), \\ x'(0)=0,\ x'(1)=0, \end{cases}$$

where $\tau_1(t) = t/3,\ \phi(x) = |x|^4x,\ p(t) = 1+t^2,\ q_1, q_2, r \in C^0[0,1]$. We will use Theorem L. Corresponding to the assumptions of Theorem L, we set

$$f(t, x_0, x_1) = \frac{[x_0]^5}{1+2[\sin x_0]^8} + q_1(t)[x_0]^5 + q_2(t)[x_1]^5 + r(t).$$

$$h(t, x_0, x_1)x_0 = \frac{x_0^6}{1+2[\sin x_0]^8} \geq \frac{1}{3}|x_0|^6,$$

and

$$g_0(t,x) = q_1(t)x^5, \quad g_1(t,y) = q_2(t)y^5$$

and $\tau_1(t) = t/3$, $\beta = 1/3$, $p(t) = 1+t^2$, $\theta = 5$.

$$\begin{aligned} & a\int_0^T f(t,a,a)dt \\ = \;& a\int_0^1 r(t)dt + a^6\int_0^1 \left(\frac{1}{1+2[\sin a]^8} + q_1(t) + q_2(t)\right)dt \\ \geq \;& a\int_0^1 r(t)dt + a^6\int_0^1 \left(\frac{1}{3} + q_1(t) + q_2(t)\right)dt. \end{aligned}$$

It follows from Theorem L that problem (13) has at least one solution if

$$\frac{1}{3} > ||q_1|| + 3^{\frac{5}{6}}||q_2||, \quad \int_0^1 \left(q_1(t) + q_2(t) + \frac{1}{3}\right)dt > 0.$$

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