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Meromorphic Functions with Three Weighted Sharing Values

Xiao-Min Li
Department of Mathematics, Ocean University of China, Qingdao, Shandong 266071, People's Republic of China
e-mail : xmli01267@gmail.com

Hong-Xun Yi
Department of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China
e-mail : hxyi@sdu.edu.cn

Abstract. In this paper, we prove some results on uniqueness of meromorphic functions with three weighted sharing values. The results in this paper improve those given by H. X. Yi, I. Lahiri, T. C. Alzahary and H. X. Yi and other authors.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [3], [4], [8]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r,h)$ any quantity satisfying

$$S(r,h)=o(T(r,h)) \quad (r\to\infty, r\notin E).$$

Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share the value 0 CM, and we say that f and g share ∞ IM, if f and g share the value 0 IM (see[9]). In this paper, we also need the following two definitions.

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Definition 1.1 ([1, Definition 1]). Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_{p)}(r, 1/(f-a))$ we denote the counting function of those zeros of $f-a$, (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r, 1/(f-a))$ we denote the corresponding reduced counting function, (ignoring multiplicities). By $N_{(p}(r, 1/(f-a))$ we denote the counting function of those zeros of $f-a$, (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r, 1/(f-a))$ we denote the corresponding reduced counting function, (ignoring multiplicities).

Definition 1.2 ([5, Definition 4]). For $a \in C \cup \{\infty\}$, we put

$$\delta_{p)}(a, f) = 1 - \limsup_{r\to\infty} \frac{N_{p)}(r, \frac{1}{f-a})}{T(r, f)},$$

where p is a positive integer.

In 1995, Yi proved the following one theorem.

Theorem A ([10, Theorem 4]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM, and let* $a\,(\neq 0, 1)$ *be a finite complex number. If*

$$N(r, \frac{1}{f-a}) \neq T(r, f) + S(r, f),$$

then a is a Picard exceptional value of f, and f and g satisfy one of the following three relations:

(i) $(f-a)(g+a-1) = a(1-a)$;

(ii) $f + (a-1)g = a$;

(iii) $f = ag$.

In 1995, H. X. Yi and C. C. Yang proved the following theorem.

Theorem B ([9, Theorem 5.13]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If f is not a fractional linear transformation* (*Möbius transformation*) *of g, then*

(i) $N_0(r, \frac{1}{f'}) = \overline{N}_0(r, \frac{1}{f'}) + S(r, f)$, $\overline{N}(r, \frac{1}{f'}) = \overline{N}_0(r, \frac{1}{f'}) + S(r, f)$, *the same identities hold for g*;

(ii) $N_{(3}(r, \frac{1}{f-a}) = S(r, f)$, $N_{(3}(r, \frac{1}{g-a}) = S(r, f)$;

(iii) $T(r, f) = \overline{N}(r, \frac{1}{g'}) + N_0(r) + S(r, f)$, $T(r, g) = \overline{N}(r, \frac{1}{f'}) + N_0(r) + S(r, f)$, $N_0(r) = \overline{N}_0(r) + S(r, f)$;

(iv) $T(r, f) = N(r, \frac{1}{f-a}) + S(r, f)$, $T(r, g) = N(r, \frac{1}{g-a}) + S(r, f)$;

(v) $T(r, f) + T(r, g) = \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-1}) + \overline{N}(r, f) + N_0(r) + S(r, f)$;

(vi) $N(r, \frac{1}{f-g}) = \overline{N}(r, \frac{1}{f-g}) + S(r, f)$;

where $N_0(r, 1/f')\,(\overline{N}_0(r, 1/f'))$ *denotes the counting function corresponding to the zeros of* f' *that are not zeros of* f *and* $f-1$ *(ignoring multiplicities) and* $N_0(r)\,(\overline{N}_0(r))$ *is the counting function of the zeros of* $f-g$ *that are not zeros of* g, $g-1$ *and* $1/g$ *(ignoring multiplicities), and* $a\,(\neq 0, 1)$ *is a finite complex number.*

Regarding Theorem A and Theorem B, it is natural to ask the following question.

Question 1.1 ([6]). Is it really possible to relax in any way the nature of sharing any one of 0, 1 and ∞ in Theorem A and Theorem B ?

In this paper, we will study Question 1.1. Next we will explain the notion of weighted sharing by the following definition.

Definition 1.3 ([5]). Let k be a nonnegative integer or infinity. For any $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all $a-$points of f, where an $a-$point of multiplicity m is counted m times if $m \leq k$, and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

Remark 1.1. Definition 1.3 implies that if f, g share a value a with weight k, then z_0 is a zero of $f-a$ with multiplicity $m\,(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity $m\,(\leq k)$, and z_0 is a zero of $f-a$ with multiplicity $m\,(> k)$, if and only if it is a zero of $g-a$ with multiplicity $n\,(> k)$, where m is not necessarily equal to n. Throughout this paper, we write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k), then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Recently, T. C. Alzahary and H. X. Yi proved the following result.

Theorem C ([1, Theorem 1])**.** *Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* $(a_1, 1)$, (a_2, ∞) *and* (a_3, ∞), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$. *Then either there exists an entire function* γ *such that* f *and* g *are given as one of the following three expressions:*

$$(a)\quad f = \frac{e^\gamma - 1}{c-1} \quad \text{and} \quad g = \frac{c(e^\gamma - 1)}{(c-1)e^\gamma};$$

$$(b)\quad f = \frac{c-1}{e^\gamma - 1} \quad \text{and} \quad g = \frac{(c-1)e^\gamma}{c(e^\gamma - 1)};$$

$$(c)\quad f = \frac{c(e^\gamma - 1)}{e^\gamma - c} \quad \text{and} \quad g = \frac{e^\gamma - 1}{e^\gamma - c};$$

for some $c \in C \setminus \{0, 1\}$, *or else for any* $a \in C \setminus \{0, 1\}$, *each of* (i)-(vi) *mentioned in Theorem B holds.*

In this paper, we will prove the following theorem, which improves Theorem C.

Theorem 1.1. *Let f and g be two distinct nonconstant meromorphic functions sharing $(a_1,1)$, (a_2,m) and (a_3,k), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, m and k are positive integers such that*

$$(m-1)(km-1)>(1+m)^2. \tag{1.1}$$

Then either there exists an entire function γ such that f and g are given as one of the (a), (b) and (c) in Theorem C for some $c\in C\setminus\{0,1\}$, or else for any $a\in C\setminus\{0,1\}$, each of (i)-(vi) *mentioned in Theorem B holds.*

From Theorem 1.1 and the conclusion (ii) in Theorem B, in the same manner as in the proof of Theorem 2 in [1] we get the following theorem, which improves Theorem A.

Theorem 1.2. *Let f and g be two distinct nonconstant meromorphic functions sharing $(a_1,1)$, (a_2,m) and (a_3,k), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, m and k are positive integers satisfying* (1.1)*, and let $a\,(\neq 0,1)$ be a finite complex number. If*

$$N_{2)}(r,\frac{1}{f-a})\neq T(r,f)+S(r,f), \tag{1.2}$$

then a is a Picard exceptional value of f, and there exists an entire function γ such that f and g are given as one of the following three expressions:

(i) $f=a(1-e^{\gamma})$ *and* $g=(1-a)(1-e^{-\gamma})$;

(ii) $f=\dfrac{a}{1-e^{\gamma}}$ *and* $g=\dfrac{a}{(a-1)(1-e^{-\gamma})}$;

(iii) $f=\dfrac{ae^{\gamma}-a}{ae^{\gamma}-1}$ *and* $g=\dfrac{e^{\gamma}-1}{ae^{\gamma}-1}$.

From Theorem 1.2 we get the following corollary.

Corollary 1.1. *Let f and g be two nonconstant meromorphic functions sharing $(a_1,1)$, (a_2,m) and (a_3,k), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, m and k are positive integers such that* (1.1) *holds, and let $a\,(\neq 0,1)$ be a finite complex number satisfying* (1.2)*. If $\sigma(f)<\infty$, where $\sigma(f)$ is the order of f such that $\sigma(f)$ is not a positive integer, then $f=g$.*

From Theorem 1.1 and in the same manner as in the proof of Theorem 3 in [1], we get the following theorem.

Theorem 1.3. *Let f and g be two distinct nonconstant meromorphic functions sharing $(a_1,1)$, (a_2,m) and (a_3,k), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, m and k are positive integers satisfying* (1.1)*. If f is not any fractional linear transformation of g, then for any $a\in C\setminus\{0,1\}$, each of* (i)-(vi) *mentioned in Theorem B still holds.*

2. Some lemmas

Lemma 2.1 ([2]). *Let f and g be two meromorphic functions sharing* 0, 1, ∞ *IM, then*

$$T(r,f) \leq 3T(r,g) + S(r,f) \quad \textit{and} \quad T(r,g) \leq 3T(r,f) + S(r,g).$$

Remark 2.1. From Lemma 2.1 we can see that if f and g share 0, 1, ∞ IM, then $S(r,f) = S(r,g) =: S(r)$.

Lemma 2.2 ([7, Lemma 2])**.** *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0,1)$, $(1,m)$ and (∞,k), where m and k are positive integers satisfying* (1.1)*. Then*

(*i*) $\overline{N}_{(2}(r,\frac{1}{f}) + \overline{N}_{(2}(r,\frac{1}{f-1}) + \overline{N}_{(2}(r,f) = S(r)$,

(*ii*) $\overline{N}_{(2}(r,\frac{1}{g}) + \overline{N}_{(2}(r,\frac{1}{g-1}) + \overline{N}_{(2}(r,g) = S(r)$.

From Lemma 2.2 we get the following result.

Lemma 2.3. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0,1)$, $(1,m)$ and (∞,k), where m and k are positive integers satisfying* (1.1)*, and let*

$$\frac{f-1}{g-1} = h_1 \tag{2.1}$$

and

$$\frac{f}{g} = h_2. \tag{2.2}$$

Then

$$\overline{N}(r,\frac{1}{h_j}) + \overline{N}(r,h_j) = S(r)\,(j=1,2). \tag{2.3}$$

Remark 2.2. Let

$$h_0 = \frac{h_1}{h_2}. \tag{2.4}$$

Then from Lemma 2.2 we get

$$T(r,\frac{h_j'}{h_j}) = S(r)\,(j=0,1,2) \quad \textit{and} \quad T(r,\frac{h_0'}{h_0}) = S(r). \tag{2.5}$$

Lemma 2.4 ([7, Lemma 5]). *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0,1)$, $(1,m)$ and (∞,k), where m and k are positive integers satisfying* (1.1). *Then for any $a\,(\neq 0,1,\infty)$,*

$$\overline{N}_{(3}(r,\frac{1}{f-a})+\overline{N}_{(3}(r,\frac{1}{g-a})=S(r).$$

3. Proof of theorems

Proof of Theorem 1.1. We discuss the following six cases.

Case 1. Suppose that f and g share $(0,1)$, $(1,m)$ and (∞,k). Since $f\not\equiv g$, thus from Lemma 2.3 we have (2.1) and (2.2), where $h_1\not\equiv 1$, $h_2\not\equiv 1$ and $h_0\not\equiv 1$. Again from (2.1), (2.2) and (2.4) we get

$$f=\frac{h_1-1}{h_0-1} \tag{3.1}$$

and

$$g=\frac{h_1^{-1}-1}{h_0^{-1}-1}. \tag{3.2}$$

We discuss the following four subcases.

Subcase 1.1. Suppose that

$$h_0=c, \tag{3.3}$$

where $c\,(\neq 0,1)$ is a finite complex number. Then from (2.1), (2.2), (2.4) and (3.3) we can get

$$\frac{g}{g-1}=\frac{cf}{f-1}. \tag{3.4}$$

Since f and g share 0, 1 and ∞ IM, thus from (3.4) we deduce that ∞ is a Picard exceptional value of f and g, and that f and g share 0 and 1 CM. So from (2.1) we let

$$h_1=e^{\gamma}, \tag{3.5}$$

where γ is an entire function. From (3.1)-(3.3) and (3.5) we obtain the expression (a) in Theorem C.

Subcase 1.2. Suppose that

$$h_1=c, \tag{3.6}$$

where $c\,(\neq 0,1)$ is a finite complex number. Since f and g share 0, 1 and ∞ IM, thus from (2.1) and (3.6) we deduce that 0 is a Picard exceptional value of f and g, and that f and g share 1 and ∞ CM. From this, (2.1), (2.2), (2.4) and (3.6) we let

$$h_0 = e^{\gamma}, \tag{3.7}$$

where γ is an entire function. From (3.1), (3.2), (3.6) and (3.7) we obtain the expression (b) in Theorem C.

Subcase 1.3. Suppose that

$$h_2 = c, \tag{3.8}$$

where $c\,(\neq 0,1)$ is a finite complex number. Since f and g share 0, 1 and ∞ IM, thus from (2.2) and (3.8) we deduce that 1 is a Picard exceptional value of f and g, and that f and g share 0 and ∞ CM. From this and (2.1) we have (3.5), from (2.4), (3.1), (3.2), (3.5) and (3.8) we obtain the expression (c) in Theorem C.

Subcase 1.4. Suppose that none of h_1, h_2 and h_0 are constants. Let

$$h = \frac{\frac{h_1'}{h_1}}{\frac{h_0'}{h_0}} = \frac{\frac{h_1'}{h_1}}{\frac{h_1'}{h_1} - \frac{h_2'}{h_2}}. \tag{3.9}$$

Then from (2.1), (2.2), (2.3) and (3.9) we deduce

$$T(r,h) = S(r). \tag{3.10}$$

If

$$\frac{h_1'}{h_1} \cdot (h-1) - h' \equiv 0,$$

then

$$h_1 = c(h-1), \tag{3.11}$$

where $c\,(\neq 0)$ is a finite complex number. From (3.10) and (3.11) we deduce

$$T(r,h_1) = S(r). \tag{3.12}$$

Again from (3.9) and (3.11) we have

$$\frac{h_0'}{h_0} = \frac{\frac{ch_1'}{h_1}}{h_1 + c} = -\frac{(ch_1^{-1}+1)'}{ch_1^{-1}+1}. \tag{3.13}$$

By integrating two sides of (3.13) we get

$$h_0 = \frac{d}{ch_1^{-1}+1}, \tag{3.14}$$

where $d\,(\neq 0)$ is a finite complex number. From (3.12) and (3.14) we get

$$T(r,h_0)=T(r,h_1)+O(1)=S(r). \tag{3.15}$$

From (3.1), (3.12) and (3.15) we get $T(r,f)=S(r)$, this is impossible. Thus

$$\frac{h_1'}{h_1}\cdot(h-1)-h'\not\equiv 0,$$

from which and (3.1) we get

$$f-h=\frac{h_1-h_0h+h-1}{h_0-1}. \tag{3.16}$$

Let

$$F=(f-h)(h_0-1)=h_1-h_0h+h-1. \tag{3.17}$$

From (3.9) and (3.17) we get

$$\frac{F'}{F}-\frac{h_1'}{h_1}=\frac{(h_1-h_0h+h-1)'-\frac{h_1'}{h_1}\cdot(h_1-h_0h+h-1)}{(f-h)(h_0-1)}=\frac{\frac{h_1'}{h_1}\cdot(h-1)-h'}{f-h},$$

from which we get

$$\frac{1}{f-h}=\frac{\frac{F'}{F}-\frac{h_1'}{h_1}}{\frac{h_1'}{h_1}\cdot(h-1)-h'}. \tag{3.18}$$

From (2.3), (3.10) and (3.18) we deduce

$$m(r,\frac{1}{f-h})=S(r) \tag{3.19}$$

and

$$N_{(2}(r,\frac{1}{f-h})=S(r). \tag{3.20}$$

From (2.1) and (3.2) we get

$$\frac{f-g}{g-1}=h_1-1 \quad \text{and} \quad g=\frac{h_1-1}{h_1-h_2}. \tag{3.21}$$

Thus

$$\frac{g'(f-g)}{g(g-1)}=\frac{(\frac{h_2'}{h_2}-\frac{h_1'}{h_1})\cdot h_1+\frac{h_1'}{h_1}\cdot h_0-\frac{h_2'}{h_2}}{h_0-1}. \tag{3.22}$$

On the other hand, from (3.9) and (3.18) we get

$$(f-h)\cdot(\frac{h_2'}{h_2}-\frac{h_1'}{h_1})=\frac{(\frac{h_2'}{h_2}-\frac{h_1'}{h_1})\cdot h_1+\frac{h_1'}{h_1}\cdot h_0-\frac{h_2'}{h_2}}{h_0-1}. \tag{3.23}$$

From (3.22) and (3.23) we get

$$-\frac{h_0'}{h_0}\cdot(f-h)=\frac{g'(f-g)}{g(g-1)}. \tag{3.24}$$

From (2.3), (2.5), (3.20), (3.21) and (3.24) we deduce

$$N(r,\frac{1}{f-h})=N_0(r)+N_0(r,\frac{1}{g'})+S(r) \tag{3.25}$$

$$N_0(r)=\overline{N}_0(r)+S(r) \tag{3.26}$$

and

$$N_0(r,\frac{1}{g'})=\overline{N}_0(r,\frac{1}{g'})+S(r). \tag{3.27}$$

From (3.27) and Lemma 2.2 we deduce

$$N_0(r,\frac{1}{g'})=\overline{N}(r,\frac{1}{g'})+S(r). \tag{3.28}$$

which implies (i) of Theorem B. From (3.10), (3.19) and (3.25) we deduce

$$T(r,f)=N_0(r)+N_0(r,\frac{1}{g'})+S(r). \tag{3.29}$$

In the same manner as above we obtain

$$N_0(r,\frac{1}{f'})=\overline{N}(r,\frac{1}{f'})+S(r) \tag{3.30}$$

and

$$T(r,g)=N_0(r)+N_0(r,\frac{1}{f'})+S(r). \tag{3.31}$$

From (3.28), (3.29), (3.30) and (3.31) we get (iii) of Theorem B. Next we denote by $\overline{N}_{(k,l)}(r,a_i)$ $(i=1,2,3)$ the counting function of those points in $\overline{N}(r,1/f-a_i)$, such that a_i is taken by f with multiplicity k, and such that a_i is taken by g with multiplicity l, and each point is counted only once. First, from Lemma 2.2 we get

$$\overline{N}(r,g)=\overline{N}_{(1,1)}(r,g)+S(r) \tag{3.32}$$

On the other hand, since

$$N(r,f-g)\leq\sum_{l\geq 1}\sum_{k\geq l}k\overline{N}_{(k,l)}(r,\infty)+\sum_{k\geq 1}\sum_{l>k}l\overline{N}_{(k,l)}(r,\infty), \tag{3.33}$$

from (3.32) and (3.33) we deduce

$$\begin{aligned} & N(r,f-g)+\overline{N}(r,g) \\ \leq\ & \sum_{l\geq 1}\sum_{k\geq l}k\overline{N}_{(k,l)}(r,\infty)+\overline{N}_{(1,1)}(r,g)+\sum_{k\geq 1}\sum_{l>k}l\overline{N}_{(k,l)}(r,\infty)+S(r) \\ \leq\ & N(r,f)+N(r,g)+S(r). \end{aligned} \tag{3.34}$$

From (3.26), (3.29), (3.34) and the second fundamental theorem we have

$$\begin{aligned} & T(r,f)+T(r,g) \\ \leq\ & T(r,f)+\overline{N}(r,\frac{1}{g})+\overline{N}(r,g)+\overline{N}(r,\frac{1}{g-1})-N_0(r,\frac{1}{g'})+S(r) \\ =\ & N_0(r)+\overline{N}(r,\frac{1}{g})+\overline{N}(r,g)+\overline{N}(r,\frac{1}{g-1})+S(r) \\ \leq\ & \overline{N}(r,\frac{1}{f-g})+\overline{N}(r,g)+S(r)\leq N(r,\frac{1}{f-g})+\overline{N}(r,g)+S(r) \\ \leq\ & T(r,f-g)+\overline{N}(r,g)+S(r) \\ \leq\ & m(r,f)+m(r,g)+N(r,f-g)+\overline{N}(r,g)+S(r) \\ \leq\ & m(r,f)+m(r,g)+N(r,f)+N(r,g)+S(r) \\ =\ & T(r,f)+T(r,g)+S(r), \end{aligned}$$

thus

$$T(r,f)+T(r,g)=N_0(r)+\overline{N}(r,\frac{1}{g})+\overline{N}(r,g)+\overline{N}(r,\frac{1}{g-1})+S(r), \tag{3.35}$$

$$N(r,\frac{1}{f-g})=\overline{N}(r,\frac{1}{f-g})+S(r) \tag{3.36}$$

From (3.35) and (3.36) we get (v) and (vi) of Theorem B. Since

$$N(r,\frac{1}{f'})=(N(r,\frac{1}{f})-\overline{N}(r,\frac{1}{f})+(N(r,\frac{1}{f-1})-\overline{N}(r,\frac{1}{f-1})+N_0(r,\frac{1}{f'}), \tag{3.37}$$

thus from (3.31), (3.35), (3.37) and the second fundamental theorem we get

$$\begin{aligned}
&2T(r,f)\\
\leq\ & N(r,\frac{1}{f})+\overline{N}(r,f)+N(r,\frac{1}{f-1})+N(r,\frac{1}{f-a})-N(r,\frac{1}{f'})+S(r)\\
=\ & \overline{N}(r,\frac{1}{f})+\overline{N}(r,f)+\overline{N}(r,\frac{1}{f-1})+N(r,\frac{1}{f-a})-N_0(r,\frac{1}{f'})+S(r)\\
=\ & \overline{N}(r,\frac{1}{f})+\overline{N}(r,f)+\overline{N}(r,\frac{1}{f-1})+N(r,\frac{1}{f-a})+N_0(r)-T(r,g)+S(r)\\
=\ & T(r,f)+N(r,\frac{1}{f-a})+S(r)\\
\leq\ & 2T(r,f)+S(r),
\end{aligned}$$

which implies that

$$N(r,\frac{1}{f-a})=T(r,f)+S(r). \tag{3.38}$$

From (3.38) we get (iv) of Theorem B.

Let z_0 is a zero of $g-a$ with multiplicity≥ 3, then z_0 is a zero of $g'(f-g)$ with multiplicity ≥ 2. From this, (2.5), (3.20) and (3.24) we obtain

$$N_{(3}(r,\frac{1}{g-a})-\overline{N}_{(3}(r,\frac{1}{g-a})=S(r). \tag{3.39}$$

Thus from (3.39) and Lemma 2.4 we get

$$N_{(3}(r,\frac{1}{g-a})=S(r). \tag{3.40}$$

In the same manner as above we get

$$N_{(3}(r,\frac{1}{f-a})=S(r). \tag{3.41}$$

From (3.40) and (3.41) we get (ii) of Theorem B. Case 1 is thus completely proved.

Case 2. Suppose that f and g share $(0,1)$, (∞,m) and $(1,k)$. Let $F=f/f-1$, $G=g/g-1$, and $b=a/a-1$. Then F and G share $(0,1)$, $(1,m)$ and (∞,k), and $F\not\equiv G$. In the same manner as in Case 1, we obtain that the conclusion of Theorem 1.1 holds for F, G and b. From this we obtain the conclusion of Theorem 1.1 holds for f, g and a.

Case 3. Suppose that f and g share $(1,1)$, $(0,m)$ and (∞,k). Let $F=1-f$, $G=1-g$, and $b=1-a$. Then F and G share $(0,1)$, $(1,m)$ and (∞,k), and $F\not\equiv G$. In the same manner as in Case 1, we obtain that the conclusion of Theorem 1.1 holds for F, G and b. From this we obtain the conclusion of Theorem 1.1 holds for f, g and a.

Case 4. Suppose that f and g share $(1,1)$, (∞,m) and $(0,k)$. Let $F=f-1/f$, $G=g-1/g$, and $b=a-1/a$. Then F and G share $(0,1)$, $(1,m)$ and (∞,k), and $F\not\equiv G$. In the same manner as in Case 1, we obtain that the conclusion of Theorem 1.1 holds for F, G and b. From this we obtain the conclusion of Theorem 1.1 holds for f, g and a.

Case 5. Suppose that f and g share $(\infty,1)$, $(0,m)$ and $(1,k)$. Let $F=1/1-f$, $G=1/1-g$, and $b=1/1-a$. Then F and G share $(0,1)$, $(1,m)$ and (∞,k), and $F\not\equiv G$. In the same manner as in Case 1, we can obtain that the conclusion of Theorem 1.1 holds for F, G and b. From this we can obtain the conclusion of Theorem 1.1 holds for f, g and a.

Case 6. Suppose that f and g share $(\infty,1)$, $(1,m)$ and $(0,k)$. Let $F=1/f$, $G=1/g$ and $b=1/a$. Then F and G share $(0,1)$, $(1,m)$ and (∞,k), and $F\not\equiv G$. In the same manner as in Case 1, we obtain that the conclusion of Theorem 1.1 for F, G and b. From this we obtain the conclusion of Theorem 1.1 for f, g and a. Theorem 1.1 is thus completely proved. □

4. On some results of Yi and I. Lahiri

In 1995, Yi and Yang proved the following result.

Theorem D (see [9, Lemma 4.5]). *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM, and let* $a\,(\neq 0,1)$ *be a finite complex number. Then* (3.41) *holds.*

In 2001, I. Lahiri proved the following result.

Theorem E ([5, Lemma 5]). *Let f and g be two distinct meromorphic functions sharing* $(0,1)$, $(1,\infty)$ *and* (∞,∞), *and let* $a(\neq 0,1)$ *be a finite complex number. Then*

$$\overline{N}_{(3}(r,\frac{1}{f-a})=S(r). \tag{4.1}$$

In 2003, I. Lahiri proved the following result.

Theorem F ([7, Lemma 5]). *Let f and g be two distinct nonconstant meromorphic functions sharing* $(a_1,1)$, (a_2,m) *and* (a_3,k), *where* $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, *$m$ and k are positive integers satisfying* (1.1), *and let* $a(\neq 0,1)$ *be a finite complex number. Then* (4.1) *holds.*

In 2004, T. C. Alzahary and H. X. Yi proved the following result.

Theorem G ([1, Theorem 4]). *Let f and g be two distinct nonconstant meromor-*

phic functions sharing $(a_1, 1)$, (a_2, ∞) *and* (a_3, ∞), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, *and let* $a\,(\neq 0, 1)$ *be a finite complex number. Then* (3.41) *holds.*

From Theorem 1.1 and in the same manner as in the proof of Theorem 4 in [1], we get the following result, which improves Theorem D, Theorem E, Theorem F and Theorem G.

Theorem 4.1. *Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* $(a_1, 1)$, (a_2, m) *and* (a_3, k), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, m *and* k *are positive integers satisfying* (1.1), *and let* $a\,(\neq 0, 1)$ *be a finite complex number. Then* (3.41) *holds.*

5. On some other results of Yi and I. Lahiri

In 2001, I. Lahiri proved the following result.

Theorem H ([5, Theorem 2]). *Let* f *and* g *be two distinct meromorphic functions sharing* $(0, 1)$, $(1, \infty)$ *and* (∞, ∞). *If* $a\,(\neq 0, 1)$ *is a finite complex number such that* $3\delta_{2)}(a, f) + 2\delta_{1)}(\infty, f) > 3$, *then* a *and* ∞ *are Picard exceptional values of* f, $1 - a$ *and* ∞ *are Picard exceptional values of* g, *and* $(f - a)(g + a - 1) = a(1 - a)$.

In 2004, T. C. Alzahary and H. X. Yi proved the following result.

Theorem K ([1, Theorem 5]). *Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* $(a_1, 1)$, (a_2, ∞) *and* (a_3, ∞), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, *and let* $a\,(\neq 0, 1)$ *be a finite complex number such that* (1.2) *holds. Then*

- (i) *If* $\overline{N}_{1)}(r, f) \neq T(r, f) + S(r, f)$, *then* a *and* ∞ *are Picard exceptional values of* f, $1 - a$ *and* ∞ *are Picard exceptional values of* g, *and* $(f - a)(g + a - 1) = a(1 - a)$;
- (ii) *If* $\overline{N}_{1)}(r, 1/f) \neq T(r, f) + S(r, f)$, *then* a *and* 0 *are Picard exceptional values of* f, $a/a - 1$ *and* 0 *are Picard exceptional values of* g, *and* $f + (a - 1)g = a$;
- (iii) *If* $\overline{N}_{1)}(r, 1/f - 1) \neq T(r, f) + S(r, f)$, *then* a *and* 1 *are Picard exceptional values of* f, $1/a$ *and* 1 *are Picard exceptional values of* g, *and* $f = ag$.

Proceeding as in the proof of Theorem 5 in [1], we get the following result, which improves Theorem H and Theorem K.

Theorem 5.1. *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(a_1, 1)$, (a_2, m) *and* (a_3, k), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, m *and* k *are positive integers satisfying* (1.1), *and let* $a\,(\neq 0, 1)$ *be a finite complex number such that* (1.2) *holds. Then the conclusions of Theorem K are valid.*

From Theorem 5.1 we obtain the following corollary, which improves Theorem H.

Corollary 5.1. *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(a_1, 1)$, (a_2, m) *and* (a_3, k), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, m *and* k *are positive* *integers such that* (1.1) *holds. If* $a\,(\neq 0, 1)$ *is a finite complex number such that* $\delta_{2)}(a, f) > 0$ *and* $\delta_{1)}(\infty, f) > 0$, *then the conclusions of Theorem H still holds.*

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