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Anti-periodic Boundary Value Problem for Impulsive Differential Equations with Delay

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Abstract. The method of upper and lower solutions coupled with monotone iterative technique is used to obtain the results of existence and uniqueness for an anti-periodic boundary value problem of impulsive differential equations with delay.

1. Introduction

We are concerned with the following anti-periodic boundary value problem (AP-BVP for short) for a first-order impulsive differential equation with delay in R

$$(1.1)\qquad \begin{cases} x'(t) = f(t, x(t), x_t), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \cdots, m, \\ x(\theta) \equiv x(0) = -x(T), & \theta \in [-\tau, 0], \end{cases}$$

where $f : J \times R \times D \to R$, $D = L^1([-r, 0], R), I_k \in C(R, R), \Delta x(t_k)$ represents the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ for all $k = 1, 2, \cdots, m, 0 < t_1 < t_2 < \cdots < t_m < T$, $\delta = \max\{t_k - t_{k-1}; k = 1, 2, \cdots, m+1\}$ here $t_0 = 0, t_{m+1} = T$; $\tau > 0, J = [0, T]$, $J' = J \backslash \{t_1, t_2, \cdots, t_m\}$; for every $t \in J, x_t \in D$ is defined by $x_t(s) = x(t+s), -\tau \le s \le 0$.

Suppose $J_0 = [-\tau, T]$. Let $PC(J_0, R) = \{x : J_0 \to R, x(t)$ is continuous for $t \in J_0, t \neq t_k$ and $x(t_k^+), x(t_k^-)$ exist and $x(t_k) = x(t_k^-)$ for $k = 1, 2, \cdots, m\}$; $PC'(J, R) = \{x : J \to R, x(t)$ is continuously differentiable for $t \in J'$ and $x(t_k^+), x(t_k^-)$ exist and $x'(t_k) = x'(t_k^-)$ for $k = 1, 2, \cdots, m\}$; $E = PC(J_0, R) \bigcap PC'(J, R)$. Obviously, for any $t \in J$ and $x \in E$,we have $x_t \in D$ and $PC(J_0, R)$ and E are Banach spaces with the norms,

$$\| x \|_{PC(J_0,R)} = \sup\{| x(t) | : t \in J_0\}, \| x \|_E = \| x \|_{PC(J,R)} + \| x' \|_{PC(J,R)},$$

where $\| x' \|_{PC(J,R)} = \sup\{| x'(t) | : t \in J\}$. By a solution of (1.1) mean $x \in E$ for which problem (1.1) is satisfied.

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Impulsive differential equations with delay have been extensively studied; see [1]-[4]. Those results are applicable in some important cases such as the initial or the periodic case. However they are not valid, for example, for anti-periodic $x(0) + x(T) = 0$. The APBVPs have been studied by many authors; see [5]-[7] and references therein.

It is the purpose of the present paper to establish the existence and uniqueness of solution for (1.1) .

2. Comparison theorems

This section is devoted to comparison theorems, which are needed for the successful employment of the monotone iterative technique.

Lemma 2.1([7]). *Let $p \in E$ such that*

$$\begin{cases} p'(t) \le -Mp(t) - N \min\limits_{s\in[-\tau,0]} p_t(s), & t \in J', \\ p(t_k^+) \le (1 - L_k)p(t_k), & k = 1, 2, \cdots, m, \\ p(0) \le p(\theta) \le 0, & \theta \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $M, N > 0, 0 \le L_k < 1$ $(k = 1, 2, \cdots, m)$ are constants such that for any positive integer $n_1, n_2 : 1 \le n_1 \le n_2 \le m$

$$\prod_{k=n_1}^{n_2} (1 - L_k) \ge Ne^{M\tau}\delta[1 + \sum_{k=n_1}^{n_2}\prod_{j=k}^{n_2}(1 - L_j)]. \tag{2.2}$$

Then $p(t) \le 0$ on J_0.

For any $p(t) \in E$, we have

$$\min_{s\in[-\tau,0]}(-p_t(s)) \le -p_t(s) \le - \min_{s\in[-\tau,0]} p_t(s).$$

Therefore, we get the following corollary.

Corollary 2.1. *If $p \in E$ such that*

$$\begin{cases} p'(t) \ge -Mp(t) - N \min\limits_{s\in[-\tau,0]} p_t(s), & t \in J', \\ p(t_k^+) \ge (1 - L_k)p(t_k), & k = 1, 2, \cdots, m, \\ p(0) = p(\theta) \ge 0, & \theta \in [-\tau, 0]. \end{cases}$$

And other conditions of Lemma 2.1 hold. Then $p(t) \ge 0$ on J_0.

Lemma 2.2([1]). *Let $p(t) \in E$ such that*

$$\begin{cases} p'(t) \le -Mp(t) + N \min\limits_{s\in[-\tau,0]} p_t(s), & t \in J', \\ p(t_k^+) \le (1 - L_k)p(t_k), & k = 1, 2, \cdots, m, \\ p(0) = p(\theta) \le p(T), & t \in [-\tau, 0], \end{cases}$$

where $M > 0, N \geq 0, 0 \leq L_k < 1, k = 1, 2, \cdots, m, \tau > 0$ *are constants such that for any positive integer:* $1 \leq n_1 \leq n_2 \leq m$,

$$c = \frac{N[\sum_{i=0}^{m-1} \prod_{t_i<t_k<T}(1 - L_i) \exp[-M(T - t_{i+1})]]}{M[1 - \prod_{k=1}^{m}(1 - L_k) \exp(-MT)]} < 1, \quad \bar{t} \in (t_1, T]. \tag{2.3}$$

Then $p(t) \leq 0$ *on* J_0.

Lemma 2.3([8]). *Let* F *be a Banach space and* $\hat{E} = C([a, b], F)$. *Let* $S : \hat{E} \to F$ *be an operator for which*

$$\| S\varphi - S\psi \|_F \leq \epsilon \| \varphi - \psi \|_{\hat{E}}, 0 \leq \epsilon < 1.$$

Then for any point $\xi \in [a, b]$ *there exist an element* $\phi \in \hat{E}$ *such that* $S\phi = \phi(\xi)$.

3. Results

In this section, we first consider the linear APBVPs

$$\begin{cases} p'(t) + Mp(t) + Np_t(s) = \sigma(t), \quad t \in J', \\ \Delta p(t_k) = L_k p(t_k) + \sigma_1(t_k), \quad k = 1, 2, \cdots, m, \\ p(0) \equiv p(\theta)), \quad \theta \in [-\tau, 0], \\ p(0) + p(T) = 0, \end{cases} \tag{3.1}$$

where $\sigma : R \to R$ continuous. From Lemma 2.1 and Corollary 2.1 in Section 2, we can get the following corollaries.

Corollary 3.1. *Assume that* (2.2) *hold. Then* (3.1) *has at most one solution.*

Theorem 3.1. *Assume that* $M, N, L_k (k = 1, 2, \cdots, m)$ *satisfy the condition* (2.2) *and* (2.3). *Then* (3.1) *possesses a unique solution.*

Proof. For any $t \in J$, we have

$$\begin{aligned} p(t) = p(0) &\prod_{0<t_k<t}(1 - L_k) \exp(-Mt) \\ &+ \int_0^t \prod_{0<t_k<t}(1 - L_k) \exp[-M(t - s)](\sigma(s) - Np_t(s))ds \\ &+ \sum_{0<t_k<t} \prod_{t_k<t_i<t}(1 - L_i) \exp[-M(t - t_k)]\sigma_1(t_k). \end{aligned}$$

Let $t = T$, we obtain

$$\begin{aligned} p(0) = [1 + &\prod_{k=1}^{m}(1 - L_k) \exp(-MT)]^{-1} \\ &\{-\int_0^T \prod_{0<t_k<T}(1 - L_k) \exp[-M(T - s)](\sigma(s) - Np_t(s))ds \\ &- \sum_{0<t_k<T} \prod_{t_k<t_i<T}(1 - L_i) \exp[-M(T - t_k)]\sigma_1(t_k)\}. \end{aligned}$$

Define the operator $B : E_0 \to R$ by the equality

$$\begin{aligned}Bp = &[1 + \textstyle\prod_{k=1}^{m}(1 - L_k)\exp(-MT)]^{-1}\\ &\{-\textstyle\int_0^T \prod_{0<t_k<T}(1 - L_k)\exp[-M(T - s)](\sigma(s) - Np_t(s))ds\\ &- \sum_{0<t_k<T} \textstyle\prod_{t_k<t_i<T}(1 - L_i)\exp[-M(T - t_k)]\sigma_1(t_k)\},\end{aligned}$$

where $E_0 = \{x \in PC(J_0, R) : x(\theta) \equiv x(0), \theta \in [-\tau, 0]\}$. For any $p, q \in E_0$ we have

$$\begin{aligned}&\mid Bp - Bq \mid \le [1 + \textstyle\prod_{k=1}^{m}(1 - L_k)\exp(-MT)]^{-1}N \mid p - q \mid_0\\ &\times \textstyle\int_0^T \prod_{0<t_k<T}(1 - L_k)\exp[-M(T - s)]ds \le c \mid p - q \mid_0,\end{aligned}$$

i.e., $|Bp - Bq| \le c|p - q|_0$, where $0 < c < 1$. By Lemma 2.3, there exists an element $p \in E_0$ such that $Bp \equiv p(0)$. This implies that there exists a solution p for (3.1). The uniqueness of solutions of (3.1) follows from Corollary 3.1. The proof of the theorem is complete. □

Now we give the definition of a pair of lower and upper related solutions.

Definition 3.1. We say $v, w \in E$ are a pair of lower and upper related solutions for (1.1) if they satisfy

$$\begin{cases} v'(t) \le f(t, v(t), v_t), & t \in J' \\ \Delta v(t_k) \le I_k(v(t_k)), & k = 1, 2, \cdots, m, \\ v(\theta) \equiv v(0) \le -w(T), & \theta \in [-\tau, 0], \end{cases} \qquad \begin{cases} w'(t) \ge f(t, w(t), w_t), & t \in J', \\ \Delta w(t_k) \ge I_k(w(t_k)), & k = 1, 2, \cdots, m, \\ w(\theta) \equiv w(0) \ge -v(T), & \theta \in [-\tau, 0]. \end{cases}$$

Let $v, w \in E$ be a pair of lower and upper related solutions of (1.1) such that

$$v \le w \quad \text{on} \quad J, \tag{3.2}$$

we define the sector $[v, w] = \{u \in E, v(t) \le u(t) \le w(t), t \in J\}$.

Let us list the following assumptions for convenience.

(H_0) $f \in C(J \times R \times D, R)$ and $I_k \in C(R, R)(k = 1, 2, \cdots, m)$;

(H_1) $v, w \in E$ are lower and upper related solutions of (1.1) satisfying (3.2);

(H_2) There exist $M, N > 0$ such that

$$\begin{aligned}&-M(\bar{u} - u) + N(\max_{s\in[-\tau,0]} \bar{\varphi}(s) - \max_{s\in[-\tau,0]} \varphi(s)) \ge f(t, \bar{u}, \bar{\varphi}) - f(t, u, \varphi)\\ &\ge -M(\bar{u} - u) - N(\max_{s\in[-\tau,0]} \bar{\varphi}(s) - \max_{s\in[-\tau,0]} \varphi(s))\end{aligned}$$

whenever $v(t) \le u \le \bar{u} \le w(t), v_t(s) \le \varphi(s) \le \bar{\varphi}(s) \le w_t(s)$ for $t \in J$ and $s \in [-\tau, 0]$, where $u, \bar{u} \in R$ and $\varphi, \bar{\varphi} \in D$;

(H_3) There exist $L_k : 0 \le L_k < 1(k = 1, 2, \cdots, m)$ such that $I_k(\bar{u}) - I_k(u) \ge -L_k(\bar{u} - u)$ whenever $v(t_k) \le u \le \bar{u} \le w(t_k)$ where $u, \bar{u} \in R$;

(H_4) $M, N, L_k(k = 1, 2, \cdots, m)$ satisfy (2.2) and (2.3).

The following theorem is the most important result that we get.

Theorem 3.2. *Suppose that* $(H_0) - (H_4)$ *hold. Then* (1.1) *has a unique solution*

$x \in [v, w]$.

Proof. We construct the sequences $\{v_n\}, \{w_n\} \subset E$ by defining $v_1 = v, w_1 = w$, and for $n > 1$, v_n and w_n are the solutions of

$$(3.3)\quad \begin{cases} v_n'(t) = f(t, v_{n-1}(t), v_{n-1_t}) - M[v_n(t) - v_{n-1}(t)] \\ -N\big[\max\limits_{s\in[-\tau,0]} v_{n_t}(s) - \max\limits_{s\in[-\tau,0]} v_{n-1_t}(s)\big], \\ \Delta v_n(t_k) = -L_k v_n(t_k) + I_k(v_{n-1}(t_k)) + L_k v_{n-1}(t_k), k = 1, 2, \cdots, m, \\ v_n(0) = -w_{n-1}(T), \\ v_n(\theta) \equiv v_n(0), \theta \in [-\tau, 0] \end{cases}$$

and

$$(3.4)\quad \begin{cases} w_n'(t) = f(t, w_{n-1}(t), w_{n-1_t}) - M[w_n(t) - w_{n-1}(t)] \\ -N\big[\max\limits_{s\in[-\tau,0]} w_{n_t}(s) - \max\limits_{s\in[-\tau,0]} w_{n-1_t}(s)\big], \\ \Delta w_n(t_k) = -L_k w_n(t_k) + I_k(w_{n-1}(t_k)) + L_k w_{n-1}(t_k), k = 1, 2, \cdots, m, \\ w_n(0) = -v_{n-1}(T), \\ w_n(\theta) \equiv w_n(0), \theta \in [-\tau, 0]. \end{cases}$$

Obviously, the existence and uniqueness of solutions for (3.3) and (3.4) are guaranteed.

We prove that $v_1 \le v_2 \le w_2 \le w_1$. In fact, if we consider $\alpha = v_1 - v_2$, by (H_2) and Lemma 2.1 we have $\alpha(t) = v_1 - v_2 \le 0$ on J_0. Analogously, $w_2 \le w_1$ on J_0. Now, let $\alpha_1 = v_2 - w_2$, using (H_2) and Lemma 2.1, $\alpha_1 = v_2 - w_2 \le 0$ on J. It is therefore easy to see that these sequences satisfy the property $v_n \le v_{n+1} \le w_{n+1} \le w_n, \quad n \ge 1$.

We have two monotone sequences that are bounded. By standard arguments[9], there exist ρ and μ with $\{v_n\} \nearrow \rho$ and $\{w_n\} \searrow \mu$ and $\rho \le \mu$. Moreover the convergence is uniform on J. Also, we obtain that the function ρ, μ satisfy

$$\begin{cases} \rho'(t) = f(t, \rho(t), \rho_t), \quad t \in J' \\ \Delta\rho(t_k) = I_k(\rho(t_k)), k = 1, 2, \cdots, m, \\ \rho(\theta) \equiv \rho(0), \theta \in [-\tau, 0] \\ \rho(0) + \mu(T) = 0, \end{cases} \qquad \begin{cases} \mu'(t) = f(t, \mu(t), \mu_t), \quad t \in J' \\ \Delta\mu(t_k) = I_k(\rho(t_k)), k = 1, 2, \cdots, m, \\ \mu(\theta) \equiv \mu(0), \theta \in [-\tau, 0] \\ \mu(0) + \rho(T)) = 0. \end{cases}$$

If we show that $\rho = \mu$, then ρ is a solution of (1.1). Consider $p = \rho - \mu$. By Lemma 2.2, we get $p(t) = \rho(t) - \mu(t) \ge 0$ for $t \in J$. Hence we have $\rho \equiv \mu$ for $t \in J$.

If $x(t) \in [v, w]$ is a solution of (1.1), one can see that $v_n \le x \le w_n, n = 1, 2, \cdots$. In fact, we suppose $v_{n_0} \le x \le w_{n_0}$. Let $\alpha_2(t) = v_{n_0+1}(t) - x(t)$, then we can get that

$$\begin{cases} \alpha_2'(t) \le -M\alpha_2(t) - N \min\limits_{s\in[-\tau,0]} \alpha_{2_t}(s), \\ \Delta\alpha_2(t_k) \le -L_k\alpha_2(t_k), k = 1, 2, \cdots, m, \\ \alpha_2(0) = -w_{n_0}(T) + x(T) \le 0, \\ \alpha_2(\theta) \equiv \alpha_2(0), \theta \in [-\tau, 0]. \end{cases}$$

By Lemma 2.1, we have $\alpha_2(t) \le 0$ on J, that is, $v_{n_0+1}(t) \le x(t)$. Analogously, $x(t) \le w_{n_0+1}(t)$ on J. Therefore, $v_n \le x \le w_n (n = 1, 2, \cdots)$ by induction. Thus, passing to the limit, we may conclude that $\rho \le x \le \mu$, that is, $\rho \equiv \mu \equiv x$. Therefore, The proof of the theorem is completed. □

Example. Consider the anti-periodic boundary value problem of the impulsive equation

$$(3.5) \qquad \begin{cases} x'(t) = -x - \frac{3}{2e}\sin x_t + (1+\frac{3}{2e})(1-t), \quad t \in [0,1], t \neq \frac{1}{2}, \\ \Delta x(\frac{1}{2}) = -\frac{1}{2}x(\frac{1}{2}), \\ x(\theta) \equiv x(0) = -x(1), \quad \theta \in [-\frac{1}{2}, 0], \end{cases}$$

where $x_t = x(t+s), s \in [-\frac{1}{2}, 0]$.

Let $v(t) = \begin{cases} -1, & 0 \le t \le \frac{1}{2}, \\ -\frac{1}{2}, & \frac{1}{2} < t \le 1, \end{cases}$ $w(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} < t \le 1. \end{cases}$ Then $v(t), w(t)$ are a pair of lower and upper related solutions for (3.5). And (3.5) satisfies all the conditions of Theorem 3.2, therefore, we have a unique solution of (3.5) by Theorem 3.2.

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