KYUNGPOOK Math. J. 48(2008), 495-502

Residual P-Finiteness of Certain Generalized Free Products of Nilpotent Groups

Goansu Kim
Department of Mathematics, Yeungnam University, Kyongsan, 712-749, Korea
e-mail : gskim@yu.ac.kr

Youngmi Lee
Department of Mathematics, Yeungnam University, Kyungsan, 712-749, Korea
e-mail : dianthus@yumail.ac.kr

James McCarron
Waterloo Maple, Inc. Waterloo, Ontario, Canada N2L 3G1
e-mail : jmccarro@maplesoft.com

Abstract. We show that free products of finitely generated and residually p-finite nilpotent groups, amalgamating p-closed central subgroups are residually p-finite. As a consequence, we are able to show that generalized free products of residually p-finite abelian groups are residually p-finite if the amalgamated subgroup is closed in the pro-p topology on each of the factors.

1. Introduction

A generalized free product of two finite p-groups need not be residually a finite p-group. (Here, and in the sequel, p is a prime number.) However, Higman [4] showed that if the amalgamated subgroup is cyclic, then the generalized free product of two finite p-groups is residually a finite p-group. Higman's result was subsequently extended to generalized free products of residually p-finite groups [5], [6], [8] where the amalgamated subgroups are cyclic.

In this paper, we consider again the generalized free product of residually p-finite groups, using a somewhat different approach than was used in [5], [6]. We introduce the idea of a p-filter, which enables us to develop criteria for generalized free products to be residually p-finite. We then apply these criteria to obtain a number of new results identifying residually p-finite generalized free products. In particular, generalized free products in which the amalgamated subgroup is normal or cyclic are considered. For example, a free product of finitely generated and

Received January 22, 2008.
2000 Mathematics Subject Classification: 20E26, 20E06, 20F05.
Key words and phrases: generalized free products, residually finite p-groups, p-closed subgroups, p-filters.
The first author was supported by the Yeungnam University research grants in 2002.

residually p-finite nilpotent groups, amalgamating a p-closed central subgroup is shown to be residually a finite p-group. As a consequence, we are able to show that a generalized free product of residually p-finite abelian groups is residually a finite p-group if the amalgamated subgroup is closed in the pro-p topology on each of the factors.

The main tool used in our results is the notion of a "p-filter". These are defined, and a number of technical results concerning them developed, in Section 3.

2. Preliminaries

Definition 2.1. A group G is said to be *residually a finite p-group* ($\mathcal{RF}p$) if, for each $1 \neq x \in G$, there exists a normal subgroup N of p-power index in G ($N \lhd_p G$) such that $x \notin N$.

If H is a subgroup of G, we denote by $\mathrm{Aut}_G(H)$ the image, in the automorphism group of H, of the normalizer of H in G, under the natural homomorphism into $\mathrm{Aut}(H)$. The following result of G. Higman will be important throughout this paper.

Theorem 2.2([4]). *Let $G = A *_H B$, where A and B are finite p-groups.*
(1) *If H is cyclic, then G is $\mathcal{RF}p$.*
(2) *If H is normal, both in A and in B, then G is $\mathcal{RF}p$ if, and only if,* $\mathrm{Aut}_A(H)$ *and* $\mathrm{Aut}_B(H)$ *generate a p-subgroup of* $\mathrm{Aut}(H)$.

Higman's theorem (1) has a several consequences in [2], [5], [6]. If $H \leq Z(A)$ then $Aut_A(H) = 1$. Hence the following is an easy consequence of (2).

Proposition 2.3([7]). *Let $G = A *_H B$, where A and B are finite p-groups. If $H \leq Z(A)$ and $H \lhd B$ then G is $\mathcal{RF}p$.*

Definition 2.4. Let G be a group. A subgroup H of G is *p-closed* in G if, for each $g \in G \setminus H$, there exists $N \lhd_p G$ such that $g \notin NH$. In particular, $\{1\}$ is p-closed in G iff G is $\mathcal{RF}p$.

Let A be $\mathcal{RF}p$. Corollary 3.5 in [5] shows that $A *_H A$ is $\mathcal{RF}p$ if, and only if, H is p-closed in A. Thus p-closedness plays an important role in the study of the residually finite p-group property of generalized free products.

Theorem 2.5. *Let $G = A *_H B$, where $H \leq Z(B)$ and $A \neq H \neq B$. If G is $\mathcal{RF}p$, then H is p-closed in A.*

Proof. Let $a \in A \setminus H$. Choose $b \in B \setminus H$. Then $[a, b] \neq 1$. Since G is $\mathcal{RF}p$, there exists $N \lhd_p G$ such that $[a, b] \notin N$. This implies $a \notin NH$, since $H \leq Z(B)$. Hence $a \notin (N \cap A)H$ and $N \cap A \lhd_p A$. Thus H is p-closed in A. □

Every element of finite order in a $\mathcal{RF}p$ group has a p-power order. The converse of this fact holds for finitely generated nilpotent groups.

Proposition 2.6([1],[3]). *Let A be finitely generated nilpotent. Then A is $\mathcal{RF}p$ if,* *and only if, the torsion subgroup of A is a finite p-group.*

Corollary 2.7. *Let G be a finitely generated nilpotent group and $H \lhd G$. If H is a finite p-subgroup of G and G/H is $\mathcal{RF}p$, then G is $\mathcal{RF}p$.*

3. p-filters

Definition 3.1. Let $G = A *_H B$. Let $\Lambda = \{(M_i, N_i) \mid i \in I\}$ be a non-empty family of pairs (M_i, N_i), where $M_i \lhd A$ and $N_i \lhd B$, satisfying the following:

$(F1)$ $M_i \cap H = N_i \cap H$ for each $i \in I$;

$(F2)$ For each $i \in I$, $A/M_i *_{\overline{H}} B/N_i$ is $\mathcal{RF}p$, where $\overline{H} = HM_i/M_i \simeq HN_i/N_i$;

$(F3)$ For each $i_1, i_2, \cdots, i_n \in I$ and $n \in \mathbb{Z}^+$, $(\cap_{k=1}^n M_{i_k}, \cap_{k=1}^n N_{i_k}) \in \Lambda$;

$(F4)$ $\cap_{i\in I} M_i H = H = \cap_{i\in I} N_i H$.

Such Λ is called a *p-filter* of generalized free product $G = A *_H B$.

In the following lemmas, we find some p-filters of generalized free products.

Lemma 3.2. *Let A and B be $\mathcal{RF}p$ and let $G = A *_{\langle c\rangle} B$, where $|c| < \infty$. Then*

$$\Lambda = \{(M, N) \mid M \lhd_p A,\ N \lhd_p B \text{ such that } M \cap \langle c\rangle = 1 = N \cap \langle c\rangle\}$$

is a p-filter of $G = A *_{\langle c\rangle} B$.

Proof. Since every element of finite order in a $\mathcal{RF}p$ group has a p-power order, let $|c| = p^\alpha$ for some α. For each $0 < i < p^\alpha$, there exists $M_i \lhd_p A$ such that $c^i \notin M_i$. Let $M = \cap M_i$. Then $M \lhd_p A$ and $\langle c\rangle \cap M = 1$. Similarly, there exists $N \lhd_p B$ such that $\langle c\rangle \cap N = 1$. Thus $(M, N) \in \Lambda$ and $\Lambda \neq \varnothing$.

By Theorem 2.2, $A/M *_{\langle \overline{c}\rangle} B/N$ is $\mathcal{RF}p$ for each $(M, N) \in \Lambda$. Hence (F2) holds. To show (F4), let $a \in A \setminus \langle c\rangle$. Then $ac^{-i} \neq 1$, for all $0 \leq i < p^\alpha$. Since A is $\mathcal{RF}p$, there exists $M \lhd_p A$ such that $ac^{-i} \notin M$ for all i and $\langle c\rangle \cap M = 1$. Then $a \notin M\langle c\rangle$. Similarly, there exists $N \lhd_p B$ such that $a \notin N\langle c\rangle$ and $\langle c\rangle \cap N = 1$. Then $(M, N) \in \Lambda$ and $a \notin \cap_{(M,N)\in\Lambda} M\langle c\rangle$. Hence $\langle c\rangle \supseteq \cap_{(M,N)\in\Lambda} M\langle c\rangle$. Thus $\cap_{(M,N)\in\Lambda} M\langle c\rangle = \langle c\rangle$. Similarly, $\cap_{(M,N)\in\Lambda} N\langle c\rangle = \langle c\rangle$. Thus (F4) holds. Since (F1) and (F3) are trivial, Λ is a p-filter of $G = A *_{\langle c\rangle} B$. □

Lemma 3.3. *Let $G = A *_{\langle c\rangle} B$, where A, B are $\mathcal{RF}p$ and $|c| = \infty$. If $\langle c\rangle$ is p-closed in both A and B, then*

$$\Lambda = \{(M, N) \mid M \lhd_p A,\ N \lhd_p B \text{ such that } M \cap \langle c\rangle = N \cap \langle c\rangle\}$$

is a p-filter of G.

Proof. We note that $\Lambda \neq \varnothing$, since $(A, B) \in \Lambda$. For $(M, N) \in \Lambda$, $A/M *_{\langle \overline{c}\rangle} B/N$ is $\mathcal{RF}p$ by Theorem 2.2. Thus (F2) holds. Let $a \in A \setminus \langle c\rangle$. Since $\langle c\rangle$ is p-closed in A, there exists $M \lhd_p A$ such that $a \notin M\langle c\rangle$. Let $M \cap \langle c\rangle = \langle c^{p^n}\rangle$ for some n. By Corollary 2.3 [5], there exists $N \lhd_p B$ such that $N \cap \langle c\rangle = \langle c^{p^n}\rangle$. Therefore $(M, N) \in \Lambda$. Since $a \notin M\langle c\rangle$, $a \notin \cap_{(M,N)\in\Lambda} M\langle c\rangle$. Hence $\cap_{(M,N)\in\Lambda} M\langle c\rangle \subseteq \langle c\rangle$. Thus $\cap_{(M,N)\in\Lambda} M\langle c\rangle = \langle c\rangle$. Similarly, $\cap_{(M,N)\in\Lambda} N\langle c\rangle = \langle c\rangle$. Thus (F4) holds. Since

(F1) and (F3) is trivial, Λ is a p-filter of $G = A *_{\langle c\rangle} B$. $\square$

Lemma 3.4. *Let A and B be $\mathcal{RF}p$ and H be normal in both A and B. If A/H and B/H are $\mathcal{RF}p$, then $\Lambda = \{(H, H)\}$ is a p-filter of $G = A *_H B$.*

Proof. Since A/H and B/H are $\mathcal{RF}p$, $A/H * B/H$ is $\mathcal{RF}p$. Clearly (F1), (F3) and (F4) hold. Hence $\{(H, H)\}$ is a p-filter of $G = A *_H B$. $\square$

Remark 3.5. Let Λ be a p-filter of $G = A *_H B$. Then, for each $x \notin H$, there exists $N \lhd_p G$ such that $x \notin N$.

Proof. Clearly $\|x\| \geq 1$. We only consider the case $x = a_1b_1 \cdots a_nb_n$, where $a_i \in A \setminus H$ and $b_i \in B \setminus H$, since the other cases are similar. By (F4), for each $1 \leq s \leq n$, there exists $(M_{i_s}, N_{i_s}) \in \Lambda$ such that $a_s \notin M_{i_s}H$. Similarly, there exists $(M'_{i_s}, N'_{i_s}) \in \Lambda$ such that $b_s \notin N'_{i_s}H$ for each $1 \leq s \leq n$. Let $M = \cap_{s=1}^{n}(M_{i_s} \cap M'_{i_s})$ and $N = \cap_{s=1}^{n}(N_{i_s} \cap N'_{i_s})$. Then $(M, N) \in \Lambda$ by (F3) and $M \cap H = N \cap H$ by (F1). Let $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/M$, $\overline{B} = B/N$ and $\overline{H} = HM/M \simeq HN/N$. Then $\|\overline{x}\| = \|x\| \geq 1$. Hence $\overline{x} \neq 1$. Since $\overline{G}$ is $\mathcal{RF}p$ by (F2), there exists $\overline{N} \lhd_p \overline{G}$ such that $\overline{x} \notin \overline{N}$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_p G$ and $x \notin N$ as required. $\square$

Theorem 3.6. *Let Λ be a p-filter of $G = A *_H B$. If, for each $1 \neq x \in H$, there exists $(M, N) \in \Lambda$ such that $x \notin M$, then G is $\mathcal{RF}p$.*

Proof. Let $1 \neq x \in G$. If $x \notin H$ then, by Remark 3.5, there exists $N \lhd_p G$ such that $x \notin N$. Thus, let $1 \neq x \in H$ and suppose $x \notin M$ for some $(M, N) \in \Lambda$. Let $\overline{G} = A/M *_{\overline{H}} B/N$, as before. Then $\overline{x} \neq 1$. Since $\overline{G}$ is $\mathcal{RF}p$ by (F2), we can find $N \lhd_p G$ such that $x \notin N$. Hence G is $\mathcal{RF}p$. $\square$

The following is a generalization of Theorem 2.2.

Corollary 3.7([5, Theorem 4.3])**.** *If A and B are $\mathcal{RF}p$ and $|c| < \infty$, then $G = A *_{\langle c\rangle} B$ is $\mathcal{RF}p$.*

Proof. By Lemma 3.2, $\Lambda = \{(M, N) \mid M \lhd_p A, N \lhd_p B$ such that $M \cap \langle c\rangle = 1 = N \cap \langle c\rangle\}$ is a p-filter of $G = A *_{\langle c\rangle} B$. Let $1 \neq x \in \langle c\rangle$. As in the proof of Lemma 3.2, there exists $(M, N) \in \Lambda$ such that $x \notin M$. Hence, by Theorem 3.6, G is $\mathcal{RF}p$. $\square$

The following is a main result in [5]. We shall prove it using Theorem 3.6.

Corollary 3.8([5, Theorem 4.2])**.** *If A and B are $\mathcal{RF}p$ and $\langle c\rangle$ is p-closed in both A and B, then $G = A *_{\langle c\rangle} B$ is $\mathcal{RF}p$.*

Proof. By Lemma 3.3, $\Lambda = \{(M, N) \mid M \lhd_p A, N \lhd_p B$ such that $M \cap \langle c\rangle = N \cap \langle c\rangle\}$ is a p-filter of G. To use Theorem 3.6, let $1 \neq x \in \langle c\rangle$. Then $x \notin \langle c^{p^n}\rangle$ for some n. By Corollary 2.3 [5], there exist $M \lhd_p A$ and $N \lhd_p B$ such that $M \cap \langle c\rangle = \langle c^{p^n}\rangle = N \cap \langle c\rangle$. Hence $(M, N) \in \Lambda$ and $x \notin M$. Thus, by Theorem 3.6, G is $\mathcal{RF}p$. $\square$

4. Main results

In this section, we shall show that certain generalized free products of $\mathcal{RF}p$ groups, amalgamating a normal subgroup, are $\mathcal{RF}p$. The following is easy to see.

Proposition 4.1. *Let $H \lhd A$. Then H is p-closed in A if, and only if, A/H is $\mathcal{RF}p$.*

By using Proposition 2.6 and 4.1, we have:

Corollary 4.2. *Let A be finitely generated nilpotent and $H \lhd A$. Then the following are equivalent:*
(1) *H is p-closed in A.*
(2) *A/H is $\mathcal{RF}p$.*
(3) *The torsion subgroup of A/H is a finite p-group.*

Theorem 4.3. *Let A and B be finite p-groups and H be normal in both A and B. If, for each $1 \neq x \in H$, there exists a normal subgroup D of both A and B such that $x \notin D$ and $D \leq H$, where H/D is cyclic, then $G = A *_H B$ is $\mathcal{RF}p$.*

Proof. By Lemma 3.4, $\Lambda = \{(H, H)\}$ is a p-filter of $G = A *_H B$. Let $1 \neq x \in G$. If $x \notin H$ then, by Remark 3.5, there exists $N \lhd_p G$ such that $x \notin N$. If $1 \neq x \in H$ then, by assumption, there exists a normal subgroup D of both A and B such that $x \notin D$, $D \leq H$, and H/D is cyclic. Let $\overline{G} = A/D *_{\overline{H}} B/D$ where $\overline{H} = H/D$ is cyclic. Since A/D and B/D are finite p-groups, $\overline{G}$ is $\mathcal{RF}p$ by Theorem 2.2. Since $x \notin D$, $\overline{x} \neq 1$. Hence there exists $\overline{N} \lhd_p \overline{G}$ such that $\overline{x} \notin \overline{N}$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_p G$ and $x \notin N$. Hence G is $\mathcal{RF}p$. □

The following example explains why we need the normal subgroup D of both A and B in the theorem above.

Example 4.4 ([5, Example 2.5])**.** Let $A_i = \langle x_i, y_i \mid x_i^4, y_i^2, (x_i y_i)^2 \rangle$, for $i = 1, 2$. Then the generalized free product $A_1 \underset{H}{*} A_2$, where $H = \langle x_1^2, y_1 \rangle \simeq \langle y_2, x_2^2 \rangle$, of A_1 and A_2 amalgamating $x_1^2 = y_2$ and $y_1 = x_2^2$ is not $\mathcal{RF}_2$ [5]. Moreover A_1, A_2 are finite 2-groups of order 8 and $H = \langle x_1^2, y_1 \rangle$ is order 4. Thus H is normal in both A_1 and A_2. Clearly $1 \neq x_1^2 \in H$. Note that H has nontrivial subgroups $\langle x_1^2 \rangle, \langle y_1 \rangle, \langle x_1^2 y_1 \rangle$. But $\langle y_1 \rangle, \langle x_1^2 y_1 \rangle$ are not normal in A_1. Therefore, there is no $D \leq H$ such that $D \lhd A_1$, $x_1^2 \notin D$ and H/D is cyclic.

Theorem 4.5. *Let A and B be $\mathcal{RF}p$ and H be normal in both A and B. Suppose that H is p-closed in A and B. If, for each $1 \neq x \in H$, there exist $M \lhd_p A$ and $N \lhd_p B$ such that $M \cap H = N \cap H$, $x \notin M$ and HM/M is cyclic, then $A *_H B$ is $\mathcal{RF}p$.*

Proof. Let $1 \neq x \in G$. Since H is a p-closed normal subgroup of A, A/H is $\mathcal{RF}p$ by Proposition 4.1. Similarly B/H is $\mathcal{RF}p$. Hence, by Lemma 3.4, $\{(H, H)\}$ is a p-filter of $G = A *_H B$. If $x \notin H$ then, by Remark 3.5, there exists $N \lhd_p G$ such that $x \notin N$. Thus, suppose $1 \neq x \in H$. By assumption, there exist $M \lhd_p A$ and $N \lhd_p B$ such that $M \cap H = N \cap H$, $x \notin M$ and HM/M is cyclic. Let $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = HM/M$. Then, by Theorem 2.2, $\overline{G}$ is $\mathcal{RF}p$ and $1 \neq \overline{x} \in \overline{G}$. Hence, as usual, we can find $N \lhd_p G$ and $x \notin N$. Hence G is $\mathcal{RF}p$. □

Theorem 4.6. *Let A and B be $\mathcal{RF}p$ and H be normal in both A and B. Suppose that H is p-closed in A and B. If, for each $1 \neq x \in H$, there exists a normal subgroup D of A and B such that $x \notin D$ and $D \subset H$, A/D and B/D are $\mathcal{RF}p$ and H/D is cyclic, then $A *_H B$ is $\mathcal{RF}p$.*

Proof. As before, we consider $1 \neq x \in H$. By assumption, there exists a normal subgroup D of A and B such that $x \notin D$ and $D \subset H$, A/D and B/D are $\mathcal{RF}p$ and H/D is cyclic. Let $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/D$, $\overline{B} = B/D$ and $\overline{H} = H/D$. Since H is p-closed in A and B, it is not difficult to see that $\overline{H}$ is p-closed in $\overline{A}$ and $\overline{B}$. This follows from Corollary 3.8 that $\overline{G}$ is $\mathcal{RF}p$. Since $1 \neq \overline{x} \in \overline{H}$, we can find $\overline{N} \lhd_p \overline{G}$ such that $\overline{x} \notin \overline{N}$. Let N be the preimage of $\overline{N}$ in G. Then $N \lhd_p G$ and $x \notin N$. Hence G is $\mathcal{RF}p$. □

Corollary 4.7. *Let A and B be finitely generated nilpotent, $\mathcal{RF}p$ groups. Let H be p-closed in both A and B such that $H \leq Z(A) \cap Z(B)$. Then $G = A *_H B$ is $\mathcal{RF}p$.*

Proof. To apply Theorem 4.6, let $1 \neq g \in H$. Since H is finitely generated abelian and $\mathcal{RF}p$, $H \simeq \mathbb{Z}_{p^{r_1}} \times \cdots \times \mathbb{Z}_{p^{r_s}} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$. Since $1 \neq g \in H$, we consider $g = (a_1, \cdots, a_s, a_{s+1}, \cdots, a_n)$ and $a_i \neq 0$ for some i. If $a_1 \neq 0$ (similarly, $a_k \neq 0$ for $2 \leq k \leq s$), let $D = \{0\} \times \mathbb{Z}_{p^{r_2}} \times \cdots \times \mathbb{Z}_{p^{r_s}} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$. Then $H/D \simeq \mathbb{Z}_{p^{r_1}}$ is cyclic and $x \notin D$. If $a_{s+1} \neq 0$ (similarly, $a_k \neq 0$ for $s+2 \leq k \leq n$), then $a_{s+1} \notin p^\alpha\mathbb{Z}$ for some α. Let $D = \mathbb{Z}_{p^{r_1}} \times \cdots \times \mathbb{Z}_{p^{r_s}} \times p^\alpha\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$. Then $H/D \simeq \mathbb{Z}_{p^\alpha}$ is cyclic and $x \notin D$. Now, A/D is finitely generated nilpotent and H/D is a finite p-subgroup of A/D and $(A/D)/(H/D) \simeq A/H$. Since H is p-closed in A, by Proposition 4.1, A/H is $\mathcal{RF}p$. Hence, A/D is $\mathcal{RF}p$ by Corollary 2.7. Similarly, B/D is $\mathcal{RF}p$. Therefore, by Theorem 4.6, G is $\mathcal{RF}p$. □

Combining Theorem 4.7, Theorem 2.5 and Corollary 4.2, we have

Corollary 4.8. *Let A and B be finitely generated abelian, $\mathcal{RF}p$ groups, and let H be a proper subgroup of both A and B. The the following are equivalent.*

1. *$A *_H B$ is $\mathcal{RF}p$;*
2. *H is p-closed in both A and B; and,*
3. *the torsion subgroups of A/H and B/H are p-groups.*

The following improves Corollary 4.7.

Theorem 4.9. *Let A and B be $\mathcal{RF}p$ groups, where A is finitely generated nilpotent. Let H be p-closed in both A and B such that $H \leq Z(A)$ and $H \lhd B$. Then $G = A *_H B$ is $\mathcal{RF}p$.*

Proof. Let $1 \neq g \in G$. As in the proof of Theorem 4.5, we suppose $1 \neq g \in H$. Since B is $\mathcal{RF}p$, there exists $M \lhd_p B$ such that $g \notin M$. Then $M \cap H \lhd_p H$ and $M \cap H \lhd A$, since $H \leq Z(A)$. Let $\overline{A} = A/(M \cap H)$. Then $\overline{H}$ is a finite p-group and $\overline{A}/\overline{H} \cong A/H$ is $\mathcal{RF}p$. This follows from Corollary 2.7 that $\overline{A}$ is $\mathcal{RF}p$. Hence there exists $\overline{N}\lhd_p\overline{A}$ such that $\overline{N}\cap\overline{H}=1$. Let N be the preimage of $\overline{N}$ in A. Then $N\lhd_p A$ and $N\cap H=M\cap H$. Let $\overline{G}=A/N *_{\overline{H}} B/M$. By Proposition 2.3, $\overline{G}$ is $\mathcal{RF}p$ and $\overline{g}\neq 1$. Hence we can find $\overline{S}\lhd_p\overline{G}$ such that $\overline{g}\notin\overline{S}$. Let S be the preimage of $\overline{S}$ in G. Then $S\lhd_p G$ and $g\notin S$, as required. Hence G is $\mathcal{RF}p$. □

We can apply this to certain tree products amalgamating a single subgroup.

Theorem 4.10. *Let $I=\{1,2,\cdots,n\}$, where $n\geq 2$. Suppose each A_i $(i\in I)$ is a finitely generated nilpotent, $\mathcal{RF}p$ group. Let $H\leq Z(A_i)$ and $H\neq A_i$ for all $i\in I$. Let G be the generalized free product $*_H A_i$ of A_i $(i\in I)$ amalgamating a single subgroup H. Then G is $\mathcal{RF}p$ if, and only if, H is p-closed in each A_i if, and only if, the torsion subgroup of each A_i/H is a finite p-group.*

Proof. Suppose H is p-closed in each A_i. Let $G_{i+1}=G_i *_H A_{i+1}$. Inductively we show that G_{i+1} is $\mathcal{RF}p$ assuming that G_i is $\mathcal{RF}p$. Clearly $H\lhd G_i$, in fact, $H\leq Z(G_i)$. Since G_i/H is a free product of $\mathcal{RF}p$ groups $A_1/H,\cdots,A_i/H$, G_i/H is $\mathcal{RF}p$. Hence H is p-closed in G_i by Proposition 4.1. Thus, by Theorem 4.9, G_{i+1} is $\mathcal{RF}p$. Therefore $G=G_n$ is $\mathcal{RF}p$.

Conversely, suppose G is $\mathcal{RF}p$. Since its subgroup $A_i *_H A_{i+1}$ is also $\mathcal{RF}p$, by Theorem 2.5, H is p-closed in A_i. □

Combining Theorem 4.10, Theorem 2.5 and Corollary 4.2, we have

Theorem 4.11. *Let $I=\{1,2,\cdots,n\}$, where $n\geq 2$. Suppose each A_i $(i\in I)$ is a finitely generated abelian, $\mathcal{RF}p$ group. Let G be the generalized free product $*_H A_i$ of A_i $(i\in I)$ amalgamating a single subgroup H. Then G is $\mathcal{RF}p$ if, and only if, H is p-closed in each A_i if, and only if, the torsion subgroup of each A_i/H is a finite p-group.*

Finally we note that the condition"p-closed" in Theorem 4.11 can not be eliminated because of the following simple example.

Example 4.12. Note that $\langle x^n\rangle$ is a p-closed subgroup of $\langle x\rangle$ if and only if $n=p^\alpha$ for some α. Hence $\langle a^2\rangle$ is a 2-closed subgroup of $\langle a\rangle$ and $\langle b^3\rangle$ is not a 2-closed subgroup of $\langle b\rangle$. Then $\langle a\rangle \underset{a^2=b^3}{*} \langle b\rangle$ is not $\mathcal{RF}_2$. In fact, $\langle a\rangle \underset{a^2=b^3}{*} \langle b\rangle$ is not $\mathcal{RF}p$ for any prime p.

References

[1] G. Baumslag,. Lecture notes on nilpotent groups, C. B. M. S. Regional Conf. Ser. in Math., Amer. Math. Soc. No. 2., (1971)

[2] D. Doniz. *Residual properties of free products of infinitely many nilpotent groups amalgamating cycles*, J. Algebra, **179**(1996), 930–935.

[3] K. W. Gruenberg. *Residual properties of infinite soluble groups*, Proc. London Math. Soc., (3), **7**(1957), 29–62.

[4] G. Higman. *Amalgams of p-groups*, J. Algebra, **1**(1964), 301–305.

[5] G. Kim and J. McCarron. *On amalgamated free products of residually p-finite groups*, J. Algebra, **162(1)**(1993), 1–11.

[6] G. Kim and C. Y. Tang. *On generalized free products of residually p-finite groups*, J. Algebra, **201**(1998), 317–327.

[7] J. McCarron. Residual nilpotence and one relator groups, PhD thesis, University of Waterloo, Canada, (1995).

[8] P. C. Wong and C. K. Tang. *Tree products of residually* p-*finite groups*, Algebra Colloq., **2(3)**(1995), 209–212.