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MV-Algebras of Continuous Functions and l-Monoids

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Abstract. A. Di Nola & S.Sessa [8] showed that two compact spaces X and Y are homeomorphic iff the MV-algebras $C(X, I)$ and $C(Y, I)$ of continuous functions defined on X and Y respectively are isomorphic. And they proved that A is a semisimple MV-algebra iff A is a subalgebra of $C(X)$ for some compact Hausdorff space X. In this paper, firstly by use of functorial argument, we show these characterization theorems. Furthermore we obtain some other functorial results between topological spaces and MV-algebras. Secondly as a classical problem, we find a necessary and sufficient condition on a given residuated l-monoid that it is segmenently embedded into an l-group with order unit.

1. Introduction

An MV-algebra is a universal algebra $(A, +, \cdot, *, 0, 1)$ of $(2, 2, 1, 0, 0)$ type such that $(A, +, 0)$ is an abelian monoid and moreover, $x+1 = 1, x^{**} = x, 0^* = 1, x+x^* = 1, x \cdot y = (x^* + y^*)^*$ and $x + x^*y = y + y^*x$ for all $x, y \in A$. By setting $x \vee y = x + x^*y$ and $x \wedge y = x(x^* + y)$ we have $(A, \vee, \wedge, 0, 1)$ as a bounded distributive lattice.

The system of MV-algebras is a kind of better system in the sense that closed under subalgebras, quotients and products and the free MV-algebra with a denumerable set of generators can be described by MV-algebras of continuous $I = [0, 1]$-valued functions on the Hilbert cube [11]. Furthermore, the variety of MV-algebras is a Malcev variety and has the congruence regularity [10].

In the first part of this paper, we establish a dual-adjunction $(\eta, \varepsilon) : S \vdash C$ from the subcategory $\mathbf{X}$ of Tychonoff spaces of $\mathbf{Top}$ into the subcategory $\mathbf{A}$ of semisimple MV-algebras of $\mathfrak{M}_v$, where $\mathbf{Top}$ and $\mathfrak{M}_v$ are the categories of topological spaces and MV-algebras respectively. We have shown that for every compact Hausdorff space X, the counit ε_X is a homeomorphism. It reduces that $C(X) \cong C(Y)$ (MV-isomorphism) for two compact Hausdorff spaces X and Y iff $X \cong Y$ (homeo-

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morphism). We also showed that if X is a compact Hausdorff space and if $X = S(A)$ for some MV-algebra then X must be a Boolean space.

Mundici [11] showed that, given an MV-algebra A, there exists an abelian lattice-ordered group G with order unit u such that $A \cong \Gamma(G,u)$ where $\Gamma(G,u) = \{x \in G | 0 \leq x \leq u\}$ and vice versa. This fact induces that these two categories are categorically equivalent.

Furthermore, given an AFC^*-algebra $\mathfrak{a}$ there exists a countable po-abelian group $K_0(\mathfrak{a})$ (=the dimension group [6] with order unit [1]. And hence we have a countable MV-algebra $\Gamma(K_0(\mathfrak{a}), [1_{\mathfrak{a}}])$ and vice versa.

In the second part of this paper, given a residuated lattice-ordered monoid M, we study that under what conditions on M, M can be segmently embedded into an l-group G with order unit u, namely, $M \cong \Gamma(G,u)$.

2. Dual adjunctions

Let $\mathbf{X}$ be the subcategory of Tychonoff spaces of the category **Top** of topological spaces. Then $\mathbf{X}$ is the epireflective hull of the unit interval $I = [0,1]$ with the ordinary topology, i.e., $X \in \mathbf{X}$ iff X admits enough morphisms in I to separate points. Let $\mathbf{A}$ be the epireflective hull of the unit interval MV-algebra $I = [0,1]$ in the category $\mathfrak{M}_v$. It is well known that an MV-algebra A is semi-simple iff A is embedded into a product of unit interval MV-algebras. Therefore, $\mathbf{A}$ is the category of semi-simple MV-algebras and their MV-homomorphisms.

Let $C : \mathbf{X^{op}} \to \mathbf{A}$ be a functor defined by, for $X \in \mathbf{X}^{op}$, $C(X) = \hom_{\mathbf{X}}(X, I)$ where I has the usual topology. Then $C(X)$ is an MV-subalgebra of $I^{|X|}$ which is the product of I's with power $|X|$.

For a morphism $f : X \to Y$ in $\mathbf{X^{op}}$, we define $C(f)(u) = uf$ for each $u \in C(Y)$. Define $S : \mathbf{A} \to \mathbf{X^{op}}$ by $S(A) = \hom_{\mathbf{A}}(A, I)$ for each $A \in \mathbf{A}$. $S(A)$ is a subspace of $(I^{|A|}, \tau_p)$, where τ_p is the product topology of I's. For $f : A \to B$ in $\mathbf{A}$, we define $S(f)(u) = uf$ for $u \in S(B)$, then $S(f)$ serves both the restriction to $S(B)$ and corestriction to $S(A)$ of the morphism $\bar{f} : I^{|B|} \to I^{|A|}$ in $\mathbf{X^{op}}$ where $\bar{f}(u) = uf$ for each $u \in I^{|B|}$. The unit η_A for $A \in \mathbf{A}$ is defined by $\eta_A(a)(u) = u(a)$ for each $u \in S(A)$ and each $a \in A$. And counit ε_X for each $X \in \mathbf{X}$ is defined by $\varepsilon_X(x)(u) = u(x)$ for each $u \in C(X)$ and each $x \in X$.

Then we have the following Theorem :

Theorem 2.1. *For the categories* $\mathbf{X}$ *and* $\mathbf{A}$ *of Tychonoff spaces and semi-simple* MV*-algebras,* C *is a right adjoint to* S *via* η *and* ε.

Proof. Straightforward. □

Let Fix$\eta = \{A \in \mathfrak{M}_v |\ \eta_A$ is an isomorphism$\}$ and Fix$\varepsilon = \{X \in \mathbf{X} | \varepsilon_X$ is a homeomorphism $\}$.

Theorem 2.2. *If* X *is a compact Hausdorff space in* $\mathbf{X}$ *then* $X \in$ Fixε.

Proof. For any $h \in SC(X), M = h^{-1}(0)$ is a maximal ideal of $C(X)$. Since X

is compact, every maximal ideal of $C(X)$ is fixed [9], i.e., there exists a point $x \in X$ such that $M = \{f \in C(X) | f(x) = 0\}$. On the other hand, h maps every constant function $\mathbf{r}$ to r for each $r \in I$. For, the identity homomorphism is the only homomorphism of I into I [9].

Clearly, for $f, g \in C(X), f \equiv g(M)$ iff $d(f,g) \in M = h^{-1}(0)$ iff $(fg^* + gf^*)(x) = 0$ iff $f(x) = g(x)$ in I.

Claim that for each $r \in I, h^{-1}(r) = \{g \in C(X) | g(x) = r\}$. Indeed, we have that $g \in h^{-1}(r)$ iff $g \equiv \mathbf{r}(M)$ iff $g(x) = \mathbf{r}(x) = r$. Now for each $f \in C(X) = \cup\{h^{-1}(r) | r \in I\}, f \in h^{-1}(r)$ for some r. Thus $f(x) = r$ and hence $h(f) = f(x)$, i.e., $h(f) = \varepsilon_X(x)(f)$. Hence $h = \varepsilon_X(x)$. Thus ε_X is surjective. But ε_X is always an embedding for each $X \in \mathbf{X}$. Since X is compact and $SC(X)$ is Hausdorff, ε_X is a homeomorphism. Thus $X \in \text{Fix}\varepsilon$. The proof is complete. □

Corollary 2.3([8, Theorem 1])**.** *Let X and Y be both compact Hausdorff spaces. Then $C(X)$ and $C(Y)$ are isomorphic iff X and Y are homeomorphic.*

For an MV-algebra A, let $\mathfrak{M}(A)$ be the maximal ideal space of A with the Zarisk topology τ_z. Let $S(A)$ be the space of all homomorphisms of A into I with the relative topology τ_p of the product topology of $I^{|A|}$. Then we have the following Lemma :

Lemma 2.4. *For $A \in \mathbf{A}$, if $\Phi : S(A) \to \mathfrak{M}(A)$ is a map defined by $\Phi(u) = u^{-1}(0)$ for each $u \in S(A)$. Then Φ is a continuous bijection.*

Proof. For $u \in S(A)$ let $u^{-1}(0) = M$. Then M is obviously an ideal of A. Thus A/M is embedded into I and hence it is locally finite. Thus M is a maximal ideal of A. Clearly Φ is a well-defined injective. To show Φ is surjective, let $M \in \mathfrak{M}(A)$. Then A/M is locally finite, and hence it is embedded into I. For this embedding i, setting $u = i\varphi$, where φ is the canonical map of A onto A/M we have that $u \in S(A)$ and $u^{-1}(0) = M$. Hence Φ is a bijection. For the continuity of Φ, let $\Phi(u) = M \in \overline{x}$ for $x \in A$, where $\overline{x} = \{M | x \notin M\} \in \tau_z$. Then $x \notin M = u^{-1}(0)$, i.e., $u(x) \neq 0$.

Consider $U = pr_x^{-1}(I - \{0\})$ which is an open set in $S(A)$, where pr_x is the x^{th}-projection. Claim that $\Phi(U) \subset \overline{x}$. Indeed, if $v \in U$ then $v(x) \in I - \{0\}, v(x) \neq 0$. Thus $x \notin v^{-1}(0) = \Phi(v)$ and hence $\Phi(v) \in \overline{x}$. Since $\{\overline{x} | x \in A\}$ is a basis for τ_z, Φ is continuous. □

Corollary 2.5. *$S(A)$ is compact in τ_p iff $S(A)$ and $\mathfrak{M}(A)$ are homeomorphic.*

Proof. By Lemma 2.4, $\Phi : S(A) \to \mathfrak{M}(A)$ is a continuous bijection. If $S(A)$ is compact, since $\mathfrak{M}(A)$ is always T_2 [2], then we have Φ is a closed map. Thus Φ is a homeomorphism. The converse is trivial. □

Corollary 2.6. *For $A \in \mathfrak{M}_v$, $C(\mathfrak{M}(A))$ is a subalgebra of $C(S(A))$.*

Proof. Let $F : C(\mathfrak{M}(A)) \to C(S(A))$ be the function defined by $F(h) = h \circ \Phi$ for each $h \in C(\mathfrak{M}(A))$, where Φ is the same Φ in above Lemma 2.4. Since Φ is a continuous bijection, F is injective. Moreover F is an MV-homomorphism. Indeed, clearly for any $h_1, h_2 \in C(\mathfrak{M}(A))$, $F(h_1 + h_2) = F(h_1) + F(h_2)$. For each

$u \in S(A)$, $F(h^*)(u) = h^*(\Phi(u)) = 1 - h(\Phi(u)) = 1 - F(h)(u) = (F(h))^*(u)$. Thus $F(h^*) = F(h)^*$ for each $h \in C(\mathfrak{M}(A))$. □

Theorem 2.7. *An MV-algebra A is semi-simple iff A is isomorphic to a subalgebra of $C(X)$ for some Tychonoff space X.*

Proof. Let A be a semi-simple MV-algebra. By [8], A is a subalgebra of $C(\mathfrak{M}(A))$ where $\mathfrak{M}(A)$ is the space of maximal ideals. By Corollary 2.6, A is a subalgebra of $C(S(A))$ and $S(A)$ is a Tychonoff space because $S(A)$ is embedded in $I^{|A|}$. Conversely if A is a subalgebra of $C(X)$ for some Tychonoff space X, since $C(X)$ is embedded into $I^{|X|}$, A is semi-simple. □

Let $S(\mathbf{A})$ be the full isomorphism closed subcategory of $\mathbf{X}$ consisting of $S(A)$ for all $A \in \mathbf{A}$, and let **CompH** be the full subcategory of $\mathbf{X}$ consisting of all compact Hausdorff spaces in $\mathbf{X}$, and let **BooSp** be the full subcategory of X consisting of all Boolean spaces in $\mathbf{X}$.

Then we have the following Theorem :

Theorem 2.8. $S(\mathbf{A}) \cap$ **CompH**=**BooSp** *via* $S \vdash C$.

Proof. Let $X \in S(A) \cap \mathbf{CompH}$. Then $X = S(A)$ for some $A \in \mathbf{A}$. Since $S(A)$ is compact in τ_p, by Corollary 2.5 $S(A)$ and $\mathfrak{M}(A)$ are homeomorphic. But since A is semi-simple, $\mathfrak{M}(A)$ is a Boolean space. Thus $X \in \mathbf{BooSp}$.

Conversely if $X \in$ Boolean Space then by Theorem 2.2, ε_X is a homeomorphism, i.e., $X \cong SC(X)$. Let $A = C(X) \in \mathbf{A}$. Then $S(A) \cong X \in \mathbf{CompH}$ and $X \in S(\mathbf{A})$. □

3. Residuated l-monoids

By a $\wedge$-semilattice-ordered monoid($= \wedge - l-$monoid), we mean a system $M = (|M|, +, \wedge, 0, 1)$ satisfying the followings :

(i) $(|M|, +, 0)$ is a commutative monoid.

(ii) $(|M|, \wedge)$ is a $\wedge$-semilattice with 0 and 1.

(iii) $x + (y \wedge z) = (x + y) \wedge (x + z)$ for any $x, y, z \in M$.

By a residuated $\wedge - l-$monoid, we mean a $\wedge - l-$monoid M in which for each $a \in M$ there exists the least element a^* of $\{x \in M | a + x = 1\}$ which satisfies $a^{**} = a$.

The dual notion of a $\wedge - l-$monoid is defined as follows :

By a $\vee$-semilattice-ordered monoid ($= \vee - l-$monoid) we mean a system $M = (|M|, \cdot, \vee, 0, 1)$ satisfying the following :

(i) $(|M|, \cdot, 1)$ is a commutative monoid.

(ii) $(M, \vee)$ is a $\vee$-semilattice with 0 and 1.

(iii) $x(y \vee z) = (xy) \vee (xz)$ for any $x, y, z \in M$.

A $\vee - l-$monoid is said to be *residuated* if for each $a \in M$ there exists the greatest element a^* of $\{x \in M | a \cdot x = 0\}$ which satisfies $a^{**} = a$.

Although we define separately $\wedge - l-$monoid and its dual notion $\vee - l-$monoid, we can show that these two systems are the same notions as long as they are both residuated, as we can show by the following :

Lemma 3.1. *If M is a residuated $\wedge - l-$monoid then M is also a residuated $\vee - l-$monoid and conversely.*

Proof. Since $x = x^{**}$ for each $x \in M$, we have $x \leq y$ (iff $x \wedge y = x$) iff $y^* \leq x^*$ by definition of $*$-operation.
Then clearly $x \vee y = (x^* \wedge y^*)^*$ is the least upper bound of x and y in M. For any $a, x \in M$ we have that $a^* \leq x$ iff $a + x = 1$ which is equivalent to that $x^* \leq a$ iff $a^* \cdot x^* = 0$, where $a^* \cdot x^* = (a+x)^*$, that is equivalent to that $y \leq a^*$ iff $ay = 0$ for any a and $y \in M$. Hence $*$-operation of M is the same $*$-operation of the $\vee - l$-monoid induced by M. The other requirments are obvious. □

In the following, we call either residuated ($\wedge$ or $\vee$) l-monoid a *residuated l-monoid* simply.

Definition 3.2. Let M be a residuated l-monoid. If M satisfies the following conditions :

(a) for $x, y \in M, x^*(x^*y)^* = y^*(y^*x)^*$

(b) for $x, y \in M, xy^* \wedge x^*y = 0$

then we say that M has *the commuting property.*

Our aim is that any residuated l-monoid satisfying the commuting property has an MV-algebra structure so that it can be segmently embedded into an l-group. The crucial argument here is that the lattice-operations of M are actually those of the MV-algebra obtained from M.

Firstly we have the following obvious Lemma :

Lemma 3.3. *If A is an MV-algebra then $M(A) = (|A|, +, \cdot, *, \vee, \wedge, 0, 1)$ forms a residuated l-monoid with the commutating property, where $x \vee y = x + x^*y$ and $x \wedge y = x(x^* + y)$.*

Conversely we have the following Theorem :

Theorem 3.4. *If M is a residuated l-monoid satisfying the commuting property, then M becomes an MV-algebra, denoted by $A(M)$, whose lattice-operations $\boxed{\vee}$ and $\boxed{\wedge}$ are actually the same as those operations $\vee$ and $\wedge$ of M, respectively.*

Proof. Evidently from the structures of M (both structures of residuated $\vee$ and $\wedge$-semi lattice ordered monoid), all the axioms of an MV-algebra hold except for the commuting property, but we assume the commuting property. Thus M forms an MV-algebra denoted by $A(M)$. So $A(M)$ has its own lattice-operations : $x \boxed{\vee} y = y \boxed{\vee} x = x + x^*y$ and $x \boxed{\wedge} y = y \boxed{\wedge} x = x(x^* + y)$ for all x and $y \in A(M)$. Thus the first part of the proof of the theorem is complete. We however note that $x \leq y$ (iff $x \vee y = y$) for $x, y \in M$ is not necessary to be the same as $x \boxed{\leq} y$ (iff $x \boxed{\vee} y = y$) for $x, y \in A(M)$ as yet.

For the proof of the second part of theorem we need the following several Lemmas.

Lemma 3.5. *In M, we have that $x \leq y$ iff $xy^* = 0$ for each $x, y \in M$.*

Proof. $x \leq y$ means $x \vee y = y$. Thus $y^*(x \vee y) = 0$ which implies $y^*x = 0$. Conversely, $y^*x = 0$ means that $x \leq y^{**}$ by definition of $*$-operation of M, i.e., $x \leq y$. □

Lemma 3.6. *In $M, x \vee y \leq x + x^*y (= x \boxed{\vee} y)$ for all x, y.*

Proof. $(x \vee y)(x + x^*y)^* = [x^*(x + y^*)](x \vee y) = [xx^*(x + y^*)] \vee [yx^*(x + y^*)] = (yx^*)(yx^*)^* = 0.$ □

Lemma 3.7. *In M, $(x + y)z \leq xz + y$ for all x, y, z.*

Proof. $[(x + y)z](xz + y)^* = [(x + y)z][(xz)^*y^*] = [y^*(x + y)][z(z^* + x^*)] = (y^* \boxed{\wedge} x) \cdot (z \boxed{\wedge} x^*) = (x \boxed{\wedge} y^*)(x^* \boxed{\wedge} z) = x(x^* + y^*) \cdot x^*(x + z) = 0.$ □

Lemma 3.8. *In M, $x + x^*y \leq x \vee y$, i.e., $x \boxed{\vee} y \leq x \vee y$ for all x, y.*

Proof.

$$\begin{aligned} &(x + x^*y)(x \vee y)^* = (x + x^*y)(x^* \wedge y^*) \\ &\leq x \cdot (x^* \wedge y^*) + x^*y \qquad \text{by Lemma 3.7,} \\ &\leq x \cdot x^* + x^*y = x^*y. \end{aligned}$$

Similarly $(y + y^*x)(x \vee y)^* \leq y^*x$. By the commuting property (a) and (b), $(x + x^*y)(x \vee y)^* \leq x^*y \wedge y^*x = 0$. Hence we have $x \boxed{\vee} y \leq x \vee y$. □

By lemmas 13 and 15, we have $x \boxed{\vee} y = x \vee y$ and dually $x \boxed{\wedge} y = x \wedge y$ in M or in $A(M)$. And hence the partial ordering $\leq$ of M is the same as that $\boxed{\leq}$ of $A(M)$. The proof of theorem is complete. □

Corollary 3.9. *A residuated l-monoid M is segmently embedded into an l-group G with order unit iff M has the commuting property.*

Consider the category $\mathfrak{M}_v$ of MV-algebras and their homomorphisms and the category $\mathcal{L}_m$ of residuated l-monoids with the commuting property and their l-monoid-homomorphisms preserving $*$-operations.

Let $\Phi : \mathfrak{M}_v \to \mathcal{L}_m$ be the functor defined by $\Phi(A) = M(A)$ for each $A \in \mathfrak{M}_v$. If f is an MV-morphism in $\mathfrak{M}_v$ then $\Phi(f)$ is clearly an l-monoid morphism preserving $*$.

Now we define $\Psi : \mathcal{L}_m \to \mathfrak{M}_v$ by $\Psi(M) = A(M)$ for each $M \in \mathcal{L}_m$. And if φ is an l-monoid-morphism preserving $*$, then obviously $\Psi(\varphi)$ is an MV-morphism.

Let $\Phi(A) = M(A)$ for an MV-algebra. Then from the above construction of $M(A)$, the underlying sets $|A|$ of A and A of $M(A)$ are the same. Similarly for

$\Psi(M) = A(M)$ for $M \in \mathcal{L}_m, |M| = |A(M)|$. Thus $|A(M(A)| = |A|$. The operations of $A(M(A))$ and A are coincide. Hence it easy to see $\Psi \circ \Phi = \mathrm{id}_{\mathfrak{M}_v}$. Similarly $\Phi \circ \Psi = \mathrm{id}_{\mathcal{L}_m}$.

It is easy to see the following Theorem :

Theorem 3.10. *The categories $\mathfrak{M}_v$ and $\mathcal{L}_m$ are categorically equivalent via Φ and Ψ.*

Corollary 3.11. *The categories $\mathcal{L}_m$ and that of l-groups with order unit and their l-group-homomorphisms preserving order units are categorically equivalent.*

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