

Linear Preservers of Perimeters of Nonnegative Real Matrices

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ABSTRACT. For a nonnegative real matrix A of rank 1, A can be factored as $\mathbf{a}\mathbf{b}^t$ for some vectors \mathbf{a} and \mathbf{b} . The *perimeter* of A is the number of nonzero entries in both \mathbf{a} and \mathbf{b} . If B is a matrix of rank k , then B is the sum of k matrices of rank 1. The perimeter of B is the minimum of the sums of perimeters of k matrices of rank 1, where the minimum is taken over all possible rank-1 decompositions of B . In this paper, we obtain characterizations of the linear operators which preserve perimeters 2 and k for some $k \geq 4$. That is, a linear operator T preserves perimeters 2 and $k(\geq 4)$ if and only if it has the form $T(A) = UAV$, or $T(A) = UA^tV$ with some invertible matrices U and V .

1. Introduction

There is much literature on the study of linear operators that preserve the ranks of matrices over several semirings([1]-[8]). Nonnegative matrices also have been the subject of research by many authors([2], [5], [6], [8]). Beasley, Gregory and Pullman [2] obtained characterizations of linear operators which preserve the rank of nonnegative real matrices. In [8], Song and Hwang characterized spanning column ranks and their preservers of nonnegative matrices. Beasley, Song, Kang and Sarma [5] treated column ranks of nonnegative real matrices and characterized their preservers.

But there are few papers on the characterizations of linear operators preserving the perimeter of matrices. Beasley et al. characterized those linear operators preserving the rank and perimeter of Boolean rank-1 matrices([1]).

In this paper, we consider the set of linear operators that preserve the perimeter of matrices of rank $k(\geq 2)$ over the nonnegative reals.

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2. Preliminaries and some results

Let $M_{m,n}(R_+)$ denote the set of all $m \times n$ matrices with entries in R_+ , the set of nonnegative reals. Addition, multiplication by scalars, and the product of matrices are also defined as if R_+ were a field. Throughout this paper, we shall adopt the convention that $1 \leq m \leq n$ unless otherwise specified.

The *rank* or *factor rank*, $r(A)$, of a nonzero matrix $A \in M_{m,n}(R_+)$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = BC$. The rank of a zero matrix is zero.

The *Boolean algebra* consists of the set $B = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 +_B 1 = 1$. If $A = [a_{ij}]$ is any matrix in $M_{m,n}(R_+)$, we define $A^* = [a_{ij}^*]$ to be the $m \times n$ Boolean matrix whose (i, j) -th entry is 1 if and only if $a_{ij} \neq 0$. Then $*$ maps $M_{m,n}(R_+)$ onto $M_{m,n}(B)$, and preserves matrix addition, product, and multiplication by scalars. That is, $*$ is a homomorphism. It follows that

$$(2.1) \quad (A + B)^* = A^* + B^* \quad \text{and} \quad (BC)^* = B^*C^*,$$

for all $A, B \in M_{m,n}(R_+)$ and all $C \in M_{n,r}(R_+)$.

If $A \in M_{m,n}(R_+)$ with $r(A) = 1$, there exist nonzero vectors $\mathbf{a} \in M_{m,1}(R_+)$ and $\mathbf{b} \in M_{n,1}(R_+)$ such that $A = \mathbf{a}\mathbf{b}^t$. But these vectors \mathbf{a} and \mathbf{b} are not uniquely determined by A . For any vector $\mathbf{u} \in M_{m,1}(R_+)$, $|\mathbf{u}|$ denote the number of nonzero entries in \mathbf{u} .

Lemma 2.1. *For any factorization $\mathbf{a}\mathbf{b}^t$ of $A \in M_{m,n}(R_+)$ with $r(A) = 1$, $|\mathbf{a}|$ and $|\mathbf{b}|$ are uniquely determined by A .*

Proof. It follows from (2.1) and the fact that $|\mathbf{a}^*|$ and $|\mathbf{b}^*|$ are uniquely determined by A^* . \square

Let A be a rank-1 matrix in $M_{m,n}(R_+)$. We define the *perimeter* of a rank-1 matrix A , $p(A)$, as $|\mathbf{a}| + |\mathbf{b}|$ for arbitrary factorization $A = \mathbf{a}\mathbf{b}^t$. By Lemma 2.1, $p(A)$ is well-defined. For any matrix A in $M_{m,n}(R_+)$, a *rank-1 decomposition* of A is a sum of matrices of rank 1 which equals A . We say that A has *1-rank* k if k is the minimal number of terms in any rank-1 decomposition of A . This shows that A has 1-rank k if and only if $r(A) = k$. If $A \in M_{m,n}(R_+)$, the *perimeter* of A , $p(A)$, is defined as

$$\min \left\{ \sum_{i=1}^t p(A_i) \mid \sum_{i=1}^t A_i \text{ is a rank-1 decomposition of } A, \quad 1 \leq t \leq m \right\}.$$

Let P_k denote the set of matrices in $M_{m,n}(R_+)$ whose perimeter is k for $k = 2, 3, \dots, m(n+1)$. Let E_{ij} denote the $m \times n$ matrix whose (i, j) th entry is 1 and whose other entries are all 0, and $E_{m,n} = \{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. We call E_{ij} a *cell* and αE_{ij} a *weighted cell* for any nonzero scalar $\alpha \in R_+$. Thus, we have $A \in P_2$ if and only if A is a weighted cell.

For any matrix $A \in M_{m,n}(R_+)$, a *line* is a row or a column of A . The set of cells is *collinear* if they are located in a same line. We denote the number of all nonzero entries in a matrix $A \in M_{m,n}(R_+)$ as $|A|$.

Let A be a matrix in $P_{2h}(1 \leq h \leq m)$ which has minimum nonzero entries. What is the value of $|A|$? The answer is $|A| = h$, and any two nonzero entries are not located in a same line. Let B be a matrix in $P_{2h+1}(1 \leq h \leq m - 1)$ which has minimum nonzero entries. Then we have $|B| = h + 1$, and h lines contain all $h + 1$ nonzero entries. Furthermore, at least one line contains two nonzero entries. This proves the following Lemma:

Lemma 2.2. *The following statements hold:*

- (1) *Let $k = 2h$ with $1 \leq h \leq m$. Then the elements of P_k with minimum weighted cells have exactly h nonzero entries, no two of them are in one line.*
- (2) *Let $k = 2h + 1$ with $1 \leq h \leq m - 1$. Then the elements of P_k with minimum weighted cells have exactly $h + 1$ nonzero entries which are located in h lines, and at least one line has two nonzero entries.*

Consider a matrix

$$A = \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & d \end{bmatrix} \in M_{3,3}(R_+),$$

where $abcd \neq 0$. Then the minimum number of lines that contain all the nonzero entries of A is 2. The Lemma 2.3(below) shows that the value of $p(A)$ is either 4 or 6. In fact, if $ad = bc$, then $p(A) = 4$, while $ad \neq bc$ implies $p(A) = 6$ from the following factorizations:

$$A = \begin{bmatrix} a \\ 0 \\ c \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{b}{a} \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} a & 0 & b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} c & 0 & d \end{bmatrix},$$

respectively.

Lemma 2.3. *Let A be a matrix in $M_{m,n}(R_+)$ with $|A| = t + 2$. If the minimum number of lines that contain all the nonzero entries of A is t , we have either $p(A) = 2t$ or $p(A) = 2t + 2$.*

Proof. Since t is the minimum number of lines that contain all nonzero entries of $A = [a_{ij}]$, without loss of generality we may assume $a_{ii} \neq 0$ for $i = 1, \dots, t$. It follows that the remaining two nonzero entries, x and y , lie on the i th row or the j th column for some $i, j = 1, \dots, t$. Then two possibilities exist: (a) x and y lie on the same line; (b) x and y lie on two different lines. For the case (a), we have $p(A) = 2t + 2$. For the case (b), if there exist distinct indices i, j in $\{1, \dots, t\}$ with $x = a_{ij}$ and $y = a_{ji}$ such that $\frac{a_{ii}}{a_{ji}} = \frac{a_{ij}}{a_{jj}}$, then we have $p(A) = 2t$. For the other cases, the value of $p(A)$ is $2t + 2$. □

3. Linear operators preserving P_2 and P_k

In this section, we will characterize those linear operators that preserve perimeters 2 and $k \geq 3$. An $n \times n$ nonnegative real matrix A is said to be *invertible* if there exists a matrix $B \in M_{n,n}(R_+)$ such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix. It is well known [2] that a square matrix A over R_+ is invertible if and only if some permutation of its rows is a diagonal matrix all of whose diagonal entries are nonzero in R_+ .

Lemma 3.1. *The perimeter of a matrix of rank 1 is preserved under pre or post-multiplication by an invertible matrix.*

Proof. If A is a matrix in $M_{m,n}(R_+)$ with $r(A) = 1$, there exist nonzero vectors $\mathbf{a} \in M_{m,1}(R_+)$ and $\mathbf{b} \in M_{n,1}(R_+)$ such that $A = \mathbf{a}\mathbf{b}^t$. Let U and V be invertible matrices in $M_{m,m}(R_+)$ and $M_{n,n}(R_+)$, respectively. Then we have

$$\begin{aligned} p(UAV) &= p(U\mathbf{a}\mathbf{b}^tV) = p((U\mathbf{a})(V^t\mathbf{b})^t) \\ &= |U\mathbf{a}| + |V^t\mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| = p(A). \end{aligned}$$

The Lemma now follows. □

A mapping $T : M_{m,n}(R_+) \rightarrow M_{m,n}(R_+)$ is called a *linear operator* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $A, B \in M_{m,n}(R_+)$ and for all $\alpha, \beta \in R_+$. For a linear operator T on $M_{m,n}(R_+)$, we say that T is a (U, V) -operator if there exist invertible matrices $U \in M_{m,m}(R_+)$ and $V \in M_{n,n}(R_+)$ such that $T(A) = UAV$ for all A in $M_{m,n}(R_+)$, or $m = n$ and $T(A) = UA^tV$ for all A in $M_{m,n}(R_+)$. A linear operator T on $M_{m,n}(R_+)$ is said to *preserve perimeter* if $p(T(A)) = p(A)$ for all $A \in M_{m,n}(R_+)$. A linear operator T on $M_{m,n}(R_+)$ *preserves P_k* if $T(A) \in P_k$ whenever $A \in P_k$.

Proposition 3.2. *If T is a (U, V) -operator on $M_{m,n}(R_+)$, then T preserves perimeter.*

Proof. Since T is a (U, V) -operator, there exist invertible matrices $U \in M_{m,m}(R_+)$ and $V \in M_{n,n}(R_+)$ such that either $T(A) = UAV$, or $m = n$ and $T(A) = UA^tV$ for all A in $M_{m,n}(R_+)$. Let $A = \sum_{i=1}^t A_i$ be a rank-1 decomposition of A .

For the case $T(A) = UAV$, we have

$$\sum_{i=1}^t p(T(A_i)) = \sum_{i=1}^t p(UA_iV) = \sum_{i=1}^t p(A_i)$$

by Lemma 3.1. This implies that $p(T(A)) = p(A)$. For the case $T(A) = UA^tV$, we can show that $p(T(A)) = p(A)$ by the similar method as above. Thus the proof is complete. □

Let $R_i = \{E_{ij} \mid 1 \leq j \leq n\}$, $C_j = \{E_{ij} \mid 1 \leq i \leq m\}$, $\mathbf{R} = \{R_i \mid 1 \leq i \leq m\}$ and $\mathbf{C} = \{C_j \mid 1 \leq j \leq n\}$. For a linear operator T on $M_{m,n}(R_+)$, define $T^*(X) =$

$[T(X)]^*$ for all X in $M_{m,n}(R_+)$. Let $T^*(R_i) = \{T^*(E_{ij}) \mid 1 \leq j \leq n\}$ for each $i = 1, \dots, m$ and $T^*(C_j) = \{T^*(E_{ij}) \mid 1 \leq i \leq m\}$ for each $j = 1, \dots, n$. Let T be a linear operator on $M_{m,n}(R_+)$ which preserves P_2 . Then T^* maps $E_{m,n}$ into $E_{m,n}$ because P_2 is the set of all weighted cells.

Lemma 3.3. *Let $4 \leq k \leq 2m$. If T is a linear operator on $M_{m,n}(R_+)$ which preserves P_2 and P_k , then T^* maps $E_{m,n}$ onto $E_{m,n}$.*

Proof. We consider two cases: (a) $k = 2h + 1$ with $2 \leq h \leq m - 1$ and (b) $k = 2h$ with $2 \leq h \leq m$.

Case a. $k = 2h + 1$ with $2 \leq h \leq m - 1$. Suppose T^* does not map $E_{m,n}$ onto $E_{m,n}$. Then there exist two distinct cells E_{ij} and E_{pq} in $E_{m,n}$ such that $T^*(E_{ij}) = T^*(E_{pq})$. By Lemma 2.2-(2), there is a matrix $X = [x_{ij}] \in P_k$ which has minimum nonzero entries such that $x_{ij} \neq 0$ and $x_{pq} \neq 0$. But then we have $T(X) \notin P_k$ because $|T(X)| < |X|$. This contradiction shows that T^* maps $E_{m,n}$ onto $E_{m,n}$.

Case b. $k = 2h$ with $2 \leq h \leq m$. Assume that $T^*(E_{ij}) = T^*(E_{pq})$ for two distinct cells E_{ij} and E_{pq} in $E_{m,n}$. If E_{ij} and E_{pq} are not collinear, by the similar argument of case a) with Lemma 2.2-(1), there exists a matrix $Y = [y_{ij}] \in P_k$ which has minimum nonzero entries such that $y_{ij} \neq 0$ and $y_{pq} \neq 0$. But then we have $T(Y) \notin P_k$ because $|T(Y)| < |Y|$. This is a contradiction. Hence E_{ij} and E_{pq} are collinear. Without loss of generality we may assume $(i, j) = (1, 1)$ and $(p, q) = (1, 2)$.

Consider a matrix $Z = E_{1,1} + E_{1,2} + E_{1,3} + \sum_{i=2}^{h-1} E_{i,i} \in M_{m,n}(R_+)$. Then we have $Z \in P_k$ with $|Z| = h + 1$. But then $|T(Z)| \leq h$ follows from $T^*(E_{ij}) = T^*(E_{pq})$. Since a matrix $A \in P_k$ with minimum nonzero entries contains h nonzero entries, and T preserves P_k , we have $|T(Z)| = h$. By Lemma 2.2-(1), any two nonzero entries in $T(Z)$ are not collinear. Without loss of generality we may assume that $T^*(E_{1,1}) = T^*(E_{1,2}) = E_{1,1}$, $T^*(E_{i,i}) = E_{i,i}$ for $2 \leq i \leq h - 1$ and $T^*(E_{1,3}) = E_{r,s}$, where $r, s \geq h$.

Let $W = E_{1,1} + E_{1,3} + E_{1,t} + \sum_{i=2}^{h-1} E_{i,i}$ be a matrix in $M_{m,n}(R_+)$ with $4 \leq t \leq n$ so that $W \in P_k$. Suppose that $T^*(E_{1,t})$ is neither $E_{i,i}$ for $1 \leq i \leq h - 1$ nor $E_{r,s}$. Then we have $p(T(W)) \geq k + 1$, a contradiction because T preserves P_k . Hence for all $4 \leq t \leq n$, $T^*(E_{1,t})$ is one of $E_{i,i}$ for $1 \leq i \leq h - 1$ or $E_{r,s}$. Let $S = E_{1,3} + E_{1,4} + E_{1,h} + \sum_{i=2}^{h-1} E_{i,i}$ be a matrix in $M_{m,n}(R_+)$ so that $S \in P_k$. But then we have $|T(S)| \leq h - 1$, and hence $T(S) \notin P_k$, a contradiction. This completes the proof. \square

Let $T : M_{m,n}(R_+) \rightarrow M_{m,n}(R_+)$ be a linear operator which preserves P_2 .

Because T^* maps $E_{m,n}$ into $E_{m,n}$, we can write for all $X = [x_{ij}] \in M_{m,n}(R_+)$,

$$T(X) = \sum_{i=1}^m \sum_{j=1}^n x_{ij} b_{ij} T^*(E_{ij}),$$

where each b_{ij} is a nonzero scalar in R_+ .

Lemma 3.4. *Let $k = 2h$ with $2 \leq h \leq m$. If T is a linear operator on $M_{m,n}(R_+)$ which preserves P_2 and P_k , then T maps lines onto lines.*

Proof. It follows from Lemma 3.3 that T^* maps $E_{m,n}$ onto $E_{m,n}$. Suppose T does not preserve lines to lines. Then there exist two distinct cells E_{ij} and E_{pq} which are not collinear such that $T(E_{ij})$ and $T(E_{pq})$ are collinear. Let

$$X = E_{ij} + E_{pq} + \sum_{t=1}^{h-2} E_{i_t j_t}$$

be a matrix in $M_{m,n}(R_+)$, where all i_t, i, p are distinct and all j_t, j, q are distinct. Then we have $p(X) = k$, while $p(T(X)) \leq 3 + 2(h-2) = 2h-1 < k$, a contradiction. Therefore T maps lines onto lines. \square

Lemma 3.5. *Let $4 \leq k \leq 2m$. If T is a linear operator on $M_{m,n}(R_+)$ which preserves P_2 and P_k , then T maps lines onto lines.*

Proof. By Lemma 3.4, we can assume that $k = 2h + 1$ with $2 \leq h \leq m - 1$. Lemma 3.3 implies that T^* maps $E_{m,n}$ onto $E_{m,n}$. If T does not map lines onto lines, without loss of generality we may assume that $T(E_{1,1})$ and $T(E_{1,2})$ are not collinear. Let $X = E_{1,1} + E_{1,2} + \sum_{i=2}^h E_{i,i}$ be a matrix in $M_{m,n}(R_+)$ so that $X \in P_k$. But then $T(X)$ is a sum of $h + 1$ weighted cells with $p(T(X)) = k$ because T preserves P_k . By Lemma 2.2-(2), all nonzero entries in $T(X)$ must be located in h line, and at least one line has two nonzero entries.

Let Y be a matrix which is the sum of h cells including $E_{1,1}$ and $E_{1,2}$ such that h is the minimum number of lines that contain all the nonzero entries of $T(Y)$. Let $Z = Y + E_{1,3} + E_{1,h}$ be a matrix in $M_{m,n}(R_+)$ so that $p(Z) = 5 + 2(h - 2) = 2h + 1 = k$. But then $T(Z)$ is the sum of $h + 2$ weighted cells, and the minimum number of lines that contain all the nonzero entries of $T(Z)$ is greater than or equal to h . By Lemma 2.3, the perimeter of $T(Z)$ is either $2h, 2h + 2, 2h + 3$ or $2h + 4$, a contradiction. Therefore T maps lines onto lines. \square

Theorem 3.6. *If T is a linear operator on $M_{m,n}(R_+)$ which preserves P_2 and P_k with $4 \leq k \leq \min(2m, n + 2)$, we have T is a (U, V) -operator.*

Proof. It follows from Lemma 3.3 that T^* maps $E_{m,n}$ onto $E_{m,n}$. By Lemma 3.5, there are two cases; (a) T^* maps \mathbf{R} onto \mathbf{R} and maps \mathbf{C} onto \mathbf{C} or (b) T^* maps \mathbf{R} onto \mathbf{C} and \mathbf{C} onto \mathbf{R} .

Case a. We note that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all i, j , where σ

and τ are permutations of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. Let P and Q be the permutation matrices corresponding to σ and τ , respectively. Then for any $E_{ij} \in E_{m,n}$, we can write $T(E_{ij}) = b_{ij}E_{\sigma(i)\tau(j)}$ for some nonzero scalar $b_{ij} \in R_+$. Now we claim that for all $i, l \in \{1, \dots, m\}$ and all $j, r \in \{1, \dots, n\}$,

$$\frac{b_{ij}}{b_{ir}} = \frac{b_{lj}}{b_{lr}}.$$

Consider a matrix

$$A = E_{ij} + E_{ir} + E_{lj} + E_{lr} + \sum_{t=1}^{k-4} (E_{ij_t} + E_{lj_t}),$$

where $j_t \neq j, r$ for all $t = 1, \dots, k - 4$. Then we have $p(A) = k$ and the image of A becomes

$$\begin{aligned} T(A) &= b_{ij}E_{\sigma(i)\tau(j)} + b_{ir}E_{\sigma(i)\tau(r)} + b_{lj}E_{\sigma(l)\tau(j)} + b_{lr}E_{\sigma(l)\tau(r)} \\ &+ \sum_{t=1}^{k-4} (b_{ij_t}E_{\sigma(i)\tau(j_t)} + b_{lj_t}E_{\sigma(l)\tau(j_t)}). \end{aligned}$$

Since $T(A)$ has perimeter k , it follows that $\frac{b_{ij}}{b_{ir}} = \frac{b_{lj}}{b_{lr}} = \frac{b_{ij_t}}{b_{lj_t}}$ for all $t = 1, \dots, k - 4$. Let $C \in M_{m,m}(R_+)$ and $D \in M_{n,n}(R_+)$ be diagonal matrices such that

$$c_{11} = 1, \quad d_{11} = b_{11}, \quad c_{ii} = \frac{b_{i1}}{b_{11}}, \quad \text{and} \quad d_{jj} = b_{1j}$$

for all $i = 2, \dots, m$ and $j = 2, \dots, n$. Then we have $b_{ij} = c_{ii}d_{jj}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Let $X = [x_{ij}]$ be any $m \times n$ matrix in $M_{m,n}(R_+)$. Then we have

$$\begin{aligned} T(X) &= T \left(\sum_{i=1}^m \sum_{j=1}^n x_{ij} E_{ij} \right) = \sum_{i=1}^m \sum_{j=1}^n x_{ij} T(E_{ij}) \\ &= \sum_{i=1}^m \sum_{j=1}^n x_{ij} b_{ij} E_{\sigma(i)\tau(j)} = \sum_{i=1}^m \sum_{j=1}^n c_{ii} x_{ij} E_{\sigma(i)\tau(j)} d_{jj} \\ &= CPXQD. \end{aligned}$$

Since $CP = U$ is an $m \times m$ invertible matrix and $QD = V$ is an $n \times n$ invertible matrix, it follows that T is a (U, V) -operator.

Case b. This is the case of $m = n$ and $T^*(R_i) = C_{\sigma(i)}$ and $T^*(C_j) = R_{\tau(j)}$ for all i and j , where σ and τ are permutations of $\{1, \dots, m\}$. By similar argument to case a), we obtain that $T(X)$ is of the form $T(X) = CPX^tQD$. Thus T is a (U, V) -operator. \square

We say that a linear operator T on $M_{m,n}(R_+)$ *strongly preserves perimeter k* if $T(A) \in P_k$ if and only if $A \in P_k$.

Theorem 3.7. *If T is a linear operator on $M_{m,n}(R_+)$ which strongly preserves P_2 and P_3 , T is a (U, V) -operator.*

Proof. It is obvious that T^* maps $E_{m,n}$ onto $E_{m,n}$ because T strongly preserves P_2 . If T does not map lines to lines, there exist two distinct cells E_{ij} and E_{pq} which are collinear such that $T(E_{ij})$ and $T(E_{pq})$ are not collinear. But then the perimeter of $E_{ij} + E_{pq}$ is 3, while that of $T(E_{ij} + E_{pq})$ is 4, a contradiction. It follows that T maps lines to lines. By the similar method in the proof of Theorem 3.6, we have T is a (U, V) -operator. \square

Corollary 3.8. *Let T be a linear operator on $M_{m,n}(R_+)$. Then the following are equivalent;*

- (1) T is a (U, V) -operator.
- (2) T preserves perimeter.
- (3) T preserves P_2 and P_k with $4 \leq k \leq \min(2m, n + 2)$.
- (4) T strongly preserves P_2 and P_k with $3 \leq k \leq \min(2m, n + 2)$.

Proof. The proof follows from Theorem 3.6 and Theorem 3.7. \square

Thus we have characterizations of the linear operators which preserve all perimeters over the nonnegative reals.

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