KYUNGPOOK Math. J. 48(2008), 443-455

Weak Normality and Strong t-closedness of Generalized Power Series Rings

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ABSTRACT. For an extension $A \subseteq B$ of commutative rings, we present a sufficient condition for the ring $[[A^{S, \leq}]]$ of generalized power series to be weakly normal (resp., strongly t-closed) in $[[B^{S, \leq}]]$, where $(S, \leq)$ be a torsion-free cancellative strictly ordered monoid. As a corollary, it can be applied to the ring of power series in infinitely many indeterminates as well as in finite indeterminates.

1. Introduction and preliminaries

Let $A \subseteq B$ be an extension of commutative rings with (the same) identity. Consider the following conditions:

(a) B is integral over A.

(b) $\mathsf{Spec}(B) \to \mathsf{Spec}(A)$ is a bijection.

(c) The residue field extensions are isomorphisms. i.e., for each $Q \in \mathsf{Spec}(B)$ the extension $A_P/PA_P \hookrightarrow B_Q/QB_Q$ is an isomorphism, where $P = Q \cap A$.

(c$'$) The residue field extensions are purely inseparable.

We first recall some special extensions satisfying two or three conditions above including the condition (a).

- R. G. Swan called the extension $A \subseteq B$ *subintegral* if (a), (b) and (c) are satisfied.

Received October 24, 2007.
2000 Mathematics Subject Classification: 13F25, 13C11.
Key words and phrases: weak normality, (strong) t-closedness, generalized power series ring.
The first author was supported by Hoseo University Research Fund, 2007. The third author was supported by Changwon National University Research Fund, 2007.

- H. Yanagihara called the extension $A \subseteq B$ *weakly subintegral* if (a), (b) and (c$'$) are satisfied.
- G. Picavet and M. Picavet-L'Hermitte called the extension $A \subseteq B$ *infra-integral* if (a) and (c) are satisfied.
- M. Picavet-L'Hermitte called the extension $A \subseteq B$ *weakly infra-integral* if (a) and (c$'$) are satisfied.

Using these extensions, they defined and characterized the *seminormalization* ${}^+_B A$ (resp., *weak normalizarion* ${}^*_B A$, *t-closure* ${}^t_B A$, *strong t-closure* ${}^\circ_B A$) of A in B as the largest subintegral (resp., weakly subintegral, infra-integral, weakly infra-integral) subextension of B over A. They also defined that A is *seminormal* (resp., *weakly normal*, *t-closed*, *strongly t-closed*) *in* B if $A = {}^+_B A$ (resp., $A = {}^*_B A$, $A = {}^t_B A$, $A = {}^\circ_B A$). When B is the quotient field of A, we use the notation ${}^+A$ (resp., ${}^*A$, ${}^tA$, ${}^\circ A$) instead of ${}^+_B A$ (resp., ${}^*_B A$, ${}^t_B A$, ${}^\circ_B A$). An integral domain A is called *seminormal* (resp., *weakly normal*, *t-closed*, *strongly t-closed*) if it is so in its quotient field.

We next recall some related results on power series rings. D. E. Dobbs and M. Roitman proved, in [3, Theorem 3], that for an extension $A \subseteq B$ of integral domains, if A is weakly normal in B, then $A[[X_1, \cdots, X_n]]$ is weakly normal in $B[[X_1, \cdots, X_n]]$. In [13, Theorem 4.18], M. Picavet-L'Hermitte showed that for an integral extension $A \subseteq B$ of commutative rings, $A[[X]]$ is strongly t-closed in $B[[X]]$ if and only if A is strongly t-closed in B.

In [4], [6], [7], [14], [15], the authors investigated the following question: If a ring A satisfies a property P and a partially ordered monoid $(S, \leq)$ satisfies a property Q, does the ring of generalized power series $[[A^{S,\leq}]]$ satisfy P? And conversely, if $[[A^{S,\leq}]]$ satisfies P, does A satisfy P and $(S, \leq)$ satisfy Q? Among other results, P. Ribenboim proved the following result which is related to seminormality: Let $(S, \leq)$ be a submonoid of a torsion-free cancellative ordered monoid $(T, \leq)$ and let A be a reduced subring of a commutative ring B. Then the generalized power series ring $[[A^{S,\leq}]]$ is seminormal in $[[B^{T,\leq}]]$ if and only if A is seminormal in B and S is seminormal in T [15, (6.7) and (6.9)]. On the other hand, the first author showed that for a torsion-free cancellative ordered monoid $(S, \leq)$ and an extension $A \subseteq B$ of commutative rings satisfying property $\mathcal{P}_1(A, B)$, the generalized power series ring $[[A^{S,\leq}]]$ is t-closed in $[[B^{S,\leq}]]$ if and only if A is t-closed in B [6, Theorem 2.3]. This paper is mainly concerned with weak normality and strong t-closedness of generalized power series rings. As an interesting corollary, it can be applied to the ring of power series in infinitely many indeterminates as well as in finite indeterminates. Thus we generalize and unify the well-known results mentioned above.

Let $(S, \leq)$ be an ordered set. We recall that $(S, \leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. It is easy to see that $(S, \leq)$ is artinian if and only if every non-empty subset of S has a minimal element. Moreover, if $\leq$ is a total order, then $(S, \leq)$ is artinian if and only if it is well-ordered.

$(S,\leq)$ is a *strictly ordered monoid* if $s,s',t\in S$ and $s<s'$ imply $s+t<s'+t$. Note that if S is cancellative or if $\leq$ is the trivial order (i.e., $x\leq y$ implies $x=y$), then $(S,\leq)$ is a strictly ordered monoid.

The following definition is due to P. Ribenboim [4]: Let $(S,\leq)$ be a strictly ordered monoid and let A be a commutative ring. Let $R=[[A^{S,\leq}]]$ be the set of all functions $f:S\to A$ such that $Supp(f)=\{s\in S\mid f(s)\neq 0\}$ is artinian and narrow. It is clear that R is an additive abelian group with pointwise addition. For every $s\in S$ and $f_1,\cdots,f_n\in R$, let $X_s(f_1,\cdots,f_n)=\{(u_1,\cdots,u_n)\in S^n\mid s=u_1+\cdots+u_n,\ u_i\in Supp(f_i)$ for each $i\}$. It follows from [4, (e) p. 368] that $X_s(f_1,\cdots,f_n)$ is finite. This fact allows one to define the operation of convolution $*$ as following;

$$(f*h)(s)=\sum_{(u,v)\in X_s(f,h)} f(u)h(v).$$

With this operation, and pointwise addition, R becomes a commutative ring with identity element e, where

$$e(s)=\begin{cases}1 & \text{if } s=0\\ 0 & \text{if } 0\neq s\in S.\end{cases}$$

We call R the *ring of generalized power series*. It should be noted that the definition of $[[A^{S,\leq}]]$ depends on the order $\leq$, for example, see [4, p.371]. Following [14, 2.5], R is an integral domain if and only if D is an integral domain, and S is torsion-free and cancellative. It follows from [4, p.368] that A is canonically embedded as a subring of $[[A^{S,\leq}]]$, and that S is canonically embedded as a submonoid of $([[A^{S,\leq}]]\setminus\{0\},*)$.

In [4], [15], there are many results on the ordered monoid and on the ring of generalized power series. The following result is well-known, which is frequently used in the sequel. If S is a torsion-free cancellative monoid and if $\leq$ is any compatible strict order on S, then there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$ (i.e., if $s,t\in S$ such that $s\leq t$, then $s\leq' t$).

General references for any undefined terminology or notation are [4], [15].

2. Weak normality

Recall from [17, Theorem 1] that A is weakly normal in B if and only if A is seminormal in B and, whenever an element b in B satisfies $b^p, pb\in A$ for some prime number p, then $b\in A$.

Lemma 2.1([3, Lemma 2])**.** *Let $A\subseteq B$ be an extension of commutative rings with identity such that A is seminormal in B. Let $m\geq 2$ be an integer. Suppose that $a\in A$ and $b\in B$ satisfy $ab, ab^m\in A$. Then $ab^i\in A$ for all $1\leq i\leq m$.*

The following definition (with notation $\mathcal{O}$) is due to [15, p.571]. If $0\neq f\in [[D^{S,\leq}]]$, we denote by $\mathcal{O}(f)$ the set of minimal elements of $Supp(f)$; then $\mathcal{O}(f)$ is a nonempty finite set, consisting of pairwise order incomparable elements. If $\mathcal{O}(f)$ consists only of one element s, then we write $\mathcal{O}(f) = s$.

The following result is a generalization of [3, Theorem 3]. Although its proof is essentially the same as that of [3, Theorem 3], we will give a proof for completeness.

Theorem 2.2. *Let $(S, \leq)$ be a torsion-free cancellative ordered monoid and let $A \subseteq B$ be an extension of commutative rings.*

(1) *If A is weakly normal in B, then the generalized power series ring $[[A^{S,\leq}]]$ is weakly normal in $[[B^{S,\leq}]]$.*

(2) ${}^{*}_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \subseteq [[({}^{*}_{B}A)^{S,\leq}]]$. *Moreover, if ${}^{*}_{B}A$ is finitely generated as an A-module (in particular, if A is Noetherian and B is a finitely generated A-module), then the equality holds*

Proof. (1) Suppose that A is weakly normal in B. Then by [17, Theorem 1], A is seminormal in B. Thus by [15, (6.9)], $[[A^{S,\leq}]]$ is seminormal in $[[B^{S,\leq}]]$. Let $f \in [[B^{S,\leq}]]$ such that $f^p, pf \in [[A^{S,\leq}]]$. Since S is torsion-free and cancellative, there exists a compatible strict total order $\leq'$ on S which is finer than $\leq$. Note that $[[A^{S,\leq}]]$ is a subring of $[[A^{S,\leq'}]]$ and $[[B^{S,\leq}]]$ is a subring of $[[B^{S,\leq'}]]$. Thus we may assume that $f \in [[B^{S,\leq'}]]$ such that $f^p, pf \in [[A^{S,\leq'}]]$. We will show that $f(u) \in A$ for each $u \in Supp(f)$.

We denote by $\mathcal{O}'(f)$ the smallest element of $Supp(f)$ in the order $\leq'$. Then we first show that $f(s) \in A$ for $\mathcal{O}'(f) = s$. Since $f^p(ps) = (f(s))^p \in A$ and $(pf)(s) = p(f(s)) \in A$, we have $f(s) \in A$ since A is weakly normal in B.

Suppose that $f(v) \in A$ for every $v \in Supp(f)$ such that $s \leq' v <' u$. We will show that $f(u) \in A$. Let

$$f_u(x) = \begin{cases} f(x) & \text{if } x <' u \\ 0 & \text{if } u \leq' x. \end{cases}$$

Then $f_u \in [[A^{S,\leq'}]]$ by hypothesis. Let $g = f - f_u$. Then $pg = p(f - f_u) = pf - pf_u \in [[A^{S,\leq'}]]$. It suffices to show that $g^p \in [[A^{S,\leq'}]]$; for then, by replacing f with g in the above argument, we have $f(u) \in A$ since $\mathcal{O}'(g) = u$, and the proof concludes by transfinite induction.

Since $f^p, pf \in [[A^{S,\leq'}]]$, applying Lemma 2.1, we have $pf^i \in [[A^{S,\leq'}]]$ for all $1 \leq i \leq p-1$. Moreover, $g^p = (f - f_u)^p = \sum_{i=0}^{p} \binom{p}{i} f^i (-f_u)^{p-i}$. Since $\binom{p}{i}$ is an integral multiple of p for $1 \leq i < p-1$ and $f^p \in [[A^{S,\leq'}]]$, we conclude that $g^p \in [[A^{S,\leq'}]]$.

Thus for every $u \in Supp(f)$, $f(u) \in A$, and so $f \in [[A^{S,\leq}]]$.

(2) By (1), $[[({}^{*}_{B}A)^{S,\leq}]]$ is weakly normal in $[[B^{S,\leq}]]$. Since $[[A^{S,\leq}]] \subseteq [[({}^{*}_{B}A)^{S,\leq}]]$, we have ${}^{*}_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \subseteq [[({}^{*}_{B}A)^{S,\leq}]]$, by the following fact that the weak normalization of a commutative ring R in a given extension commutative ring T is the smallest ring D such that $R \subseteq D \subseteq T$ and D is weakly normal in T [3, Lemma

1(iv)]. To prove "Moreover" part, let $b_1, \cdots, b_n$ be the generators of ${}^*_BA$ as an A-module. Then $[[({}^*_BA)^{S,\leq}]] = [[(\sum_{i=1}^n Ab_i)^{S,\leq}]] = \sum_{i=1}^n [[A^{S,\leq}]]b_i \subseteq {}^*_{[[B^{S,\leq}]]}[[A^{S,\leq}]]$. □

Corollary 2.3. *Let* $(S_1, \leq_1), \cdots, (S_m, \leq_m)$ *be torsion-free cancellative ordered monoids. Denote by* (lex $\leq_i$) *and* (revlex $\leq_i$) *the lexicographic order, the reverse lexicographic order, respectively, on the monoid* $S_1 \times \cdots \times S_m$*. Let* $A \subseteq B$ *be rings such that* A *is weakly normal in* B*. Then* $[[A^{S_1\times\cdots\times S_m,(\text{lex}\leq_i)}]]$ *and* $[[A^{S_1\times\cdots\times S_m,(\text{revlex}\leq_i)}]]$ *are weakly normal in*

$$[[B^{S_1\times\cdots\times S_m,(\text{lex}\leq_i)}]] \quad \textit{and} \quad [[B^{S_1\times\cdots\times S_m,(\text{revlex}\leq_i)}]],$$

respectively.

Let S be a torsion-free cancellative monoid with quotient group G. Denote $S^c = \{g \in G \mid \text{there exists } s \in S \text{ such that } s + ng \in S \text{ for all } n \geq 1\}$, called the *complete integral closure* of S. We say that S is *completely integrally closed* if $S^c = S$. For an integral domain R, let R' denote the integral closure of R in its quotient field.

Corollary 2.4. *Let* A *be an integral domain. Let* $(S, \leq)$ *be a torsion-free cancellative completely integrally closed subtotally ordered monoid. Then* ${}^*[[A^{S,\leq}]] \subseteq [[({}^*A)^{S,\leq}]]$*. Moreover, if* A *is weakly normal, then* $[[A^{S,\leq}]]$ *is also weakly normal.*

Proof. Let K be the quotient field of A. Then it follows from [15, (5.2)] or [6, Theorem 2.5] that $[[K^{S,\leq}]]$ is completely integrally closed. Thus we have $[[A^{S,\leq}]]' \subseteq [[K^{S,\leq}]]$. By [3, Lemma 1(ii)], ${}^*[[A^{S,\leq}]] = {}^*_{[[A^{S,\leq}]]'}[[A^{S,\leq}]] \subseteq {}^*_{[[K^{S,\leq}]]}[[A^{S,\leq}]]$. Thus the assertions follow from Theorem 2.2. □

As mentioned in [5], there are at least three distinct rings of power series in infinitely many indeterminates $\{X_\lambda\}_{\lambda\in\Lambda}$ over a ring A in the literature. That is,

- $A[[\{X_\lambda\}]]_1 := \varinjlim_{F\subset\Lambda} A[[\{X_\lambda\}_{\lambda\in F}]]$, where F is a finite subset of Λ.
- $A[[\{X_\lambda\}]]_2 :=$ the completion of $A[\{X_\lambda\}_{\lambda\in\Lambda}]$ for the $(\{X_\lambda\}_{\lambda\in\Lambda})$-adic topology.
- $A[[\{X_\lambda\}_{\lambda\in\Lambda}]]_3 :=$ the set of all functions $f : \oplus_\Lambda\mathbb{N} \to A$ with pointwise addition and the convolution $*$, which is called the *full* ring of power series.

Note from [5, p. 543] that $A[[\{X_\lambda\}]]_1 \subseteq A[[\{X_\lambda\}]]_2 \subseteq A[[\{X_\lambda\}]]_3$ within isomorphism, and that each of these containments is proper (if and only if Λ is infinite).

The following observation is due to [5]: Let $S = \oplus_\Lambda\mathbb{N}$, where the indexing set Λ has infinite cardinality. For $s = (n_\lambda)_{\lambda\in\Lambda} \in S$, we define $\sigma(s) = \sum_\lambda n_\lambda$. Given a well-ordering on the set Λ, we well-order the set S as follows: If $s = (m_\lambda)$ and $t = (n_\lambda)$ are distinct elements of S, then $s <_\sigma t$ if $\sigma(s) < \sigma(t)$ or if $\sigma(s) = \sigma(t)$ and $m_\lambda < n_\lambda$ for the first $\lambda \in \Lambda$ such that $m_\lambda \neq n_\lambda$. Then the order $<_\sigma$ is compatible with the semigroup operation. Also $(S, <_\sigma)$ is a torsion-free cancellative strictly totally ordered monoid. Let A be an integral domain. Then $A[[\{X_\lambda\}_{\lambda\in\Lambda}]]_3 = [[A^{S,\leq_\sigma}]]$

In [5, section 2], it was shown that for any integral domain A, $A[[\{X_\lambda\}]]_3 \cap K_i = A[[\{X_\lambda\}]]_i$, where K_i denotes the quotient field of $A[[\{X_\lambda\}]]_i$.

Corollary 2.5. *Let $A \subseteq B$ be an extension of integral domains.*

(1) *If A is weakly normal in B, then $A[[\{X_\lambda\}_{\lambda\in\Lambda}]]_i$ is weakly normal in $B[[\{X_\lambda\}_{\lambda\in\Lambda}]]_i$ for each $i = 1, 2, 3$.*

(2) *If A is weakly normal, then $A[[\{X_\lambda\}_{\lambda\in\Lambda}]]_i$ is weakly normal for each $i = 1, 2, 3$.*

Proof. (1) The case $i = 3$ follows Theorem 2.2. Thus the cases $i = 1$ and $i = 2$ follow easily from the fact that $A[[\{X_\lambda\}]]_3 \cap K_i = A[[\{X_\lambda\}]]_i$, where K_i denotes the quotient field of $A[[\{X_\lambda\}]]_i$ and [16, Proposition 2].

(2) This follows from Corollary 2.4. $\square$

Let $(S, \leq)$ be a nonempty set S, endowed with an order relation $\leq$. We shall say that the order is *trivial* when $x \leq y$ implies $x = y$. It should be noted that if the order is trivial, then the ring $[[A^{S,\leq}]]$ of generalized power series is equal to the monoid ring $A[S]$. For, if $\leq$ is the trivial order, then $(Supp(f), \leq)$ is narrow if and only if $Supp(f)$ is finite.

Corollary 2.6. *Let S be a torsion-free cancellative monoid.*

(1) *Let $A \subseteq B$ be an extension of integral domains. Then ${}^*_{B[S]}A[S] = {}^*_{B}A[S]$.*

(2) *If $A \subseteq B$ is an extension of integral domains such that A is weakly normal in B, then $A[S]$ is weakly normal in $B[S]$.*

(3) *If S is integrally closed, then ${}^*(A[S]) = {}^*A[S]$ for each integral domain A.*

(4) *If A is weakly normal integral domain and if S is integrally closed, then $A[S]$ is weakly normal.*

Proof. (1) By [3, Lemma 1] and Theorem 2.2, ${}^*_{B[S]}A[S] = {}^*_{[[B^{S,\leq}]]}A[S] \cap B[S] \subseteq {}^*_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \cap B[S] \subseteq [[({}^*_{B}A)^{S,\leq}]] \cap B[S] = {}^*_{B}A[S]$. Thus ${}^*_{B[S]}A[S] \subseteq {}^*_{B}A[S]$. By [3, Lemma 1(ii)], ${}^*_{B}A[S] \subseteq {}^*_{B[S]}A[S]$, so (1) holds.

(2) This follows from (1).

(3) Note that if K denotes the quotient field of A, then ${}^*(A[S]) = {}^*_{K[S]}A[S]$ since $K[S]$ is integrally closed.

(4) This follows from (3). $\square$

For an ideal I of a commutative ring A, let $[[I^{S,\leq}]] := \{f \in [[A^{S,\leq}]] \mid f(s) \in I$ forevery $s \in S\}$; then $[[I^{S,\leq}]]$ is an ideal of $[[A^{S,\leq}]]$ and $[[A^{S,\leq}]]/[[I^{S,\leq}]] \cong [[(A/I)^{S,\leq}]]$ by [14, (2.2)].

The following result can be easily verified along the lines of the proof of [2, Proposition 13] by using the remark just above and Theorem 2.2.

Corollary 2.7. *Let $A \subseteq B$ be commutative rings with a common ideal I and let $(S, \leq)$ be a torsion-free cancellative ordered monoid. Then the following assertions are equivalent:*

(1) *A is weakly normal in B.*

(2) *A/I is weakly normal in B/I.*

(3) *$[[A^{S,\leq}]]$ is weakly normal in $[[B^{S,\leq}]]$.*

(4) *$[[(A/I)^{S,\leq}]]$ is weakly normal in $[[(B/I)^{S,\leq}]]$.*

3. t-closedness

In [9], the authors introduced t-closedness and investigated the links between t-closedness and quasinomality (an integral domain R is *quasinormal* if $Pic(R) \cong Pic(R[X, X^{-1}])$). We recall from [9] that an extension $A \subseteq B$ of commutative rings is said to be *t-closed (in B)* if whenever $b^2 - ab,\ b^3 - ab^2 \in A$ for $a \in A$ and $b \in B$, then $b \in A$. Note from [12, Théorème 3.3] that A is t-closed in B if and only if ${}^t_BA = A$. That A is t-closed in B implies that A is seminormal in B (i.e., whenever $b^2, b^3 \in A$ for $b \in B$, then $b \in A$). For some information about the historical development and numerous results of t-closedness, one may consult [12].

Let $A \subseteq B$ be an extension of commutative rings and $n \geq 1$. Recall from [1] that property $\mathcal{P}_n(A, B)$ holds if for each $a \in A$ and $b \in B$ such that $nab \in A$, we have $nab^2 \in A$. In [1], D. F. Anderson, D. E. Dobbs, and M. Roitman investigated property $\mathcal{P}_n(A, B)$ because of its relationship to root closedness of power series ring. Among other results, it was shown that a commutative ring A is p-injective (i.e., each principal ideal of A is an annihilator of some subset of A) if and only if property $\mathcal{P}_1(A, B)$ is satisfied for any ring extension B of A ([1, Corollary 1.15]), and that a reduced commutative ring A is von Neumann regular if and only if property $\mathcal{P}_1(A, B)$ is satisfied for any ring extension B of A ([1, Proposition 1.14]).

Let $A \subseteq B$ be an extension of commutative rings. Then property $\mathcal{P}_1(A, B)$ is satisfied if either of the following conditions is satisfied:

- B is an integral extension of A and A is seminormal in B [2, Lemma 6].
- A is an integral domain with quotient field K and $K \cap B = A$ [2, Corollary 8].

Let A be a subring of a commutative ring B such that A is an integral domain with quotient field K. Then $K \cap B = A$ if each principal ideal of A is contracted from B (i.e., $aB \cap A = aA$ for each $a \in A$), in particular, if B is a faithfully flat A-module.

Theorem 3.1([6, Theorem 2.3])**.** *Let $(S, \leq)$ be a torsion-free cancellative ordered monoid and let $A \subseteq B$ be commutative rings satisfying property $\mathcal{P}_1(A, B)$. If A is t-closed in B, then the generalized power series ring $[[A^{S,\leq}]]$ is t-closed in $[[B^{S,\leq}]]$.*

Corollary 3.2. *Let $(S, \leq)$ be a torsion-free cancellative ordered monoid and let $A \subseteq B$ be an extension of commutative rings satisfying property $\mathcal{P}_1(A, B)$. Then ${}^{t}_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \subseteq [[({}^{t}_{B}A)^{S,\leq}]]$.*

Proof. By Theorem 3.1, $[[({}^{t}_{B}A)^{S,\leq}]]$ is t-closed in $[[B^{S,\leq}]]$. Since $[[A^{S,\leq}]] \subseteq [[({}^{t}_{B}A)^{S,\leq}]]$, we have ${}^{t}_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \subseteq [[({}^{t}_{B}A)^{S,\leq}]]$, by the following fact that the t-closure ${}^{t}_{B}A$ of a commutative ring A in a given extension ring B is the smallest A-subalgebra C of B such that C is t-closed in B [11, Théorème 3.5(2)]. □

We collect in Lemma 3.3 some basic properties of t-closure, which is the analogue of [3, Lemma 1] for t-closure.

Lemma 3.3.

(1) *For any integral domains $A \subseteq B$, we have ${}^{t}_{B}({}^{t}_{B}A) = {}^{t}_{B}A$.*

(2) *For any integral domains $A \subseteq B$ and $C \subseteq D$ such that $A \subseteq C$ and $B \subseteq D$, we have ${}^{t}_{B}A \subseteq {}^{t}_{D}C$.*

(3) *For any integral domains $A \subseteq B \subseteq C$, we have ${}^{t}_{B}A = {}^{t}_{C}A \cap B$.*

(4) *The t-closure of integral domain A in a given extension domain B is the smallest ring S such that $A \subseteq S \subseteq B$ and S is t-closed in B.*

Proof. (1) This follows from the fact that ${}^{t}_{B}A$ is t-closed in B.

(2) This follows from [11, Théorème 2.5].

(3) The inclusion ${}^{t}_{B}A \subseteq {}^{t}_{C}A \cap B$ follows from (2). On the other hand, ${}^{t}_{C}A \cap B$ is an infra-integral extension of A and so ${}^{t}_{C}A \cap B$ is contained in ${}^{t}_{B}A$, which is the largest infra-integral extension of A contained in B.

(4) This is exactly [11, Théorème 3.5](2). Another proof of this is as follows: By (1), ${}^{t}_{B}A$ is t-closed in B. If S is a domain such that $A \subseteq S \subseteq B$ and S is is t-closed in B, then by (2), ${}^{t}_{B}A \subseteq {}^{t}_{B}S = S$. □

Corollary 3.4. *Let S be a torsion-free cancellative monoid and let $A \subseteq B$ be an extension of integral domains satisfying property $\mathcal{P}_1(A, B)$.*

(1) ${}^{t}_{B[S]}A[S] = {}^{t}_{B}A[S]$.

(2) *If A is t-closed in B, then $A[S]$ is t-closed in $B[S]$.*

(3) *If S is integrally closed, then ${}^{t}(A[S]) = {}^{t}A[S]$.*

(4) *If A is t-closed and if S is integrally closed, then $A[S]$ is t-closed.*

Proof. (1) By Lemma 3.3 and Theorem 3.1, ${}^{t}_{B[S]}A[S] = {}^{t}_{[[B^{S,\leq}]]}A[S] \cap B[S] \subseteq {}^{t}_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \cap B[S] \subseteq [[({}^{t}_{B}A)^{S,\leq}]] \cap B[S] = {}^{t}_{B}A[S]$. Thus ${}^{t}_{B[S]}A[S] \subseteq {}^{t}_{B}A[S]$. By Lemma 3.3 (2), ${}^{t}_{B}A[S] \subseteq {}^{t}_{B[S]}A[S]$, so (1) holds.

(2) This follows from (1).

(3) Note that if K denotes the quotient field of A, then ${}^t(A[S]) = {}^t_{K[S]}A[S]$ since $K[S]$ is integrally closed.

(4) This follows from (3). □

Another proof of (4) can be given as follows: Indeed, if we denote by G the quotient group of S, then $A[G]$ is t-closed by [8, Proposition 2]. Since S is integrally closed, $K[S]$ is integrally closed, and so is t-closed. Thus $A[S]$ is t-closed, since $A[S] = A[G] \cap K[S]$.

4. Strong t-closedness

We recall the following characterization of strong t-closedness from [13, Theorem 4.18]: For an integral extension $A \subseteq B$ of commutative rings, $A[[X]]$ is strongly t-closed in $B[[X]]$ if and only if A is strongly t-closed in B.

The following result is a generalization of [13, Theorem 4.18].

Theorem 4.1. *Let $A \subseteq B$ be an extension of rings satisfying property $\mathcal{P}_1({}^t_BA, B)$ and let S be a torsion-free cancellative ordered monoid. Then ${}^\circ_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \subseteq [[({}^\circ_BA)^{S,\leq}]]$. Hence if A is strongly t-closed in B, then $[[A^{S,\leq}]]$ is strongly t-closed in $[[B^{S,\leq}]]$.*

Proof. Note that ${}^t_BA$ is t-closed in B [11, Corollaire 3.4]. Thus by [6, Theorem 2.3], $[[({}^\circ_BA)^{S,\leq}]]$ is t-closed in $[[B^{S,\leq}]]$. Hence it follows from [11, Théorème 3.5] that ${}^t_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \subseteq [[({}_B{}^tA)^{S,\leq}]]$. Applying [13, Theorem 3.5] and Theorem 2.2, we have the following inclusions:

$$\begin{aligned}
{}^\circ_{[[B^{S,\leq}]]}[[A^{S,\leq}]] &= {}^*_{[[B^{S,\leq}]]}\left({}^t_{[[B^{S,\leq}]]}[[A^{S,\leq}]]\right) \\
&\subseteq {}^*_{[[B^{S,\leq}]]}\left([[({}^t_BA)^{S,\leq}]]\right) \\
&\subseteq [[{}^*_B({}^t_BA)^{S,\leq}]] \\
&= [[({}^\circ_BA)^{S,\leq}]]
\end{aligned}$$

Thus we have ${}^\circ_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \subseteq [[({}^\circ_BA)^{S,\leq}]]$. The rest is clear. □

We collect in Lemma 4.2 some basic properties of strong t-closure, which is the analogue of [3, Lemma 1] for strong t-closure.

Lemma 4.2.

(1) *For any integral domains $A \subseteq B$, we have ${}^\circ_B({}^\circ_BA) = {}^\circ_BA$.*

(2) *For any integral domains $A \subseteq B$ and $C \subseteq D$ such that $A \subseteq C$ and $B \subseteq D$, we have ${}^\circ_BA \subseteq {}^\circ_DC$.*

(3) *For any integral domains $A \subseteq B \subseteq C$, we have ${}^\circ_BA = {}^\circ_CA \cap B$*

(4) *The strong t-closure of integral domain A in a given extension domain B is the smallest ring S such that $A \subseteq S \subseteq B$ and S is strongly t-closed in B.*

Proof. (i) This follows from the fact that ${}^{\circ}_{B}A$ is strongly t-closed in B.

(ii) This follows from [13, Theorem 3.15].

(iii) The inclusion ${}^{\circ}_{B}A \subseteq {}^{\circ}_{C}A \cap B$ follows from (2). On the other hand, ${}^{\circ}_{C}A \cap B$ is a weakly infra-integral extension of A and so ${}^{\circ}_{C}A \cap B$ is contained in ${}^{\circ}_{B}A$, which is the largest weakly infra-integral extension of A contained in B.

(iv) This is exactly [13, Theorem 4.4]. Another proof of this is as follows: By (1), ${}^{\circ}_{B}A$ is strongly t-closed in B. If S is a domain such that $A \subseteq S \subseteq B$ and S is is strongly t-closed in B, then by (2), ${}^{\circ}_{B}A \subseteq {}^{\circ}_{B}S = S$. □

Corollary 4.3. *Let S be a torsion-free cancellative monoid and let $A \subseteq B$ be an extension of integral domains satisfying property $\mathcal{P}_1(A, B)$.*

(1) *Let $A \subseteq B$ be integral domains. Then ${}^{\circ}_{B[S]}A[S] = {}^{\circ}_{B}A[S]$.*

(2) *If $A \subseteq B$ are integral domains such that A is strongly t-closed in B, then $A[S]$ is strongly t-closed in $B[S]$.*

(3) *If S is integrally closed, then ${}^{\circ}(A[S]) = {}^{\circ}A[S]$ for each integral domain A.*

(4) *If A is strongly t-closed integral domain and if S is integrally closed, then $A[S]$ is strongly t-closed.*

Proof. (1) By Lemma 4.2 and Theorem 4.1, ${}^{\circ}_{B[S]}A[S] = {}^{\circ}_{[[B^{S,\leq}]]}A[S] \cap B[S] \subseteq {}^{\circ}_{[[B^{S,\leq}]]}[[A^{S,\leq}]] \cap B[S] \subseteq [[({}^{\circ}_{B}A)^{S,\leq}]] \cap B[S] = {}^{\circ}_{B}A[S]$. Thus ${}^{\circ}_{B[S]}A[S] \subseteq {}^{\circ}_{B}A[S]$. By Lemma 4.2 (2), ${}^{\circ}_{B}A[S] \subseteq {}^{\circ}_{B[S]}A[S]$, so (1) holds.

(2) This follows from (1).

(3) Note that if K denotes the quotient field of A, then ${}^{\circ}(A[S]) = {}^{\circ}_{K[S]}A[S]$ since $K[S]$ is integrally closed.

(4) This follows from (3). □

Corollary 4.4. *Let $(S, \leq)$ be a torsion-free cancellative ordered monoid and let $A \subseteq B$ be an extension of commutative rings. Assume that A is strongly t-closed in B. Then the generalized power series ring $[[A^{S,\leq}]]$ is strongly t-closed in $[[B^{S,\leq}]]$ if one of the following conditions is satisfied:*

(1) *A is von Neumann regular.*

(2) *A is p-injective.*

(3) *B is an integral extension of A.*

(4) *A is an integral domain with quotient field K and $K \cap B = A$.*

Corollary 4.5. *Let $A \subseteq B$ be an extension of integral domains satisfying property $\mathcal{P}_1(A, B)$. Assume that A is strongly t-closed in B. Then $A[[\{X_\lambda\}_{\lambda \in \Lambda}]]_i$ is strongly t-closed in $B[[\{X_\lambda\}_{\lambda \in \Lambda}]]_i$ for each $i = 1, 2, 3$.*

The following result can be easily verified along the lines of the proof of [2, Proposition 13] by Theorem 4.1.

Corollary 4.6. *Let $A \subseteq B$ be commutative rings with a common ideal I and let $(S, \leq)$ be a torsion-free cancellative ordered monoid. Then:*

(1) *$[[A^{S,\leq}]]$ is strongly t-closed in $[[B^{S,\leq}]]$ if and only if $[[(A/I)^{S,\leq}]]$ is strongly t-closed in $[[(B/I)^{S,\leq}]]$.*

(2) *If I is a common maximal ideal, then the following assertions are equivalent:*

(i) *A is strongly t-closed in B.*

(ii) *A/I is strongly t-closed in B/I.*

(iii) *$[[A^{S,\leq}]]$ is strongly t-closed in $[[B^{S,\leq}]]$.*

(iv) *$[[(A/I)^{S,\leq}]]$ is strongly t-closed in $[[(B/I)^{S,\leq}]]$.*

The following three results are extensions of theorems for formal power series in [12, Proposition 2.22 and Proposition 2.25], and [13, Proposition 5.7].

For $a \in A$, define $c_a \in [[A^{S,\leq}]]$ as follows;

$$c_a(s) = \begin{cases} a & \text{if } s = 0 \\ 0 & \text{if } 0 \neq s \in S. \end{cases}$$

Proposition 4.7. *Let A be a reduced ring and let $(S, \leq)$ be a torsion-free cancellative ordered monoid satisfying $s \geq 0$ for all $s \in S$. Then A is t-closed in $[[A^{S,\leq}]]$. In particular, if $[[A^{S,\leq}]]$ is t-closed, then so is A.*

Proof. May assume that the order $\leq$ is total. Let $r, x, y \in A$ and $f \in [[A^{S,\leq}]]$ such that the following relation (*) is satisfied.

$$(*) : f^2 - c_r f = c_x \qquad \text{and} \qquad f^3 - c_r f^2 = c_y.$$

Then $fc_x = c_y$, and so $xf(s) = 0$ for all $s > 0$.

We first claim that $f(0)f(s) = 0$ for all $s > 0$. Let s_1 be the least nonzero element of $Supp(f)$. Then from the relation $(*)$ we obtain that $2f(0)f(s_1) = rf(s_1)$ and $2rf(0)f(s_1) = 3f(0)^2 f(s_1)$. Thus we get $(f(0)f(s_1))^2 = 0$, and so $f(0)f(s_1) = 0$ since A is reduced. Assume that we have that $f(0)f(v) = 0$ for all $v \in Supp(f)$ such that $s_1 \leq v < u$. Considering $f(0)(f^2 - c_r f)(u) = f(0)c_x(u)$, we deduce that

$$f(0)\Big(\sum_{(z_1,z_2)\in X_u} f(z_1)f(z_2)\Big) - f(0)rf(u)) = 0.$$

By induction hypothesis, we obtain that $2f(0)^2 f(u) = rf(0)f(u)$. Thus we get $f(0)f(u) = 0$ from the relations: $rf(0) = f(0)^2 - x$ and $xf(v) = 0$. Therefore, we have $f(0)f(s) = 0$ for all $s > 0$.

Let $w > 0$ be the least element of $Supp(f)$. Then from the relations $f(0)f = f(0)^2$, $f(0)^2 - rf(0) = 0$, and $f^2 - c_r f = c_x$, we deduce that $r(f - c_{f(0)}) = (f - c_{f(0)})^2$. Evaluating $r(f - c_{f(0)}) = (f - c_{f(0)})^2$ at s such that $w \leq s < 2w$ and at $2w$, we have that $rf(s) = 0$ for all $s \in S$ such that $w \leq s < 2w$ and $f(w)^2 = rf(2w)$. Thus we get $rf(w) = 0$, and so $rf(2w) = 0$. Hence $f(w) = 0$. This contradicts the choice of $w \in Supp(f)$. Thus $f = c_{f(0)} \in A$. □

Proposition 4.8. *Let A be a Noetherian integral domain such that its integral closure is a Noetherian integral domain and let $(S, \leq)$ be a torsion-free cancellative completely integrally closed ordered monoid.*

(1) *If A is t-closed, then so is $[[A^{S,\leq}]]$.*

(2) *If A is strongly t-closed, then so is $[[A^{S,\leq}]]$.*

Proof. (1) By the hypothesis, the integral closure A' is completely integrally closed, and so is $[[(A')^{S,\leq}]]$ [6, Theorem 2.5]. Thus $[[(A')^{S,\leq}]]$ is t-closed. If A is t-closed, then A is t-closed in A'. Hence by [6, Corollary 2.4], $[[A^{S,\leq}]]$ is t-closed in $[[(A')^{S,\leq}]]$. Thus $[[A^{S,\leq}]]$ is t-closed by [12, Proposition 1.6].

(2) This follows from (1) and Corollary 2.4. □

Proposition 4.9. *Let A be a reduced ring and let $(S, \leq)$ be a torsion-free cancellative ordered monoid satisfying $s \geq 0$ for all $s \in S$. Then A is strongly t-closed in $[[A^{S,\leq}]]$.*

Proof. By Proposition 4.7, A is t-closed in $[[A^{S,\leq}]]$. Now it suffices to show that A is weakly normal in $[[A^{S,\leq}]]$. Suppose that $f \in [[A^{S,\leq}]]$ satisfies $f^p, pf \in A$ for some prime p. Let $g := f - c_{f(0)}$. Then $pf(s) = 0$ for all $s \neq 0$ and $pg = 0$. Moreover, $f^p = \sum_{i=0}^{p} \binom{p}{i} (c_{f(0)})^i g^{p-i}$ gives $f^p = (c_{f(0)})^p + g^p$ since $\binom{p}{i}$ is an integral multiple of p for $1 \leq i < p-1$. Thus $g^p \in A$. We claim that $g = 0$. Deny and let $\mathcal{O}(g) = u \neq 0$. Since $pu \neq 0$, we have $(g(u))^p = g^p(pu) = 0$, and so $g(u) = 0$, a contradiction. It follows that $f = c_{f(0)} \in A$. Thus A is weakly normal in $[[A^{S,\leq}]]$. □

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