KYUNGPOOK Math. J. 48(2008), 433-442

Real Hypersurfaces in Complex Hyperbolic Space with Commuting Ricci Tensor

U-Hang Ki
The National Academy of Sciences, Seoul 137-044, Korea
e-mail : uhangki2005@yahoo.co.kr

Young Jin Suh
Department of Mathematics, Kyungpook National University, Daegu 702-701, Korea
e-mail : yjsuh@knu.ac.kr

Abstract. In this paper we consider a real hypersurface M in complex hyperbolic space $H_n\mathbb{C}$ satisfying $S\phi A = \phi AS$, where ϕ, A and S denote the structure tensor, the shape operator and the Ricci tensor of M respectively. Moreover, we give a characterization of real hypersurfaces of type A in $H_n\mathbb{C}$ by such a commuting Ricci tensor.

0. Introduction

A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. As is well-known, a connected complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0, c = 0$ or $c < 0$.

Let M be a real hypersurface in $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kaehler structure J and the Kaehlerian metric G of $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ holds on M, where A denotes the shape operator of M in $M_n(c)$ and $\alpha = \eta(A\xi)$. A real hypersurface is said to be a *Hopf hypersurface* if the structure vector field ξ of M is principal. For examples of such kind of Hopf hypersurfaces in $P_n\mathbb{C}$ we give some homogeneous real hypersurfaces which are represented as orbits under certain subgroup of the projective unitary group $PU(n+1)$ ([8]).

Berndt [1] showed that all real hypersurfaces with constant principal curvature of a complex hyperbolic space $H_n\mathbb{C}$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in

Received October 11, 2007.
2000 Mathematics Subject Classification: 53C40, 53C15.
Key words and phrases: real hypersurface, Ricci tensor, Hopf hypersurface.
The second author was supported by grant Proj. No. R17-2008-001-01001 from Korea Science & Engineering Foundation and by grant Proj. No. KRF-2007-313-C00058 from Korea Research Foundation

$H_n\mathbb{C}$ they said to be of type (A_0), (A_1), (A_2) and (B). He proved the following :

Theorem B ([1]). Let M be a real hypersurface in $H_n\mathbb{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following :
(A_0) a self-tube, that is, a horosphere,
(A_1) a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,
(A_2) a tube over a totally geodesic $H_k\mathbb{C}$ $(1 \leq k \leq n-2)$,
(B) a tube over a totally real hyperbolic space $H_n\mathbb{R}$.

On the other hand, we remark that every homogeneous real hypersurface in $P_n\mathbb{C}$ was proved a Hopf hypersurface (cf. [2], [8]). However, in $H_n\mathbb{C}$ there exists some kinds of homogeneous real hypersurfaces, called ruled real hypersurfaces, which are not Hopf hypersurfaces (see [6]).

Let M be a real hypersurface of type (A_0), (A_1) or (A_2) in a complex hyperbolic space $H_n\mathbb{C}$. Now, hereafter unless otherwise stated, such hypersurfaces are said to be of *type* A for our convenience sake. Now we introduce a theorem due to Montiel and Romero [7] as follows:

Theorem MR ([7]). *If the shape operator A and the structure operator ϕ commute to each other, then a real hypersurface of a complex hyperbolic space $H_n\mathbb{C}$ is locally congruent to be of type* A.

Now let us denote by S the Ricci tensor of M in a complex space form $M_n(c)$. Then in a paper due to Kwon and the second author [3], they considered a real hypersurface M in a complex space form $M_n(c)$ with $\mathcal{L}_\xi S = \nabla_\xi S$, where $\mathcal{L}_\xi$ and ∇_ξ respectively denotes the Lie derivative and the covariant derivative along the direction of the structure vector ξ of M. Then it was proved that $\mathcal{L}_\xi S = \nabla_\xi S$ is equivalent to the condition $S\phi A = \phi AS$.

In such a case we say that M has *commuting Ricci tensor*. That is, the Ricci tensor S of M in $M_n(c)$ commutes with the tensor ϕA.

Now let us consider a real hypersurface M in $M_n(c)$ with $S\phi A - \phi AS = 0$. Then we have (see [5])

$$\|S\phi - \phi S\|^2 + \frac{3}{2}c\|\phi A\xi\|^2 = 0.$$

From this naturally M becomes a Hopf hypersurface if $c > 0$. In the case where $c < 0$, by using the method of $A^2\xi \equiv 0$ (mod $\xi, A\xi$), Kwon and the second author ([3]) proved the following :

Theorem KS ([3]). *Let M be a real hypersurface in $H_n\mathbb{C}$, $n \geq 3$, with commuting Ricci tensor. If the structure vector field ξ is principal, then M is locally congruent to of type* A.

Then we want to make a generalization of Theorem KS without the assumption that the structure vector field ξ is principal. In this paper we have introduced a certain vector U defined by $U = \nabla_\xi \xi$ and have applied such a vector to the expression of $A^2\xi \equiv 0$ (mod $\xi, A\xi$), and finally proved that the structure vector ξ is principal. Namely, we prove the following

Theorem. *Let M be a real hypersurface in a complex hyperbolic space $H_n\mathbb{C}$, $n \geq 3$, with commuting Ricci tensor. Then M becomes a Hopf hypersurface. Further, M is locally congruent to one of the following spaces :*
(A_0) *a self-tube, that is, a horosphere,*
(A_1) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*
(A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ $(1 \leq k \leq n-2)$.*

1. Preliminaries

Let M be a real hypersurface immersed in a complex space form $(M_n(c), G)$ with almost complex structure J of constant holomorphic sectional curvature c, and let C be a unit normal vector field on M. The Riemannian connection $\tilde{\nabla}$ in $M_n(c)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M :

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)C, \tag{1.1}$$

$$\tilde{\nabla}_X C = -AX, \tag{1.2}$$

where g denotes the Riemannian metric on M induced from that G of $M_n(c)$ and A is the shape operator of M in $M_n(c)$. A characteristic vector X of the shape operator of A is called a principal curvature vector. Also an eigenvalue λ of A is called a principal curvature. It is known that M has an almost contact metric structure induced from the almost complex structure J on $M_n(c)$, that is, we define a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, C)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \phi\xi = 0. \tag{1.3}$$

From (1.1) we see that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X \xi = \phi AX. \tag{1.5}$$

Since the ambient space is of constant holomorphic sectional curvature c, equations of the Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X,Y)Z = &\frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{1.7}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M. We shall denote the Ricci tensor of type (1,1) by S. Then it follows from (1.6) that

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X, \tag{1.8}$$

where $h = \text{trace } A$.

To write our formulas in convention forms, we denote $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, $\gamma = \eta(A^3\xi)$, $\delta = \eta(A^4\xi)$, $\mu^2 = \beta - \alpha^2$ and ∇f by the gradient vector field of a function f on M. In the following, we use the same terminology and notation as above unless otherwise stated.

If we put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector field ξ. Then using (1.3) and (1.5), we see that

$$\phi U = -A\xi + \alpha\xi, \tag{1.9}$$

which shows that $g(U, U) = \beta - \alpha^2$. By the definition of U, (1.3) and (1.5) it is verify that

$$g(\nabla_X \xi, U) = g(A^2\xi, X) - \alpha g(A\xi, X). \tag{1.10}$$

Now, differentiating (1.9) covariantly along M and using (1.4) and (1.5), we find

$$\begin{aligned} &\eta(X)g(AU + \nabla\alpha, Y) + g(\phi X, \nabla_Y U) \\ &\quad = g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y), \end{aligned} \tag{1.11}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \tag{1.12}$$

because of (1.7). From (1.11) we also have

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, \tag{1.13}$$

where we have used (1.3), (1.5) and (1.10).

We put

$$A\xi = \alpha\xi + \mu W, \tag{1.14}$$

where W is a unit vector field orthogonal to ξ. Then from (1.9) it is seen that $U = \mu\phi W$ and hence $g(U, U) = \mu^2$, and W is also orthogonal to U. Thus, we see, making use of (1.5), that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{1.15}$$

2. Real hypersurfaces satisfying $A^2\xi \equiv 0 \pmod{\xi, A\xi}$

Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. If it satisfies $A^2\xi \equiv 0 \pmod{\xi, A\xi}$. So we can put

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi \tag{2.1}$$

for a certain scalar ρ.

Hereafter, unless otherwise stated, let us assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector field and we put $\Omega = \{p \in M | \mu(p) \neq 0\}$. Then Ω is an open subset of M, and from now on we discuss our arguments on Ω.

From (1.14) and (2.1), we see that

$$AW = \mu\xi + (\rho - \alpha)W \tag{2.2}$$

and hence

$$A^2W = \rho AW + (\beta - \rho\alpha)W \tag{2.3}$$

because $\mu \neq 0$.

Now, differentiating (2.2) covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \tag{2.4}$$

By taking an inner product with W in the last equation, we obtain

$$g((\nabla_X A)W, W) = -2g(AX, U) + X\rho - X\alpha \tag{2.5}$$

since W is a unit vector field orthogonal to ξ. We also have by applying ξ to (2.4)

$$\mu g((\nabla_X A)W, \xi) = (\rho - 2\alpha)g(AU, X) + \mu(X\mu), \tag{2.6}$$

where we have used (1.15), which together with the Codazzi equation (1.7) gives

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - \frac{c}{2}U + \mu\nabla\mu, \tag{2.7}$$

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu. \tag{2.8}$$

Replacing X by ξ in (2.4) and taking account of (2.8), we find

$$\begin{aligned}(\rho - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu + \mu\{A\nabla_\xi W - (\rho - \alpha)\nabla_\xi W\} \\ = \mu(\xi\mu)\xi + \mu^2U + \mu(\xi\rho - \xi\alpha)W.\end{aligned} \tag{2.9}$$

On the other hand, from $\phi U = -\mu W$ we have

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

Replacing X by ξ in this and using (1.9) and (1.13), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \tag{2.10}$$

which implies

$$W\alpha = \xi\mu. \tag{2.11}$$

From the last three equations, it follows that

$$\begin{aligned} &3A^2U - 2\rho AU + A\nabla\alpha + \frac{1}{2}\nabla\beta - \rho\nabla\alpha + (\alpha\rho - \beta - \frac{c}{4})U \\ &= 2\mu(W\alpha)\xi + \mu(\xi\rho)W - (\rho - 2\alpha)(\xi\alpha)\xi, \end{aligned} \tag{2.12}$$

which enables us to obtain

$$\xi\beta = 2\alpha(\xi\alpha) + 2\mu(W\alpha). \tag{2.13}$$

Differentiating (2.1) covariantly and making use of (1.5), we get

$$\begin{aligned} &(\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - \rho A\phi AX \\ &= (X\rho)A\xi + \rho(\nabla_X A)\xi + X(\beta - \rho\alpha)\xi + (\beta - \rho\alpha)\phi AX, \end{aligned} \tag{2.14}$$

which together with (1.7) implies that

$$\begin{aligned} &\frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y)\} + \frac{c}{2}(\rho - \alpha)g(\phi Y, X) - g(A^2\phi AX, Y) \\ &+ g(A^2\phi AY, X) + 2\rho g(\phi AX, AY) - (\beta - \rho\alpha)\{g(\phi AY, X) - g(\phi AX, Y)\} \\ &= g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Y\rho)g(A\xi, X) - (X\rho)g(A\xi, Y) \\ &\quad + Y(\beta - \rho\alpha)\eta(X) - X(\beta - \rho\alpha)\eta(Y), \end{aligned} \tag{2.15}$$

where we have defined a 1-form u by $u(X) = g(U, X)$ for any vector field X. If we replace X by μW to the both sides of (2.15) and take account of (1.12), (2.2), (2.3), (2.6) and (2.7), we obtain

$$\begin{aligned} &(3\alpha - 2\rho)A^2U + 2(\rho^2 + \beta - 2\rho\alpha + \frac{c}{4})AU + (\rho - \alpha)(\beta - \rho\alpha - \frac{c}{2})U \\ &= \mu A\nabla\mu + (\alpha\rho - \beta)\nabla\alpha - \frac{1}{2}(\rho - \alpha)\nabla\beta + \mu^2\nabla\rho \\ &\qquad - \mu(W\rho)A\xi - \mu W(\beta - \rho\alpha)\xi. \end{aligned} \tag{2.16}$$

Using (1.14), we can write the equation (2.14) as

$$\begin{aligned} &A(\nabla_X A)\xi + (\alpha - \rho)(\nabla_X A)\xi + \mu(\nabla_X A)W \\ &= (X\rho)A\xi + X(\beta - \rho\alpha)\xi + (\beta - \rho\alpha)\phi AX + \rho A\phi AX - A^2\phi AX. \end{aligned}$$

Thus, from this, by replacing X by $\alpha\xi + \mu W$ and making use of (1.5), (1.12), (1.14) and (2.5) $\sim$ (2.7), we find

$$\begin{aligned}(2.17)\qquad & 2\rho A^2U + 2(\alpha\rho - \beta - \rho^2 - \frac{c}{4})AU + (\rho^2\alpha - \rho\beta + \frac{c}{2}\rho - \frac{3}{4}c\alpha)U \\ & = g(A\xi, \nabla\rho)A\xi - \frac{1}{2}A\nabla\beta + \frac{1}{2}(\rho - 2\alpha)\nabla\beta + \beta\nabla\alpha \\ & - \mu^2\nabla\rho + g(A\xi, \nabla(\beta - \rho\alpha))\xi.\end{aligned}$$

3. Real hypersurfaces in $H_n\mathbb{C}$ with commuting Ricci tensor

Let us consider a real hypersurface M in complex hyperbolic space $H_n\mathbb{C}$ with negative constant holomorphic sectional curvature $c < 0$. If M satisfies $S\phi A - \phi AS = 0$, we say that M has *commuting Ricci tensor*. In this section we consider a real hypersurface M in $H_n\mathbb{C}$ with commuting Ricci tensor. Then by (1.8) we have

$$(3.1)\qquad h(A\phi A - \phi A^2) + \phi A^3 - A^2\phi A + \frac{3}{4}c\eta \otimes U = 0,$$

where we have used (1.5). Taking the transpose of this, we find

$$(3.2)\qquad h(A\phi A - A^2\phi) + A^3\phi - A\phi A^2 + \frac{3}{4}cU \otimes \xi = 0.$$

Transforming (3.1) by A to the left, and (3.2) to the right respectively, and combining to these two equations, we obtain

$$\eta \otimes AU + \xi \otimes \eta(A\phi A) = 0,$$

which implies

$$(3.3)\qquad AU = 0.$$

If we take an inner product (3.2) with ξ and make use of (3.3), then we have

$$(3.4)\qquad A\phi A^2\xi = 0.$$

Taking an inner product (3.1) with ξ and using (3.3) and the last equation, we also find

$$\phi(A^3\xi - hA^2\xi) + \frac{3}{4}cU = 0.$$

If we apply this by ϕ and take account of (1.9), then we get

$$A^3\xi - hA^2\xi = (\gamma - \beta h + \frac{3}{4}c\alpha)\xi - \frac{3}{4}cA\xi,$$

which tells us that

$$(3.5)\qquad A^4\xi - hA^3\xi = (\gamma - \beta h + \frac{3}{4}c\alpha)A\xi - \frac{3}{4}cA^2\xi.$$

Next, applying (3.1) by $A\xi$ and making use of (3.3) and (3.4), we have

$$\phi(A^4\xi - hA^3\xi) = \frac{3}{4}c\alpha U,$$

which implies that

$$A^4\xi - hA^3\xi = -\frac{3}{4}c\alpha(A\xi - \alpha\xi) + (\delta - h\gamma)\xi.$$

This, together with (3.5) implies that

$$\frac{3}{4}cA^2\xi = (\gamma - \beta h + \frac{3}{2}c\alpha)A\xi + (h\gamma - \delta - \frac{3}{4}c\alpha^2)\xi. \tag{3.6}$$

Thus, it follows that

$$\frac{3}{4}c(\beta - \alpha^2) = \alpha(\gamma - \beta h) + h\gamma - \delta. \tag{3.7}$$

Therefore (3.6) is reformed as

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi, \tag{3.8}$$

where the function ρ is defined in such a way that

$$\frac{3}{4}c\rho = \gamma - \beta h + \frac{3}{2}c\alpha. \tag{3.9}$$

Accordingly the formulas stated in Section 2 are established.

Now, we are going to prove our Main Theorem.

Transforming (2.12) by U and using (3.3), we find

$$\frac{1}{2}U\beta - \rho(U\alpha) = (\beta - \rho\alpha + \frac{c}{4})\mu^2. \tag{3.10}$$

Similarly, from (2.16) and (2.17) we have respectively

$$(\alpha\rho - \beta)U\alpha - \frac{1}{2}(\rho - \alpha)U\beta + (\beta - \alpha^2)U\rho = (\rho - \alpha)(\beta - \rho\alpha - \frac{c}{2})\mu^2, \tag{3.11}$$

$$\frac{1}{2}(\rho - 2\alpha)U\beta + \beta(U\alpha) - (\beta - \alpha^2)U\rho = (\rho^2\alpha - \rho\beta + \frac{c}{2}\rho - \frac{3}{4}c\alpha)\mu^2. \tag{3.12}$$

Differentiating (3.3) covariantly along Ω, we find

$$(\nabla_X A)U + A\nabla_X U = 0.$$

If we put $X = \xi$ in this and take account of (1.13) and (3.3), we obtain

$$(\nabla_\xi A)U + \alpha A^2\xi - \beta A\xi + \alpha A\phi\nabla\alpha = 0,$$

which shows that

$$\phi(\nabla_\xi A)U = (\beta - \rho\alpha)U - \alpha\phi A\phi\nabla\alpha,$$

where we have used (3.8). From this and (1.7), it follows that

$$\phi(\nabla_U A)\xi = (\beta - \rho\alpha + \frac{c}{4})U - \alpha\phi A\phi\nabla\alpha. \tag{3.13}$$

On the other hand, from $\nabla_X\xi = \phi AX$ and $U = \nabla_\xi\xi$, we see that

$$\nabla_X U = \phi(\nabla_X A)\xi + \alpha AX - g(A^2X, \xi)\xi + \phi A\phi AX$$

by virtue of (1.4). Replacing X by U in this and making use of (3.3), we obtain $\nabla_U U = \phi(\nabla_U A)\xi$, which together with (3.13) implies that

$$\nabla_U U = (\beta - \rho\alpha + \frac{c}{4})U - \alpha\phi A\phi\nabla\alpha.$$

If we take an inner product with U to the last equation and use (1.9), (3.3) and $\mu^2 = \beta - \alpha^2$, then we get

$$\frac{1}{2}U\beta - \alpha(U\alpha) = (\beta - \rho\alpha + \frac{c}{4})\mu^2. \tag{3.14}$$

This, together with (3.10), implies that

$$(\rho - \alpha)U\alpha = -2(\beta - \rho\alpha + \frac{c}{4})\mu^2. \tag{3.15}$$

Combining (3.12) to (3.14), we find

$$U\rho = U\alpha - \frac{c}{4}(\rho - \alpha). \tag{3.16}$$

Substituting (3.14), (3.15) and (3.16) into (3.12), we obtain $\rho - \alpha = 0$ and hence $\beta - \alpha^2 + \frac{c}{4} = 0$ by virtue of (3.15). Thus, (3.8) becomes $A^2\xi = \alpha A\xi - \frac{c}{4}\xi$, which tells us that $\gamma = \alpha^3 - \frac{c}{2}\alpha$. Then it follows

$$\delta = \alpha^4 - \frac{3}{4}c\alpha^2 + (\frac{c}{4})^2.$$

Using above facts, (3.7) turns out to be

$$\alpha h = \alpha^2 + \frac{c}{2}. \tag{3.17}$$

Since $\rho = \alpha$, (3.9) becomes $\gamma - \beta h = -\frac{3}{4}c\alpha$, which implies that $\alpha^3 - h(\alpha^2 - \frac{c}{4}) = -\frac{c}{4}\alpha$. This, together with (3.17), yields $c = 0$, a contradiction. Hence $\Omega = \emptyset$. Thus, the subset Ω (of M) on which $A\xi - \eta(A\xi)\xi \neq 0$ is an empty set, namely in $H_n\mathbb{C}$ every real hypersurface satisfying $S\phi A = \phi AS$ is a Hopf hypersurface. Then, by Theorem KS we complete the proof of our Main Theorem.

References

[1] J. Berndt, *Real hypersurfaces with constant principal curvatures in a complex hyperbolic space*, J. Reine Agnew. Math. **395**(1989), 132-141.

[2] T. E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269**(1982), 481-499.

[3] J.-H. Kwon and Y.J. Suh, *A new characterization of homogeneous real hypersurfaces in complex space forms*, Nihonkai Math. J. **9**(1998), 77-90.

[4] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296**(1986), 137-149.

[5] M. Kimura and S. Maeda, *Lie derivatives on real hypersurfaces in a complex projective space*, Czechoslovak Math. J. **45**(1995), 135-148.

[6] M. Lohnherr and H. Reckziegel, *On ruled real hypersurfaces in complex space forms*, Geom. Dedicata **74**(1999), 267-286.

[7] S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata **20**(1986), 245-261.

[8] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10**(1973), 495-506.