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Norm and Numerical Radius of 2-homogeneous Polynomials on the Real Space l_p^2, $(1 < p < \infty)$

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Abstract. In this note, we present some inequalities for the norm and numerical radius of 2-homogeneous polynomials from the 2-dimensional real space l_p^2, $(1 < p < \infty)$ to itself in terms of their coefficients. We also give an upper bound for $n^{(k)}(l_p^2), (k = 2, 3, \cdots)$.

1. Introduction

In this paper, we consider only real Banach spaces. Given a Banach space E we write B_E for its unit ball and S_E for its unit sphere. The dual space of E is denoted by E^* and let

$$\Pi(E) = \{(x, x^*) \ : \ x \in S_E, \ x^* \in S_{E^*}, x^*(x) = 1\}.$$

A mapping $P : E \to E$ is called a (continuous) k-homogeneous polynomial if there is a (continuous) k-linear mapping $A : E \times \cdots \times E \to E$ such that $P(x) = A(x, \cdots, x)$ for every $x \in E$. Let $\mathcal{P}(^kE : E)$ denote the Banach space of all k-homogeneous polynomials from E to itself, endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. We refer to the book [5] by Dineen for background on polynomials. It is natural to generalize the concepts of numerical range and numerical radius of linear operators to homogeneous polynomials. The *numerical range* of $P \in \mathcal{P}(^kE : E)$ is defined to be the set of scalars

$$V(P) := \{x^*(Px) : (x, x^*) \in \Pi(E)\}$$

and the *numerical radius* of P is defined by

$$v(P) := \sup \ \{|\lambda| : \lambda \in V(P)\}.$$

Clearly, $v(\cdot)$ is a semi-norm on $\mathcal{P}(^kE : E)$, and $v(P) \le \|P\|$ for every $P \in \mathcal{P}(^kE : E)$. It was shown by B. Glickfeld [9] (and essentially by H. Bohnenblust and S. Karlin [3]) that if E is a complex Banach space, then $\frac{1}{e}\|T\| \le v(T)$ for every $T \in \mathcal{B}(E) =$

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$\mathcal{P}(^1E:E)$, so that for complex spaces $v(\cdot)$ is always a norm and it is equivalent to the operator norm $\|\cdot\|$. An extension of Glickfeld's result for linear operators was given by L. Harris ([11], Theorem 1): for complex Banach spaces, the numerical radius is always an equivalent norm in the space of k-homogeneous polynomials. In real case, it was known that Harris' result is false.

As in the linear case, the author *et al.* [4] introduced the concept of the *polynomial numerical index of order* k of E to be the constant

$$\begin{aligned} n^{(k)}(E) &:= \inf\{v(P) : P \in \mathcal{P}(^kE:E), \|P\| = 1\} \\ &= \sup\{M \geq 0 : \|P\| \leq \frac{1}{M} v(P) \text{ for all } P \in \mathcal{P}(^kE:E)\}. \end{aligned}$$

Of course, $n^{(1)}(E)$ coincides with the usual numerical index of the space E. Note that $0 \leq n^{(k)}(E) \leq 1$, and $n^{(k)}(E) > 0$ if and only if $v(\cdot)$ is a norm on $\mathcal{P}(^kE:E)$ equivalent to the usual norm. It is obvious that if E_1, E_2 are isometrically isomorphic Banach spaces, then $n^{(k)}(E_1) = n^{(k)}(E_2)$.

The concept of the numerical index was first suggested by G. Lumer in 1968 (see [14]). At that time, it was known that if E is a complex Hilbert space (with $dimE > 1$), then $n^{(1)}(E) = 1/2$ and if it is real, then $n^{(1)}(E) = 0$. J. Duncan, C. McGregor, J. Pryce, and A. White [6] determined the range of values of the numerical index as follows:

$$\{n^{(1)}(E) : E \text{ real Banach space}\} = [0, 1],$$
$$\{n^{(1)}(E) : E \text{ complex Banach space}\} = [e^{-1}, 1].$$

C. Finet, M. Martin, and R. Paya [8] studied the values of the numerical index from the isomorphic point of view. G. Lopez, M. Martin, and R. Paya [15] studied some real Banach spaces with numerical index 1. In fact, they proved that an infinite dimensional real Banach space with numerical index 1 satisfying the Radon-Nikodym property contains l_1. M. Martin and R. Paya [17] proved that if K is a compact Hausdorff space and μ is a positive measure, then the Banach spaces $C(K, X)$ and $L_1(\mu, X)$ have the same numerical index as the Banach space X. Recently, Ed-dari [7] gave a partial answer to the problem of computing the numerical index of l_p-space $(1 < p < \infty)$. In fact, it was shown: Let $1 < p < \infty$. Then $n^{(1)}(l_p) = \lim_{m\to\infty} n^{(1)}(l_p^m)$ when l_p is real or complex. He also proved that for any positive measure μ, $n^{(1)}(L_p(\mu)) \geq n^{(1)}(l_p)$. Recently, the author *et al.* [4] introduced and studied the concept of the polynomial numerical index of order k of a Banach space, generalizing to k-homogeneous polynomials the classical numerical index. In fact, they proved $n^{(k)}(C(K)) = 1$ for every $k \in \mathbb{N}$ and

$$n^{(k)}(E) \leq n^{(k-1)}(E) \leq \frac{k^{k+\frac{1}{k-1}}}{(k-1)^{k-1}}\, n^{(k)}(E)$$

for every Banach space E. It was shown $n^{(k)}(E^{**}) \leq n^{(k)}(E)$ and that $k^{k/1-k}$ is a lower bound for $n^{(k)}(E)$ for every Banach space E and it is sharp. Very recently, the author *et al.* [13] compute that $\frac{1}{2} = n^{(2)}(c_0) = n^{(2)}(l_1) = n^{(2)}(l_\infty)$.

For general information and background on numerical ranges, we refer to the books by F. Bonsall and J. Duncan ([1], [2]). Further developments in the Hilbert space may be found in [10]. For recent progress and open questions on the numerical index of Banach spaces, we refer to the survey articles by M. Martin, and by V. Kadet, M. Martin and R. Paya ([16], [12]).

In this paper, we present some inequalities for the norm and numerical radius of 2-homogeneous polynomials from the 2-dimensional real space l_p^2, $(1 < p < \infty)$ to itself in terms of their coefficients. We also give an upper bound for $n^{(k)}(l_p^2), (k = 2, 3, \cdots)$.

2. Main results

Proposition 2.1. *Let* $1 < p < \infty, 1 = \dfrac{1}{p} + \dfrac{1}{q}$ *and* $P(x, y) = (a_1x^2 + b_1y^2 + c_1xy, a_2x^2 + b_2y^2 + c_2xy) \in \mathcal{P}(^2l_p^2 : l_p^2)$. *Then*

$$\|P\| \leq [(|a_1| + |a_2| + \frac{1}{2}(|c_1| + |c_2|))^q + (|b_1| + |b_2| + \frac{1}{2}(|c_1| + |c_2|))^q]^{1/q}.$$

Proof. It follows that

$$\begin{aligned}
\|P\| &= \sup_{(x,y)\in S_{l_p^2}} \|(a_1x^2 + b_1y^2 + c_1xy, a_2x^2 + b_2y^2 + c_2xy)\|_p \\
&= \sup_{(x,y)\in S_{l_p^2}} (|a_1x^2 + b_1y^2 + c_1xy|^p + |a_2x^2 + b_2y^2 + c_2xy|^p)^{1/p} \\
&\leq \sup_{(x,y)\in S_{l_p^2}} |a_1x^2 + b_1y^2 + c_1xy| + |a_2x^2 + b_2y^2 + c_2xy| \\
&\leq \sup_{(x,y)\in S_{l_p^2}} (|a_1| + |a_2|)|x|^2 + (|b_1| + |b_2|)|y|^2 + (|c_1| + |c_2|)|x||y| \\
&\leq \sup_{(x,y)\in S_{l_p^2}} (|a_1| + |a_2|)|x|^2 + (|b_1| + |b_2|)|y|^2 + (|c_1| + |c_2|)\frac{|x|^2 + |y|^2}{2} \\
&= \sup_{(x,y)\in S_{l_p^2}} (|a_1| + |a_2| + \frac{1}{2}(|c_1| + |c_2|))|x|^2 + (|b_1| + |b_2| + \frac{1}{2}(|c_1| + |c_2|))|y|^2 \\
&\leq \sup_{(x,y)\in S_{l_p^2}} (|a_1| + |a_2| + \frac{1}{2}(|c_1| + |c_2|))|x| + (|b_1| + |b_2| + \frac{1}{2}(|c_1| + |c_2|))|y| \\
&= \|(|a_1| + |a_2| + \frac{1}{2}(|c_1| + |c_2|), |b_1| + |b_2| + \frac{1}{2}(|c_1| + |c_2|))\|_q \\
&= [(|a_1| + |a_2| + \frac{1}{2}(|c_1| + |c_2|))^q + (|b_1| + |b_2| + \frac{1}{2}(|c_1| + |c_2|))^q]^{1/q}.
\end{aligned}$$

□

Note that if $P(x,y)=(x^2,0)$ or $P(x,y)=(xy,0)$, then the equality of Proposition 2.1 holds.

Proposition 2.2. *Let* $1<p<\infty$ *and* $P(x,y)=(a_1x^2+b_1y^2+c_1xy, a_2x^2+b_2y^2+c_2xy)\in\mathcal{P}({}^2l_p^2:l_p^2)$. *Then*

$$\begin{aligned}(a)\quad v(P) &\geq \sup_{0\leq t\leq 1}\frac{1}{4(1+t^p)^{(1+\frac{1}{p})}}[(t+t^p)|c_1+c_2|-t^2(|a_1+b_2|+|a_2+b_1|)\\ &- (|a_1+b_1|+|a_2+b_2|)].\\ (b)\quad v(P) &\geq \sup_{0\leq t\leq 1}\frac{(|c_1|+|c_2|)t^p-(|a_1|+|b_1|+|a_2|+|b_2|)}{2(1+t^p)^{(1+\frac{1}{p})}}.\end{aligned}$$

Proof. Let $0\leq t\leq 1$ and $\epsilon=\pm1$. Put $y=\dfrac{e_1+\epsilon te_2}{(1+t^p)^{\frac{1}{p}}}$ and $y^*=\dfrac{e_1^*+\epsilon t^{p-1}e_2^*}{(1+t^p)^{\frac{1}{q}}}$, where $1=\dfrac{1}{p}+\dfrac{1}{q}$. It follows that

$$\begin{aligned}v(P) &\geq |y^*(P(y))|\\ &\geq \sup_{\epsilon=\pm1,0\leq t\leq 1}\frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|a_1+\epsilon c_1t+\epsilon a_2t^{p-1}+c_2t^p+\epsilon b_2t^{p+1}+b_1t^2|.\end{aligned}$$

Thus

$$v(P)\geq \sup_{0\leq t\leq 1}\frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|a_1+c_1t+a_2t^{p-1}+c_2t^p+b_2t^{p+1}+b_1t^2|$$

and

$$v(P)\geq \sup_{0\leq t\leq 1}\frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|a_1-c_1t-a_2t^{p-1}+c_2t^p-b_2t^{p+1}+b_1t^2|.$$

By triangle inequality, we have

$$v(P)\geq \sup_{0\leq t\leq 1}\frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|a_1+c_2t^p+b_1t^2| \tag{1}$$

and

$$v(P)\geq \sup_{0\leq t\leq 1}\frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|c_1t+a_2t^{p-1}+b_2t^{p+1}|. \tag{2}$$

Put $z=\dfrac{\epsilon te_1+e_2}{(1+t^p)^{\frac{1}{p}}}$ and $z^*=\dfrac{\epsilon t^{p-1}e_1^*+e_2^*}{(1+t^p)^{\frac{1}{q}}}$. It follows that

$$\begin{aligned}v(P) &\geq |z^*(P(z))|\\ &\geq \sup_{\epsilon=\pm1,0\leq t\leq 1}\frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|b_1+\epsilon c_2t+\epsilon b_2t^{p-1}+c_1t^p+\epsilon a_1t^{p+1}+a_2t^2|.\end{aligned}$$

By triangle inequality, we have

$$v(P) \geq \sup_{0\leq t\leq 1} \frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|b_1 + c_1 t^p + a_2 t^2| \tag{3}$$

and

$$v(P) \geq \sup_{0\leq t\leq 1} \frac{1}{(1+t^p)^{(1+\frac{1}{p})}}|c_2 t + b_2 t^{p-1} + a_1 t^{p+1}|. \tag{4}$$

The proof of (a): Adding (1), (2), (3), and (4), we get

$$\begin{aligned}
v(P) \quad \geq & \quad \sup_{0\leq t\leq 1} \frac{1}{4(1+t^p)^{(1+\frac{1}{p})}} \times \\
& [(t^p+t)|c_1+c_2| - (|(a_1+b_1)+(a_2+b_1)t^2| + t^{p-1}|(a_2+b_2)+(a_1+b_2)t^2|)] \\
\geq & \quad \sup_{0\leq t\leq 1} \frac{1}{4(1+t^p)^{(1+\frac{1}{p})}} \times \\
& [(t^p+t)|c_1+c_2| - (|(a_1+b_1)+(a_2+b_1)t^2| + |(a_2+b_2)+(a_1+b_2)t^2|)] \\
& \text{(because of } t^p \leq 1) \\
\geq & \quad \sup_{0\leq t\leq 1} \frac{1}{4(1+t^p)^{(1+\frac{1}{p})}} \times \\
& [(t^p+t)|c_1+c_2| - (t^2(|a_1+b_2|+|a_2+b_1|) + |a_1+b_1| + |a_2+b_2|)] \\
& \text{(by triangle inequality)}
\end{aligned}$$

getting the inequality (a).

The proof of (b): By the inequality (1), we have

$$\begin{aligned}
v(P) \quad & \geq \quad \sup_{0\leq t\leq 1} \frac{|a_1 + c_2 t^p + b_1 t^2|}{(1+t^p)^{(1+\frac{1}{p})}} \\
& \geq \quad \sup_{0\leq t\leq 1} \frac{|c_2|t^p - |a_1| - |b_1|}{(1+t^p)^{(1+\frac{1}{p})}}.
\end{aligned} \tag{5}$$

By the inequality (2), we have

$$\begin{aligned}
v(P) \quad & \geq \quad \sup_{0\leq t\leq 1} \frac{|c_1 t + a_2 t^{p-1} + b_2 t^{p+1}|}{(1+t^p)^{(1+\frac{1}{p})}} \\
& \geq \quad \sup_{0\leq t\leq 1} \frac{|c_1|t^p - |a_2| - |b_2|}{(1+t^p)^{(1+\frac{1}{p})}}.
\end{aligned} \tag{6}$$

By the inequality (3), we have

$$\begin{aligned}
v(P) \quad & \geq \quad \sup_{0\leq t\leq 1} \frac{|b_1 + c_1 t^p + a_2 t^2|}{(1+t^p)^{(1+\frac{1}{p})}} \\
& \geq \quad \sup_{0\leq t\leq 1} \frac{|c_1|t^p - |a_2| - |b_1|}{(1+t^p)^{(1+\frac{1}{p})}}.
\end{aligned} \tag{7}$$

By the inequality (4), we have

$$
\begin{aligned}
(8) \qquad v(P) &\geq \sup_{0\leq t\leq 1} \frac{|c_2 t + b_2 t^{p-1} + a_1 t^{p+1}|}{(1+t^p)^{(1+\frac{1}{p})}} \\
&\geq \sup_{0\leq t\leq 1} \frac{|c_2| t^p - |a_1| - |b_2|}{(1+t^p)^{(1+\frac{1}{p})}}.
\end{aligned}
$$

Adding (5), (6), (7) and (8), we get the inequality (b). $\square$

Proposition 2.3. *For every $k \in \mathbb{N}$ and every $1 < p < \infty$ we have*

$$n^{(k)}(l_p^2) \leq (\frac{p-1}{k+p-1})^{\frac{1}{q}} \ (\frac{k}{k+p-1})^{\frac{k}{p}},$$

where $1 = \dfrac{1}{p} + \dfrac{1}{q}$. Then $\lim_{k\to\infty} n^{(k)}(l_p^2) = 0$.

Proof. Let $P(x_1, x_2) = (x_2^k, 0)$ for $x = (x_1, x_2) \in l_p^2$. Then $P \in \mathcal{P}(^k l_p^2 : l_p^2)$ and $\|P\| = 1$. Put $f(t) = t^{p-1} \ (1-t^p)^{\frac{k}{p}}$ for $0 \leq t \leq 1$. It is easy to show that f has its maximum $(\dfrac{p-1}{k+p-1})^{\frac{1}{q}} \ (\dfrac{k}{k+p-1})^{\frac{k}{p}}$ at $t = (\dfrac{p-1}{k+p-1})^{\frac{1}{p}}$. It follows that

$$
\begin{aligned}
0 &\leq n^{(k)}(l_p^2) \leq v(P) \\
&= \sup \ \{ \ | < (y_1, y_2), P(x_1, x_2) > | \ : \ (y_1, y_2) \in S_{l_q^2}, \ (x_1, x_2) \in S_{l_p^2}, \ \sum_{i=1}^{2} x_i y_i = 1 \ \} \\
&= \max \ \{ |y_1| \ |x_2|^k \ : \ 1 = |x_1|^p + |x_2|^p = |y_1|^q + |y_2|^q = x_1 y_1 + x_2 y_2 \ \} \\
&= \max \ \{ |y_1| \ |x_2|^k \ : \ y_1 = \mathrm{sign}(x_1^p) x_1^{p-1}, \ 1 = |x_1|^p + |x_2|^p \ \} \\
&= \max_{0\leq x\leq 1} \ \{ \ x^{p-1}(1-x^p)^{\frac{k}{p}} \ \} = (\frac{p-1}{k+p-1})^{\frac{1}{q}} \ (\frac{k}{k+p-1})^{\frac{k}{p}} \\
&\leq (\frac{p-1}{k+p-1})^{\frac{1}{q}} \to 0 \text{ as } k \to \infty,
\end{aligned}
$$

which completes the proof. $\square$

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