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On Generalized Integral Operator Based on Salagean Operator

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ABSTRACT. Let $A(p)$ be the class of functions $f : z^p + \sum_{j=1}^{\infty} a_j z^{p+j}$ analytic in the open unit disc E. Let, for any integer $n > -p$, $f_{n+p-1}(z) = z^p + \sum_{j=1}^{\infty}(p+j)^{n+p-1} z^{p+j}$. We define $f^{(-1)}_{n+p-1}(z)$ by using convolution $*$ as $f_{n+p-1} * f^{-1}_{n+p-1} = \frac{z^p}{(1-z)^{n+p}}$. A function p, analytic in E with $p(0) = 1$, is in the class $P_k(\rho)$ if $\int_0^{2\pi} \left| \frac{\text{Re } p(z) - \rho}{p - \rho} \right| d\theta \le k\pi$, where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \rho \le p$. We use the class $P_k(\rho)$ to introduce a new class of multivalent analytic functions and define an integral operator $L_{n+p-1}(f) = f^{-1}_{n+p-1} * f$ for $f(z)$ belonging to this class. We derive some interesting properties of this generalized integral operator which include inclusion results and radius problems.

1. Introduction

Let $A(p)$ denote the class of functions f given by

$$f(z) = z^p + \sum_{j=1}^{\infty} a_j z^{p+j}, \qquad p \in N = \{1, 2, \cdots\}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. The Hadamard product or convolution $(f * g)$ of two functions with

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{j,1} z^{p+j} \text{ and } g(z) = z^p + \sum_{j=1}^{\infty} a_{j,2} z^{p+j}$$

is given by

$$(f * g)(z) = z^p + \sum_{j=1}^{\infty} a_{j,1} a_{j,2} z^{p+j}.$$

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Denote by $D^n : A(p) \to A(p)$ the Salagean operator defined as follows [6].

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= zf'(z) \\ D^n f(z) &= D\big(D^{n-1} f(z)\big). \end{aligned}$$

Several classes of analytic functions, defined by using this operator, have been studied by many authors [1],[2].

In the present paper, we introduce integral operator based on Salagean operator as follows.

Let $f_{n+p-1}(z) = z^p + \sum_{j=1}^{\infty} (p+j)^{n+p-1} z^{p+j}$. For any integer n greater than $-p$, let $f_{n+p-1}^{(-1)}(z)$ be defined such that

$$f_{n+p-1}(z) * f_{n+p-1}^{(-1)}(z) = \frac{z^p}{(1-z)^{p+1}}. \tag{1.1}$$

Then

$$\begin{aligned} L_{n+p-1} f(z) &= f_{n+p-1}^{-1}(z) * f(z) \\ &= \left[z^p + \sum_{j=1}^{\infty} (p+j)^{n+p-1} z^{p+j} \right]^{-1} * f(z). \end{aligned} \tag{1.2}$$

From (1.1) and (1.2) and a well known identity for the Salagean derivative, it follows that

$$z\big(L_{n+p} f(z)\big)' = L_{n+p-1} f(z). \tag{1.3}$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{p - \rho} \right| d\theta \le k\pi \tag{1.4}$$

where $z = re^{i\theta}, k \ge 2$ and $0 \le \rho < p$. For $p = 1$, this class was introduced in [3] and for $\rho = 0$. For $\rho = 0$, $k = 2$, we have the well known class P of functions with positive real part and the class $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ. Also from (1.4), we note that $p \in P_k(\rho)$ if and only if there exist $p_1, p_2 \in P_k(\rho)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

It is known [3] that the class $P_k(\rho)$ is a convex set.

Definition 1.1. Let $f \in A(p)$. Then $f \in R_k(\alpha, p, n, \rho)$ if and only if

$$\left[(1-\alpha)\frac{L_{n+p-1}f(z)}{z^p} + \alpha\frac{L_{n+p}f(z)}{z^p}\right] \in P_k(\rho)$$

for $\alpha \geq 0,\quad n > -p,\quad 0 \leq \rho < p,\quad k \geq 2$ and $z \in E$.

2. Preliminary results

Lemma 2.1. *Let* $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in P(\rho)$. *Then*

$$Re\ p(z) \geq 2\rho - 1 + \frac{2(1-\rho)}{1+|z|}.$$

This result is well known.

Lemma 2.2 ([5]). *If* $p(z)$ *is analytic in* E *with* $p(0) = 1$ *and if* λ_1 *is a complex number satisfying* $Re\ \lambda_1 \geq 0,\quad (\lambda_1 \neq 0)$, *then* $Re\{p(z)+\lambda_1 z p'(z)\} > \beta \quad (0 \leq \beta < p)$ *implies*

$$Re\ p(z) > \beta + (1-\beta)(2\gamma_1 - 1),$$

where γ_1 *is given by*

$$\gamma_1 = \int_0^1 \left(1 + t^{\mathrm{Re}\,\lambda_1}\right)^{-1} dt.$$

Lemma 2.3 ([7]). *If* $p(z)$ *is analytic in* E, $p(0) = 1$ *and* $Re\ p(z) > \frac{1}{2},\quad z \in E,$ *then for any function* F *analytic in* E, *the function* $p * F$ *takes values in the convex hull of the image* E *under* F.

3. Main results

Theorem 3.1. *Let* $f \in R_k(\alpha, p, n, \rho_1)$ *and* $g \in R_k(\alpha, p, n, \rho_2)$, *and let* $F = f * g$. *Then* $F \in R_k(\alpha, p, n, \rho_3)$ *where*

$$\rho_3 = 1 - 2(1-\rho_1)(1-\rho_2) \tag{3.1}$$

The result is sharp.

Proof. Since $f \in R_k(\alpha, p, n, \rho_1)$, it follows that

$$H(z) = \left[(1-\alpha)\frac{L_{n+p-1}f(z)}{z^p} + \alpha\frac{L_{n+p}f(z)}{z^p}\right] \in P_k(\rho_1)$$

and so using (1.3), we have

$$L_{n+p}f(z) = \frac{1}{1-\alpha} z^{\frac{-\alpha}{1-\alpha}} \int_0^z t^{\frac{\alpha}{1-\alpha}+p-1} H(t)dt. \tag{3.2}$$

Similarly,

$$L_{n+p}g(z) = \frac{1}{1-\alpha} z^{\frac{-\alpha}{1-\alpha}} \int_0^z t^{\frac{\alpha}{1-\alpha}+p-1} H^*(t)dt, \tag{3.3}$$

where $H^* \in P_k(\rho_2)$. Using (3.1) and (3.2), we have

$$L_{n+p}F(z) = \frac{1}{1-\alpha} z^{\frac{-\alpha}{1-\alpha}} \int_0^z t^{\frac{\alpha}{1-\alpha}+p-1} Q(t)dt, \tag{3.4}$$

where $Q(z) = \left(\frac{k}{4}+\frac{1}{2}\right) q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) q_2(z)$. Then

$$L_{n+p}F(z) = \frac{1}{1-\alpha} z^{\frac{-\alpha}{1-\alpha}} \int_0^z t^{\frac{\alpha}{1-\alpha}+p-1} (H * H^*)(t)dt. \tag{3.5}$$

Now

$$\begin{aligned} H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right) h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2(z) \\ H^*(z) &= \left(\frac{k}{4}+\frac{1}{2}\right) h_1^*(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2^*(z) \end{aligned} \tag{3.6}$$

where $h_i \in P_k(\rho_1)$ and $h_i^* \in P_k(\rho_2),\ i = 1, 2$. Since

$$p_i^*(z) = \frac{h_i^*(z) - \rho_2}{2(1-\rho_2)} + \frac{1}{2} \in P\left(\frac{1}{2}\right), \quad i = 1, 2,$$

we obtain $(h_i * p_i^*)(z) \in P(\rho_1)$, by using the Herglotz formula. Thus

$$h_i * h_i^*(z) \in P(\rho_3) \tag{3.7}$$

with

$$\rho_3 = 1 - 2(1-\rho_1)(1-\rho_2).$$

Using (3.4), (3.5), (3.6), (3.1) and Lemma 2.1, we have

$$\begin{aligned} \text{Re } q_i(z) &= \frac{1}{1-\alpha} \int_0^1 u^{\frac{\alpha}{1-\alpha}+p-1} \text{Re}(h_i * h_i^*)(uz) \Big\} du \\ &\geq \frac{1}{1-\alpha} \int_0^1 u^{\frac{\alpha}{1-\alpha}+p-1} \left(2\rho_3 - 1 + \frac{2(1-\rho_3)}{1+u|z|}\right) du \\ &> \frac{1}{1-\alpha} \int_0^1 u^{\frac{\alpha}{1-\alpha}+p-1} \left(2\rho_3 - 1 + \frac{2(1-\rho_3)}{1+u}\right) du \\ &= \delta - 4(1-\rho_1)(1-\rho_2) \left[\delta - \frac{1}{1-\alpha} \int_0^1 \frac{u^{\frac{\alpha}{1-\alpha}+p-1}}{1+u} du\right], \end{aligned}$$

where $\delta = 1/\big(\alpha + p(1-\alpha)\big)$.

From this, we conclude that $F \in R_k(\alpha, p, n, \rho_3)$, where ρ_3 is given by (3.1).

We discuss the sharpness as follows. We take

$$\begin{aligned} H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_1)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_1)z}{1+z} \\ H^*(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_2)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_2)z}{1+z}. \end{aligned}$$

Since

$$\left(\frac{1+(1-2\rho_1)z}{1-z}\right) * \left(\frac{1+(1-2\rho_2)z}{1-z}\right) = 1-4(1-\rho_1)(1-\rho_2) + \frac{4(1-\rho_1)(1-\rho_2)}{1-z},$$

it follows from (3.5) that

$$\begin{aligned} q_i(z) &= \frac{1}{1-\alpha}\int_0^1 u^{\frac{\alpha}{1-\alpha}+p-1}\left\{1-4(1-\rho_1)(1-\rho_2) + \frac{4(1-\rho_1)(1-\rho_2)}{1-uz}\right\} du \\ &\to \delta - 4(1-\rho_1)(1-\rho_2)\left\{\delta - \frac{1}{1-\alpha}\int_0^1 \frac{u^{\frac{\alpha}{1-\alpha}+p-1}}{1+u} du\right\} \quad \text{as } z \to 1. \end{aligned}$$

This completes the proof. □

We define $J_c : A(p) \to A(p)$ as follows.

$$J_c(f) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \tag{3.8}$$

where c is real and $c > -p$.

Theorem 3.2. *Let $f \in R_k(\alpha, p, n, \rho)$ and $J_c(f)$ be given by* (3.8). *If*

$$\left[(1-\alpha)\frac{L_{n+p}f(z)}{z^p} + \alpha\ \frac{L_{n+p}J_c(f)}{z^p}\right] \in P_k(\rho), \tag{3.9}$$

then

$$\left\{\frac{L_{n+p}J_c(f)}{z^p}\right\} \in P_k(\gamma), \quad z \in E$$

and

$$\begin{aligned} \gamma &= \rho + (1-\rho)(2\sigma - 1) \\ \sigma &= \int_0^1 \left[1 + t^{\mathrm{Re}\frac{1-\alpha}{c+p}}\right]^{-1} dt. \end{aligned} \tag{3.10}$$

Proof. From (3.8), we have

$$(c+p)L_{n+p}f(z) = cL_{n+p}J_c(f) + z\big(L_{n+p}J_c(f)\big)'.$$

Let

$$H_c(z) = \left(\frac{k}{4}+\frac{1}{2}\right) s_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) s_2(z) = \frac{L_{n+p}\, J_c(f)}{z^p}. \tag{3.11}$$

From (3.9), (3.10) and (3.11), we have

$$\left[(1-\alpha)\frac{L_{n+p}f(z)}{z^p} + \alpha\frac{L_{n+p}J_c(f)}{z^p}\right] = \left[H_c(z) + \frac{1-\alpha}{c+p} zH_c'(z)\right]$$

and consequently

$$\left[s_i(z) + \frac{1-\alpha}{c+p} zs_i'(z)\right] \in P(\rho), \quad i = 1, 2.$$

Using Lemma 2.2, we have $\mathrm{Re}\{s_i(z)\} > \gamma$, where γ is given by (3.10). Thus

$$H_c(z) = \frac{L_{n+p}J_c(z)}{z^p} \in P_k(\gamma)$$

and this completes the proof. □

Let

$$J_0\big(f(z)\big) = J_0(f) = p\int_0^z \frac{f(t)}{t} dt. \tag{3.12}$$

Then

$$L_{n+p-1}J_0(f) = pL_{n+p}f$$

and we have the following.

Theorem 3.3. *Let $f \in R_k(\alpha, p, n+1, \rho)$. Then $J_0(f) \in R_k(\alpha, p, n, \rho)$ for $z \in E$.*

Theorem 3.4. *Let $\phi \in C_p$, where C_p is the class of p-valent convex functions, and let $f \in R_k(\alpha, p, n, \rho)$. Then $\phi * f \in R_k(\alpha, p, n, \rho)$ for $z \in E$.*

Proof. Let $G = \phi * f$. Then

$$\begin{aligned}
(1-\alpha)\frac{L_{n+p-1}G(z)}{z^p} &+ \alpha\frac{L_{n+p}G(z)}{z^p} \\
&= (1-\alpha)\frac{L_{n+p-1}(\phi * f)(z)}{z^p} + \alpha\frac{L_{n+p}(\phi * f)(z)}{z^p} \\
&= \frac{\phi(z)}{z^p} * \left[(1-\alpha)\frac{L_{n+p-1}f(z)}{z^p} + \alpha\frac{L_{n+p}f(z)}{z^p}\right] \\
&= \frac{\phi(z)}{z^p} * H(z), \quad H \in P_k(\rho) \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\rho)\left(\frac{\phi(z)}{z^p} * h_1(z)\right) + \rho\right\} \\
&- \left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\rho)\left(\frac{\phi(z)}{z^p} * h_2(z)\right) + \rho\right\}, \qquad h_1, h_2 \in P.
\end{aligned}$$

Since $\phi \in C_p$, $\operatorname{Re} \dfrac{\phi(z)}{z^p} > \dfrac{1}{2}$, $z \in E$ and so using Lemma 2.3, we conclude that $G \in R_k(\alpha, p, n, \rho)$. □

Corollary 3.5. *Let* $\phi_c = \displaystyle\sum_{m=p}^{\infty} \frac{p+c}{m+c} z^m, (c > -p)$ *is in* C_p*, then* $J_c(f) = \phi_c * f \in R_k(\alpha, p, n, \rho)$.

Corollary 3.6. *Let* $J_0 f$*, defined by* (3.12) *belong to* $R_k(\alpha, p, n, \rho)$*. Then* $f \in R_k(\alpha, p, n, \rho)$ *for* $|z| < r_0 = \dfrac{1}{2+\sqrt{3}}$.

Proof. $J_0(f) = \psi_0(z) * f(z)$ where $\psi_0(z) = \dfrac{z^p}{(1-z)^2} \in C_p$ for $|z| < r_0$. Since $L_{n+p-1} J_0(f) = \psi_0 * L_{n+p-1} f$, therefore we have the result using Theorem 3.4. □

Theorem 3.7. *For* $0 \le \alpha_2 < \alpha_1$, $R_k(\alpha_1, p, n, \rho) \subset R_k(\alpha_2, p, n, \rho), z \in E$.

Proof. For $\alpha_2 = 0$, the proof is immediate. Let $\alpha_2 > 0$ and let $f \in R_k(\alpha_1, p, n, \rho)$. Then

$$\begin{aligned}
&(1-\alpha_2)\frac{L_{n+p-1}f(z)}{z^p} + \alpha_2 \frac{L_{n+p}f(z)}{z^p} \\
= \;& \frac{\alpha_2}{\alpha_1}\left[\left(\frac{\alpha_1}{\alpha_2} - 1\right)\frac{L_{n+p-1}f(z)}{z^p} + (1-\alpha_1)\frac{L_{n+p-1}f(z)}{z^p} + \alpha_1 \frac{L_{n+p}f(z)}{z^p}\right] \\
= \;& \left(1 - \frac{\alpha_2}{\alpha_1}\right) H_1(z) + \frac{\alpha_2}{\alpha_1} H_2(z), H_1, H_2 \in P_k(\rho).
\end{aligned}$$

Since $P_k(\rho)$ is a convex set, we conclude that $f \in R_k(\alpha_2, p, n, \rho)$ for $z \in E$. □

Theorem 3.8. *Let* $f \in R_k(0, p, n, \rho)$*. Then*

$$f \in R_k(\alpha, p, n, \rho) \text{ for } |z| < r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}} \qquad \alpha \neq \frac{1}{2},\ 0 < \alpha < 1.$$

Proof. Let

$$\begin{aligned}
\psi_\alpha(z) &= (1-\alpha)\frac{z^p}{1-z} + \alpha \frac{z^p}{(1-z)^2} \\
&= z^p + \sum_{m=2}^{\infty} \big(1 + (m-1)\alpha\big) z^{m+p-1}, \quad \phi_\alpha \in C_p \quad \text{for } |z| < r_\alpha \\
&= \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}.
\end{aligned}$$

We can write

$$\left[(1-\alpha)\frac{L_{n+p-1}f(z)}{z^p} + \alpha \frac{L_{n+p}f(z)}{z^p}\right] = \frac{\psi_\alpha}{z^p} * f.$$

Applying Corollary 3.6, we see that $f \in R_k(\alpha, p, n, \rho)$ for $|z| < r_\alpha$. □

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