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Another Look at Average Formulas of Nevanlinna Counting Functions of Holomorphic Self-maps of the Unit Disk

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Abstract. This is an extended version of the paper [K] of the author. The average formulas on the circles and disks around arbitrary points of Nevanlinna counting functions of holomorphic self-maps of the unit disk, given in terms of the boundary values of the self-maps, are shown to give another characterization of the whole class or a special subclass of inner functions in terms of Nevanlinna counting function in addition to the previous applications to Rudin's orthogonal functions.

1. Introduction

For a holomorphic self-map φ of the unit disk D on the complex plane, the Nevanlinna counting function N_φ is defined by

$$N_\varphi(w) = \begin{cases} \sum\limits_{\varphi(z)=w} \log \dfrac{1}{|z|} & , \text{ if } w \in \varphi(D), \\ 0 & , \text{ if } w \notin \varphi(D). \end{cases}$$

It plays a very important role in the holomorphic change of variables by $w = \varphi(z)$ in the integral representations and in the study of the composition operator $C_\varphi(f) = f \circ \varphi$ [Sh]. The average formulas of N_φ on the circles and disks around the origin are given and exploited to the explicit representation of the Nevanlinna counting functions of Rudin's orthogonal functions in [K]. In this paper, we compute the averages of N_φ on the circles and disks around arbitrary points in the unit disk in terms of the boundary values of φ and add another application of the average formulas for the characterization of the inner functions as well as a special class of inner functions. See Theorems 2.3 and 2.4. We also clarify the results in [K] on the Nevanlinna counting function of an orthogonal function φ and the essential norm of the corresponding composition operator C_φ. See Theorem 4.1.

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2. $\overline{N_\varphi}(a) := N_\varphi(a) + \mu_a(\partial D)$

For a holomorphic self-map φ and $a \in D$, the Frostman shift $\tau_a \circ \varphi := \dfrac{a - \varphi(z)}{1 - \overline{a}\varphi(z)}$ has the canonical factorization as follows:

$$\frac{a - \varphi(z)}{1 - \overline{a}\varphi(z)} = B_a(z) S_a(z) F_a(z),$$

where B_a is the Blaschke product

$$B_a(z) = \prod_{\varphi(z_i)=a} \frac{|z_i|}{z_i} \frac{z_i - z}{1 - \overline{z_i} z}, \text{ (multiplicities counted)}$$

S_a is the singular inner function

$$S_a(z) = \exp\left(- \int_{\partial D} \frac{\zeta + z}{\zeta - z} d\mu_a(\zeta) \right)$$

with the associated positive Borel measure μ_a singular with respect to the normalized Lebesgue measure $d\sigma$ on the boundary ∂D of D, and F_a is the outer function given by

$$F_a(z) = e^{i\gamma} \exp\left(\int_{\partial D} \frac{\zeta + z}{\zeta - z} \log \left| \frac{a - \varphi^*(\zeta)}{1 - \overline{a}\varphi^*(\zeta)} \right| d\sigma(\zeta) \right),$$

with γ real, and $\varphi^*(\zeta) = \lim_{r \nearrow 1} \varphi(r\zeta)$ which exists almost every $\zeta \in \partial D$. See [G] for the canonical factorization. By Applying Jensen's formula [R2, p.307; G, p.54] to the Frostman shift on the circle $|z| = r$ and letting $r \nearrow 1$, we get a formula as in [K]

$$\begin{aligned} \overline{N_\varphi}(a) :&= N_\varphi(a) + \mu_a(\partial D) \\ &= -\log \left| \frac{a - \varphi(0)}{1 - \overline{a}\varphi(0)} \right| + \int_{\partial D} \log \left| \frac{a - \varphi^*(\zeta)}{1 - \overline{a}\varphi^*(\zeta)} \right| d\sigma(\zeta) \\ &= -\log |a - \varphi(0)| + \int_{\partial D} \log |a - \varphi^*(\zeta)| \, d\sigma(\zeta). \end{aligned} \tag{2.1}$$

We know that $\mu_a(\partial D) = 0$ nearly all $a \in D$, i.e., for all $a \in D$ except for a set of logarithmic capacity zero and $N_\varphi(a) \to 0$ as $|a| \nearrow 1$. See [F], [K], [R3], [R4]. The following average formulas are given in terms of the boundary values of φ and can be computed as in [K], where the average formulas of N_φ are given for the circles and disks only around the origin. The usefulness of the formulas is exhibited in [K] but another view will be given in Theorem 2.3 and 2.4. We note that the sets of capacity zero have the linear as well as area measure zero. Therefore, the averages for N_φ and $\overline{N_\varphi}$ coincide because they are the same for nearly all $a \in D$.

Theorem 2.1.(Average formulas) *Let φ be a holomorphic self-map of D.*

(*a*) *For* $0 < r < 1 - |a|$,

$$\begin{aligned}
(2.2)\qquad \int_{\partial D} N_\varphi(a + r\eta)d\sigma(\eta) &= \int_{\partial D} \overline{N_\varphi}(a + r\eta)d\sigma(\eta) \\
&= -\log^+ \frac{|a - \varphi(0)|}{r} + \int_{\partial D} \log^+ \frac{|a - \varphi^*(\zeta)|}{r} d\sigma(\zeta) \\
&= -\log^+ \frac{|a - \varphi(0)|}{r} + \int_r^{||a-\varphi^*||_\infty} \frac{\sigma\{\zeta \in \partial D : |a - \varphi^*(\zeta)| > t\}}{t} dt \\
&= -\log^+ \frac{|a - \varphi(0)|}{r} - \int_r^{||a-\varphi^*||_\infty} \log \frac{t}{r} d\alpha(t),
\end{aligned}$$

where $\alpha(t) = \sigma\{\zeta \in \partial D : |a - \varphi^*(\zeta)| > t\}$, $\log^+ x = \max(\log x, 0)$ *and* $||a - \varphi^*||_\infty = \operatorname{esssup}\{|a - \varphi^*(\zeta)| : \zeta \in \partial D\}$.

(*b*) *For* $D(a; R) \subset D \setminus \varphi(0)$, *where* $D(a; R)$ *is the open disk around* a *of radius* R,

$$\begin{aligned}
(2.3)\qquad \frac{1}{A(D(a;R))} \iint_{D(a;R)} N_\varphi(\omega)dA(\omega) &= \frac{1}{A(D(a;R))} \iint_{D(a;R)} \overline{N_\varphi}(\omega)dA(\omega) \\
= -\log|a - \varphi(0)| &+ \int_{\partial D} \log|a - \varphi^*(\zeta)|d\sigma(\zeta) \\
&+ \frac{1}{2} \int_{|\varphi^*(\zeta)-a|<R} \left\{ \left(\frac{|a - \varphi^*(\zeta)|}{R} \right)^2 - 1 - \log \left(\frac{|a - \varphi^*(\zeta)|}{R} \right)^2 \right\} d\sigma(\zeta),
\end{aligned}$$

where $dA = 2\rho d\rho d\sigma(\zeta)$ *denotes the normalized are a measure.*

Proof. The proof for the case $a = 0$ is given in [K], but we will give a complete proof of the average formulas around arbitrary point $a \in D \setminus \varphi(0)$ for the convenience of the reader. Let us recall the well known integral

$$(2.4)\qquad \int_{\partial D} \log|r\eta - \omega|d\sigma(\eta) = \log^+ \frac{|\omega|}{r} + \log r.$$

(*a*) Applying (2.4) to (2.1) with $\omega = a - \varphi(0)$ and with $\omega = a - \varphi^*(\zeta)$, we get (2.3) using Fubini's theorem. The other identities below (2.3) follow by applying Theorem 8.16 in [R2] and by applying integration by parts in [R1, Ex 17, p.141].

(b) We integrate (2.2) with respect to $\frac{2r}{R^2}dr$. The first integral is

$$\begin{aligned}
(2.5)\qquad & \frac{1}{R^2}\int_0^R 2r\log^+\frac{|a-\varphi(0)|}{r}dr \\
&= \frac{1}{R^2}\int_0^R 2r\log|a-\varphi(0)|dr - \frac{1}{R^2}\int_0^R 2r\log r dr \\
&= \log\frac{|a-\varphi(0)|}{R}+\frac{1}{2}\ .
\end{aligned}$$

The second integral becomes by Fubini's theorem

$$\begin{aligned}
(2.6)\quad & \frac{1}{R^2}\int\limits_{\partial D}\int_0^R 2r\log^+\frac{|a-\varphi^*(\zeta)|}{r}drd\sigma(\zeta) \\
&= \frac{1}{R^2}\int\limits_{|a-\varphi^*(\zeta)|\le R}\left[\int_0^{|a-\varphi^*(\zeta)|}2r\log|a-\varphi^*(\zeta)|dr-\int_0^{|a-\varphi^*(\zeta)|}2r\log r dr\right]d\sigma(\zeta) \\
&\quad +\frac{1}{R^2}\int\limits_{|a-\varphi^*(\zeta)|>R}\left[\int_0^R 2r\log|a-\varphi^*(\zeta)|dr-\int_0^R 2r\log r dr\right]d\sigma(\zeta) \\
&= \frac{1}{R^2}\int\limits_{|a-\varphi^*(\zeta)|\le R}\frac{|a-\varphi^*(\zeta)|^2}{2}d\sigma(\zeta)+\int\limits_{|a-\varphi^*(\zeta)|>R}\left(\log\frac{|a-\varphi^*(\zeta)|}{R}+\frac{1}{2}\right)d\sigma(\zeta) \\
&= \frac{1}{2}\int\limits_{|a-\varphi^*(\zeta)|\le R}\left(\frac{|a-\varphi^*(\zeta)|^2}{R}\right)d\sigma(\zeta)+\int\limits_{\partial D}\log\frac{|a-\varphi^*(\zeta)|}{R}d\sigma(\zeta) \\
&\quad -\int\limits_{|a-\varphi^*(\zeta)|\le R}\log\frac{|a-\varphi^*(\zeta)|}{R}d\sigma(\zeta)+\frac{1}{2}-\frac{1}{2}\sigma\{\zeta\in\partial D:|a-\varphi^*(\zeta)|\le R\} \\
&= \int\limits_{\partial D}\log\frac{|a-\varphi^*(\zeta)|}{R}d\sigma(\zeta)+\frac{1}{2} \\
&\quad +\frac{1}{2}\int\limits_{|a-\varphi^*(\zeta)|\le R}\left\{\left(\frac{a-\varphi^*(\zeta)}{R}\right)^2-1-\log\left(\frac{|a-\varphi^*(\zeta)|}{R}\right)^2\right\}d\sigma(\zeta).
\end{aligned}$$

Therefore, we have (2.4) from (2.6), (2.7). □

The sub-averaging property of the Nevanlinna counting function [Sh, p.190] follows as a corollary.

Corollary 2.2. (Sub-averaging property of N_φ and $\overline{N_\varphi}$)

$$\begin{aligned} N_\varphi(a) \quad \leq \quad \overline{N_\varphi}(a) \quad &\leq \quad \frac{1}{R^2} \iint\limits_{D(a;R)} N_\varphi(\omega) dA(\omega) \\ &= \quad \frac{1}{R^2} \iint\limits_{D(a;R)} \overline{N_\varphi}(\omega) dA(\omega) \end{aligned}$$

for $D(a;R) \subset D \backslash \varphi(0)$.

Proof. It follows from (b) if we note that $x - 1 \geq \log x$ $(x > 0)$ and

$$\begin{aligned} \overline{N_\varphi}(a) \quad &= \quad N_\varphi(a) + \mu_a(\partial D) \\ &= \quad -\log|a - \varphi(0)| + \int\limits_{\partial D} \log|a - \varphi^*(\zeta)| d\sigma(\zeta). \end{aligned}$$

□

We apply the average formula (b) to prove the following characterization of inner functions.

Theorem 2.3. *Let φ be a nonconstant holomorphic self-map of D. Then the following are equivalent:*

(a) *φ is an inner function.*

(b) $\dfrac{1}{A(D(a;R))} \iint\limits_{D(a;R)} \overline{N_\varphi}(\omega) dA(\omega) = \overline{N_\varphi}(a)$ *for every* $D(a;R) \subset D \backslash \varphi(0)$.

(c) $\overline{N_\varphi}$ *is harmonic in* $D \backslash \varphi(0)$.

This theorem gives a characterization of the special subclass of inner functions whose Frostman shifts $\tau_a \circ \varphi$ is a Blaschke product for every $a \neq \varphi(0)$ in terms of the Nevanlinna counting functions.

Theorem 2.4. *Let φ be a nonconstant holomorphic self-map of D. Then the following are equivalent:*

(a) *φ is an inner function such that the Frostman shift $\tau_a \circ \varphi$ is a Blaschke product for every $a \in D \backslash \varphi(0)$.*

(b) *φ is an inner function with $\mu_a(\partial D) = 0$ for every $a \in D \backslash \varphi(0)$.*

(c) $\dfrac{1}{A(D(0;R))} \iint\limits_{D(a;R)} N_\varphi(a + \omega) dA(\omega) = N_\varphi(a)$ *for every* $D(a;R) \subset D \backslash \varphi(0)$.

(d) *N_φ is harmonic in $D \backslash \varphi(0)$.*

(e) *N_φ is the Green function for D with pole at $\varphi(0)$.*

Theorem 2.4 follows easily from Theorem 2.3 if we note

$$\overline{N_\varphi}(a) = N_\varphi(a) + \mu_a(\partial D) = -\log\left|\frac{a-\varphi(0)}{1-\overline{a}\varphi(0)}\right|$$

for an inner function φ. We only prove Theorem 2.3.

Proof of Theorem 2.3. (c) $\Rightarrow$ (b) : It is known that any harmonic function satisfies the area mean-value property. See [R4, Ex9, p.250].
(b) $\Rightarrow$ (a) : We note that if $\varphi^*(\zeta) = \varphi(0)$ on a set $E \subset \partial D$ of positive measure then $\varphi(z) = \varphi(0)$ for all $z \in D$; a contradiction to the assumption that φ is a nonconstant map. We can write $D\backslash\varphi(0) = \bigcup_{n=1}^{\infty} D(a_n; R_n)$, a countable union of the disks, by the Lindeloef property. The condition (b) now implies

$$\int_{|\varphi^*(\zeta)-a_n|<R_n} \left\{\left(\frac{|a-\varphi^*(\zeta)|}{R_n}\right)^2 - 1 - \log\left(\frac{|a-\varphi^*(\zeta)|}{R_n}\right)^2\right\} d\sigma(\zeta) = 0.$$

Since the integrand is nonnegative, the set $E_n := \{\zeta \in \partial D \mid \varphi^*(\zeta) \in D(a; R_n)\}$ has measure zero for every n. Therefore, $|\varphi^*(\zeta)| = 1$ a.e. on ∂D; that is φ is an inner function.
(a) $\Rightarrow$ (c) : For an inner function φ, it is known from (2.1)

$$\overline{N_\varphi}(a) = N_\varphi(a) + \mu_a(\partial D) = \log\left|\frac{a-\varphi(0)}{1-\overline{a}\varphi(0)}\right|,$$

which is obviously harmonic in $D\backslash\varphi(0)$. □

3. Examples

It would be interesting to have more intrinsic conditions for the inner functions in Theorem 2.4. Suppose φ is an inner function with $\mu_a(\partial D) = 0$ for all $a \in D\backslash\varphi(0)$. Then

$$\sum_{\varphi(z_i)=a} \log\frac{1}{|z_i|} = N_\varphi(a) = -\log\left|\frac{a-\varphi(0)}{1-\overline{a}\varphi(0)}\right|, \tag{3.1}$$

$$i.e., \prod_{\varphi(z_i)=a} |z_i| = \left|\frac{a-\varphi(0)}{1-\overline{a}\varphi(0)}\right| \quad \text{for all } a \in D.$$

We note that no singular inner function can satisfy (3.1). Therefore, such inner function must be of the form $\varphi = B \cdot S_\mu$ with B a nontrivial Blaschke product and S_μ a singular inner function. For the Blaschke products (the case $\mu = 0$), no intrinsic characterization (for example, the conditions on zeros) is known to have $\mu_a(\partial D) = 0$ for all a, or to have all Frostman shifts as Blaschke products. Every

finite Blaschke products obviously have $\mu_a(\partial D) = 0$ for all $a \in D$ and the Blaschke products with zeros in a finite union of Stolz angles are known to have $\mu_a(\partial D) = 0$ for all $a \in D$. See [MN] for example. For the case $\mu \neq 0$, (3.1) forces $\varphi(0) = 0$. So, $\varphi(z)$ is of the form $\varphi(z) = z^m B_1(z) S_\mu(z)$ with $m \geq 1$ and B_1 another Blaschke product. We do not have a complete characterization of such φ but we have the following example by modifying Example 2 in [GI].

Example 1. For the inner function $\varphi(z) = ze^{-\frac{1+z}{1-z}}$, we have $\mu_a(\partial D) = 0$ for any $a \neq 0 (= \varphi(0))$. In fact, if $a \neq 0$ the Frostman shift $\dfrac{a - \varphi(z)}{1 - \overline{a}\varphi(z)}$ cannot have the radial limit zero at any boundary point $\zeta \in \partial D$; so it is a Blaschke product by Theorem 6.2 in [G].

4. Rudin's orthogonal functions

An holomorphic self-map φ of the unit disk D is called a Rudin's orthogonal function if the sequence of powers $\varphi^n, n = 0, 1, 2, \cdots$, is orthogonal in the Hardy space H^2, that is, if

$$\int_{\partial D} \varphi^{*n}(\zeta)\overline{\varphi^{*m}(\zeta)} d\sigma(\zeta) = 0,$$

whenever $n \neq m$. It is known that if φ is orthogonal then $\varphi(0) = 0$ and its pullback measure, $\mu_\varphi(E) = \sigma(\varphi^{*-1}(E))$, is radial, that is, $\mu_\varphi(E) = \mu_\varphi(\zeta E), \zeta \in \partial D$. See [B], [S] for the recent developments of the Rudin's orthogonal functions. In this section, we clarify the results in [K] on the Nevanlinna counting function of an orthogonal function φ and the essential norm of the corresponding composition operator C_φ.

Theorem 4.1. *For an orthogonal holomorphic self-map φ of D, we have*

(a) $\overline{N_\varphi}(a) = -\int_{|a|}^1 \log \dfrac{t}{|a|} d\alpha(t)$, *where* $\alpha(t) = \sigma\{\zeta \in \partial D : |\varphi^*(\zeta)| > t\}$, *and*

(b) $||C_\varphi||_e = \sigma\{\zeta \in \partial D : |\varphi^*(\zeta)| = 1\}$, *where* $||C_\varphi||_e$ *denotes the essential norm of* C_φ *in* H^2.

Proof. (a) From (2.1) and the radial symmetry of the pullback measure μ_φ, we see that $\overline{N_\varphi}$ is also radial:

$$\begin{aligned}
\overline{N_\varphi}(a) &= -\log|a| + \int_{\partial D} \log|a - \varphi^*(\zeta)| d\sigma(\zeta) \\
&= -\log|a| + \int_{\overline{D}} \log|a - z| d\mu_\varphi(z) \\
&= -\log|a| + \int_{\overline{D}} \log||a| - z| d\mu_\varphi(z) \\
&= \overline{N_\varphi}(|a|).
\end{aligned}$$

Therefore, $\overline{N_\varphi}(a)$ is the same as its radialization as follows:

$$\begin{aligned}\overline{N_\varphi}(a) &= \int_{\partial D} \overline{N_\varphi}(|a|\zeta)d\sigma(\zeta)\\ &= -\int_{|a|}^{1} \log\frac{t}{|a|}d\alpha(t)\end{aligned}$$

from Theorem 2.1(a) or Theorem 3.1(a) [K].
(b) follows from the facts:

$$\begin{aligned}||C_\varphi||_e &= \overline{\lim_{|a|\to 1}} N_\varphi(a)/\log\frac{1}{|a|} \quad \text{[Sh]};\\ N_\varphi(a) &\le \overline{N_\varphi}(a) \text{ for all } a\in D;\\ N_\varphi(a) &= \overline{N_\varphi}(a) \text{ nearly all } a\in D;\\ \lim_{|a|\to 1}\overline{N_\varphi}(a)/\log\frac{1}{|a|} &= \lim_{|a|\to 1}\frac{1}{\log\frac{1}{|a|}}\int_{|a|}^{1}\ln\frac{t}{|a|}d\alpha(t)\\ &= \sigma\{\zeta\in\partial D : |\varphi^*(\zeta)| = 1\},\end{aligned}$$

as was shown in [K]. □

For example, if φ is the orthogonal function of Sundberg [S], then

$$\overline{N_\varphi}(a) = \begin{cases}\dfrac{1}{2}\log\dfrac{1}{2} + \log\dfrac{1}{|a|}, & |a|\le\dfrac{1}{2};\\[2ex] \dfrac{1}{2}\log\dfrac{1}{|a|}, & \dfrac{1}{2}\le a<1,\end{cases} \tag{4.1}$$

and $||C_\varphi||_e = 1/2$; so C_φ is not compact. On the other hand, if φ is the orthogonal function of Bishop [B] with $\mu_\varphi(\rho\zeta) = 2\rho d\rho d\sigma(\zeta)$, then

$$\overline{N_\varphi}(a) = \frac{1}{2}(2-|a|^2)\log\frac{1}{|a|} - \frac{1}{4}(1-|a|^2) \tag{4.2}$$

and $||C_\varphi||_e = 0$; that is, C_φ is a compact operator.

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References

[B] C. J. Bishop, *Orthogonal functions in H^∞*, Pacific J. Math, **228**(2005), 1-31.

[F] O. Frostman, Potentiel d´équilibre et capacité des ensembles, Lunds Univ. Mat. Sem. 3, 1935.

[G] J. B. Garnett, Bounded analytic functions, Pure and Applied Mathematics, 96. Academic Press, New York-London 1981.

[GI] P. Gorkin and K. Izuch, *Some counter examples in subalgebras of $L^\infty(D)$*, Indiana Univ. Math. J., **40**(1991), 1301-1313.

[K] H. O. Kim, *Averages of Nevanlinna counting functions of holomorphic self-maps of the unit disk*, Hokkaido Math. J., **33**(2004), 697-706.

[MN] R. Mortini and A. Nicolau, *Frostman shifts of inner functions*, J. D'Analyse Math., **92**(2004), 285-326.

[R1] W. Rudin, Principles of Mathematical Analysis, 3rd ed., McGraw-Hill Book Co., New York-Auckland-Dusseldorf 1976.

[R2] W. Rudin, Real and Complex Analysis, 3d ed., McGraw-Hill Book Co., New York 1987.

[R3] W. Rudin, Function theory in polydiscs, W. A. Benjamin, Inc., New York-Amsterdam 1969.

[R4] W. Rudin,*A generalization of a theorem of Frostman*, Math. Scand., **21**(1967), 136-143.

[S] C. Sundberg, *Measures induced by analytic functions and a problem of Walter Rudin*, J. AMS, **16**(2003), 68-90.

[Sh] J. H. Shapiro, Composition operators and classical function theory, Universitext: Tracts in Mathematics. Springer-Verlag, New York 1993.