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Finitely Generated Modules over Semilocal Rings and Characterizations of (Semi-)Perfect Rings

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Abstract. Lomp [9] has studied finitely generated projective modules over semilocal rings. He obtained the following: finitely generated projective modules over semilocal rings are semilocal. We shall give necessary and sufficient conditions for finitely generated modules to be semilocal modules. By using a lifting property, we also give characterizations of right perfect (semiperfect) rings. Our main results can be summarized as follows:

(1) Let M be a finitely generated module. Then M has finite hollow dimension if and only if M is weakly supplemented if and only if M is semilocal.

(2) A ring R is right perfect if and only if every flat right R-module is lifting and every right R-module has a flat cover if and only if every quasi-projective right R-module is lifting.

(3) A ring R is semiperfect if and only if every finitely generated flat right R-module is lifting if and only if R_R satisfies the lifting property for simple factor modules.

1. Introduction

In this note, all rings R considered are associative rings with identity and all modules are unital right R-modules unless indicated otherwise. For a module M, $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$, $E(M)$, $\mathrm{End}_R(M)$ are the (*Jacobson*) *radical*, *socle*, *injective hull* and *endomorphism ring* of M, respectively. Let M be a module and let K be a submodule of M. K is called *small* submodule (or *superfluous* submodule) of M, abbreviated $K \ll M$, if, for every submodule $L \leq M$, $K + L = M$ implies $L = M$. Let $N_1 \leq N_2 \leq M$. N_1 is a *co-essential* submodule of N_2 in M, abbreviated $N_1 \leq_c N_2$ in M, if $N_2/N_1 \ll M/N_1$. A submodule N of M is said to be *co-closed* in M (or a *co-closed* submodule of M), if N has no proper co-essential submodule in M. i.e., $N' \leq_c N$ in M implies $N = N'$. Let $N_1 \leq N_2 \leq M$. N_1 is said to be a *co-closure* of N_2 in M if N_1 is a co-closed submodule of M with $N_1 \leq_c N_2$ in M. Any submodule of a module has a closure. However, a co-closure does not exist in general, for example, $2\mathbb{Z}$ does not have a co-closure in $\mathbb{Z}_{\mathbb{Z}}$.

Let M be a module and let N and L be submodules of M. N is called a *supplement* of L if $M = N + L$ and $N \cap L \ll N$. Note that any supplement

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submodule (hence any direct summand) of a module M is co-closed in M. Following [12], a module M is *supplemented* if every submodule of M has a supplement. N is called a *weak supplement* of L if $M = N + L$ and $N \cap L \ll M$. A module M is *weakly supplemented* if every submodule of M has a weak supplement. Let M and N be modules. An epimorphism $g : M \to N$ is called a *small cover* of N if Ker $g \ll M$. M is called a *flat cover* (resp. *projective cover*) of N if M is a small cover of N and M is a flat (resp. projective) module. Flat covers and projective covers do not exist in general. For example, $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ does not have a flat cover. We know that if a module has a projective cover, it is unique up to isomorphism. However, this is not the case for the flat covers (cf., [1, Example 2.1]). A ring R is called *semiperfect* (resp. *right perfect*) if every finitely generated right R-module (resp. right R-module) has a projective cover. A ring R is said to be *semilocal* if $R/J(R)$ is left (or right) semisimple ring.

2. Preliminaries

Theorem 2.1 ([3, Theorem 1.1.24]). *For a module M, the following hold:*
(a) *If M is a quasi-injective module, then M is a fully invariant submodule of $E(M)$.*
(b) *If M is a quasi-injective module, then any direct decomposition $E(M) = E_1 \oplus \cdots \oplus E_n$ induces $M = (M \cap E_1) \oplus \cdots \oplus (M \cap E_n)$.*
(c) *If M is a quasi-projective module with a projective cover $\varphi : P \to M$, Ker φ is a fully invariant submodule of P; whence any endomorphism of P induces an endomorphism of M.*
(d) *If M is a quasi-projective module with a projective cover $\varphi : P \to M$, then any direct decomposition $P = P_1 \oplus \cdots \oplus P_n$ induces $M = \varphi(P_1) \oplus \cdots \oplus \varphi(P_n)$.*

Lemma 2.2. *Let R be a ring such that every maximal right ideal of R is a direct summand of R_R. Then R is semisimple.*

Proof. Assume that $\mathrm{Soc}(R_R) \leq R_R$. By [2, Theorem 2.8], there is a maximal submodule I_R such that $\mathrm{Soc}(R_R) \subseteq I_R$. By hypothesis, there exists a decomposition $R_R = I \oplus X$. Then, since X is a simple submodule of R_R, we see $X \subseteq \mathrm{Soc}(R_R) \subseteq I$, which is a contradiction. Hence $R = \mathrm{Soc}(R_R)$. □

Lemma 2.3 (cf., [4] and [8]). *A ring R is right perfect (semiperfect) if and only if every (finitely generated) projective right R-module is lifting.*

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be an R-homomorphism. Put $\langle M_1 \xrightarrow{\varphi} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called the *graph* with respect to φ. Note that $M = M_1 \oplus M_2 = \langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$ (cf., [7]).

Proposition 2.4. *Let R be a right perfect ring. Suppose that P is a projective module and $P_1, \cdots, P_n$ are indecomposable direct summands of P such that $P = P_1 + \cdots + P_n$ and $\overline{P_1} \oplus \cdots \oplus \overline{P_n}$. Then $P = P_1 \oplus \cdots \oplus P_n$.*

Proof. By Lemma 2.3, P is lifting. First we show $P_1 \oplus P_2 \leq_{\oplus} P$. Since $P_1 \leq_{\oplus} P$, there exists a decomposition $P = P_1 \oplus P_1^*$. Let $\pi_{P_1} : P \to P_1$ and $\pi_{P_1^*} : P \to P_1^*$ be projections, respectively. We consider $\pi_{P_1^*}|_{P_2} : P_2 \to P_1^*$. Then $\pi_{P_1^*}(P_2)$ is not small in P_1^*. As P_1^* is lifting, there is a decomposition $P_1^* = \overline{P_1^*} \oplus \overline{\overline{P_1^*}}$ such that $\pi_{P_1^*}(P_2) \leq_c \overline{P_1^*}$ in P_1^*. Then $\pi_{P_1^*}(P_2) = \overline{P_1^*} \oplus (\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}})$. Since $(\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}}) \ll P_1^* \leq P$, $(\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}}) \ll P$. Hence $(\pi_{P_1^*}(P_2) \cap \overline{\overline{P_1^*}}) \subseteq \mathrm{Rad(P)}$. On the other hand,

$$P = P_1 + P_2 = P_1 \oplus P_1^* = P_1 \oplus \overline{P_1^*} \oplus \overline{\overline{P_1^*}}.$$

Let $\pi_{\overline{P_1^*}} : P \to \overline{P_1^*}$ and $\pi_{\overline{\overline{P_1^*}}} : P \to \overline{\overline{P_1^*}}$ be projections, respectively. Then $\pi_{P_1^*}(P_2) = \pi_{\overline{P_1^*}}(P_2) \oplus \pi_{\overline{\overline{P_1^*}}}(P_2)$ and $\pi_{\overline{P_1^*}}(P_2) = \overline{P_1^*}$. Since $\overline{P_1^*}$ is projective, the sequence $P_2 \overset{\pi_{\overline{P_1^*}}}{\to} \pi_{\overline{P_1^*}}(P_2) \to 0$ splits. Thus Ker $(\pi_{\overline{P_1^*}}) \leq_\oplus P_2$. Since P_2 is indecomposable, Ker $(\pi_{\overline{P_1^*}}) = 0$. Hence $P_2 \overset{\pi_{\overline{P_1^*}}}{\simeq} \pi_{\overline{P_1^*}}(P_2)$. Now, we define a map $\varphi : \pi_{\overline{P_1^*}}(P_2) \to P_1 \oplus \overline{\overline{P_1^*}}$ by $\pi_{\overline{P_1^*}}(p_2) \to \pi_{P_1}(p_2) + \pi_{\overline{\overline{P_1^*}}}(p_2)$. Then φ is well-defined. Since $P_2 \subseteq \langle \overline{P_1^*} \overset{\varphi}{\to} P_1 \oplus \overline{\overline{P_1^*}} \rangle$, $\langle \overline{P_1^*} \overset{\varphi}{\to} P_1 \oplus \overline{\overline{P_1^*}} \rangle = P_2 \oplus X$ for some X. Hence we get $P_1 + P_2 = P_1 \oplus P_2 \leq_\oplus P$. We put $P_1 \oplus P_2 = Q$. Using the case $n-1$, we obtain

$$P = P_1 + \cdots + P_n = Q \oplus P_3 \oplus \cdots \oplus P_n.$$

Thus, the induction works. □

Lemma 2.5 (cf., [10])**.** *Let P be a projective module. Then the following statements are equivalent:*
(i) *Every factor module of P has a projective cover;*
(ii) *P is lifting.*

Proposition 2.6. *Let R be a ring such that A is a right ideal of R. If $R/(A+J(R))$ has a projective cover, then so does R/A.*

Proof. Consider the canonical epimorphisms $R \overset{\pi_A}{\to} R/A \overset{\pi_{(A+J(R))}}{\to} R/(A+J(R))$. Then, by [2, Lemma 17.17], we can take an idempotent $e \in R$ for which $\pi_{(A+J(R))}\pi_A|_{eR} : eR \to R/(A+J(R))$ is a projective cover. Hence Ker $(\pi_{(A+J(R))}\pi_A|_{eR}) \ll eR$. Since $R = eR + A + J(R)$, we obtain $R = eR + A$. Thus $\pi_A|_{eR} : eR \to R/A$ is an epimorphism. Since Ker $(\pi_A|_{eR}) \subseteq$ Ker $(\pi_{(A+J(R))}\pi_A|_{eR}) \ll eR$, $\pi_A|_{eR} : eR \to R/A$ is a projective cover. □

Proposition 2.7 (cf., [3])**.** *Let R be a ring such that $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$. Then R_R satisfies the lifting property for simple factor modules.*

Proposition 2.8. *Let R be a ring such that R_R satisfies the lifting property for simple factor modules. Then R_R is a lifting module. In other words, if every simple right R-module has a projective cover, then every cyclic right R-module has a projective cover.*

Proof. Let $A_R \leq R_R$. We show that R/A has a projective cover. By Proposition 2.5, we may assume that $J(R) \subseteq A$. By [2, Theorem 2.8] and Lemma 2.2, $R/J(R)$ is semisimple. By [2, Theorem 9.6], $(R/J(R))/(A/J(R)) \simeq R/A$, we see that R/A can be expressed as a direct sum of simple submodules. Since any simple right R-module has a projective cover, R/A has a projective cover. □

We recall that a module M is called *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple.

Lemma 2.9 (cf., [9] and [12, 21.6(4)]). *Let R be a semilocal ring and let P be a finitely generated projective module. Then the following hold:*
(*a*) *$Rad(P) \ll P$.*
(*b*) *P is semilocal.*
(*c*) *$End_R(P)$ is semilocal.*
(*d*) *P is weakly supplemented.*

Corollary 2.10. *Let R be a ring. Then the following conditions are equivalent:*
(i) *R is semilocal;*
(ii) *Every finitely generated projective right R-module is semilocal.*

Proof. (ii)⟹(i) is obvious.
(i)⟹(ii) Let P be a finitely generated projective right R-module. Then there exists $\oplus_F R_i \xrightarrow{f} P \to 0$, where $R_i = R$ and F is a finite set. As R_R is weakly supplemented, $\oplus_F R_i$ is weakly supplemented. Since a weakly supplemented module is closed under a homomorphic image, P is weakly supplemented. Hence P is finitely generated projective weakly supplemented. Then $\mathrm{Rad}(P) \ll P$. By Lemma 2.9(b), P is semilocal. □

Lemma 2.11. *Let N be a module and let M be a lifting module. Suppose Ker $g \ll M \xrightarrow{g} N \to 0$ with $K \ll N$. Then $g^{-1}(K) \ll M$.*

Proof. Since M is lifting, there exists a decomposition $M = M^* \oplus M^{**}$ such that $M^* \leq_c g^{-1}(K)$ in M. Moreover, $N = g(M^*) + g(M^{**}) = K + g(M^{**}) = g(M^{**})$. Then $M = \mathrm{Ker}\ g + M^{**}$. Since Ker $g \ll M$, $M = M^{**}$. Thus $M^* = 0$. Hence $g^{-1}(K) \ll M$. □

3. Finitely generated modules over semilocal rings

Recall that a module H is *hollow* if every proper submodule is small in H. A module M is said to have *finite hollow dimension* (or *finite dual Goldie dimension*) if there exists an exact sequence $M \xrightarrow{f} \oplus_{i=1}^{n} H_i \to 0$, where all the H_i are hollow and Ker $f \ll M$. Then n is called the *hollow dimension* of M.

Proposition 3.1 (cf., [9, Theorem 2.7]). *Let M be a finitely generated module. Then the following statements are equivalent:*
(i) *M has finite hollow dimension;*
(ii) *M is weakly supplemented;*
(iii) *M is semilocal.*

Proof. (i)$\Longrightarrow$(ii) By assumption, there exists $M \xrightarrow{f} \oplus_{i=1}^n H_i \to 0$, where all the H_i are hollow and Ker $f \ll M$. Then $\oplus_{i=1}^n H_i$ is weakly supplemented. Since a small cover of a weakly supplemented module is weakly supplemented, M is weakly supplemented.

(ii)$\Longrightarrow$(iii) Suppose that M is weakly supplemented. Since M is weakly supplemented, for any $L \leq M$ such that $\mathrm{Rad}(M) \subseteq L$, there exists a weak supplement L of M such that $M = K + L$ and $K \cap L \ll M$. Hence $K \cap L \subseteq \mathrm{Rad}(M)$. Then $L/\mathrm{Rad}(M) \oplus (K + \mathrm{Rad}(M))/\mathrm{Rad}(M) = M/\mathrm{Rad}(M)$. Thus every submodule of $M/\mathrm{Rad}(M)$ is a direct summand.

(iii)$\Longrightarrow$(i) Consider the canonical epimorphism $M \xrightarrow{f} M/\mathrm{Rad}(M)$. Since $M/\mathrm{Rad}(M)$ is semisimple, there exists a decomposition $M/\mathrm{Rad}(M) = M_1/\mathrm{Rad}(M) \oplus \cdots \oplus M_n/\mathrm{Rad}(M)$, where $M_i/\mathrm{Rad}(M)$ is simple, $i = 1, \cdots, n$. Since M is finitely generated, $\mathrm{Rad}(M) \ll M$. Hence $M \xrightarrow{f} M/\mathrm{Rad}(M)$ is a small cover.

(iii)$\Longrightarrow$(ii) Assume that M is semilocal. Since $M/\mathrm{Rad}(M)$ is semisimple, for any $L \leq M$, there exists a decomposition $M/\mathrm{Rad}(M) = (L + \mathrm{Rad}(M))/\mathrm{Rad}(M) \oplus T/\mathrm{Rad}(M)$. Then $M = L + \mathrm{Rad}(M) + T = L + T$. Moreover, $L \cap T \ll M$. Thus M is weakly supplemented. □

By Corollary 2.10 and Proposition 3.1, we get the following:

Corollary 3.2 (cf., [5, 18.10] or [9, Theorem 3.5])**.** *Let R be a semilocal ring and let M be a finitely generated module. Then the following statements are equivalent:*
(i) *M has finite hollow dimension;*
(ii) *M is weakly supplemented;*
(iii) *M is semilocal.*

Lemma 3.3. *Let R be a semilocal ring and let M be a finitely generated module. Then every supplement in M is weakly supplemented. Moreover, every co-closed submodule of M is weakly supplemented.*

Proof. Let N be a supplement submodule of M. Then there exists a submodule K of M such that $M = K + N$ and $K \cap N \ll N$. Since $M/K \simeq N/(K \cap N)$ is weakly supplemented, $N \xrightarrow{f} N/(K \cap N)$ is a small cover. Hence N is weakly supplemented. Finally, let K be a co-closed in M. Since $\mathrm{Rad}(K) = K \cap \mathrm{Rad}(M)$, $K/\mathrm{Rad}(K) \simeq (K + \mathrm{Rad}(M))/\mathrm{Rad}(M) \leq M/\mathrm{Rad}(M)$. Since $M/\mathrm{Rad}(M)$ is semisimple, $K/\mathrm{Rad}(K)$ is semisimple. □

Let L and M be modules. Following [5], L is *small M-projective* if the canonical epimorphism $g : M \to M/K$ such that $K \ll M$ and any homomorphism $f : L \to M/K$, there exists a homomorphism $h : L \to M$ such that $gh = f$. L is *Rad M-projective* if the canonical epimorphism $g : M \to M/K$ such that $K \subseteq \mathrm{Rad}(M)$ and any homomorphism $f : L \to M/K$, there exists a homomorphism $h : L \to M$ such that $gh = f$. M is *small self-projective* if it is small M-projective and is Rad self-projective if it is Rad M-projective. It is easy to see that Rad M-projective modules are small M-projective.

Theorem 3.4. *Let R be a semilocal ring and let M be a finitely generated module. Then the following conditions are equivalent:*
(i) *M is Rad-M-projective (i.e., M is Rad-self-projective);*
(ii) *M is M-projective (i.e.,M is self-projective);*
(iii) *M is small M-projective (i.e., M is small self-projective).*

Proof. (ii)$\Longrightarrow$(iii) is obvious.
(iii)$\Longrightarrow$(ii) Suppose that M is small M-projective. Let K be a submodule of M and let $f : M \to M/K$ be any homomorphism and $\pi : M \to M/K$ be the canonical epimorphism. For $(K + \mathrm{Rad}(M))/\mathrm{Rad}(M) \leq M/\mathrm{Rad}(M)$, there exists a direct summand $T/\mathrm{Rad}(M)$ of $M/\mathrm{Rad}(M)$ such that $M/\mathrm{Rad}(M) = (K + \mathrm{Rad}(M))/\mathrm{Rad}(M) \oplus T/\mathrm{Rad}(M)$, as $M/\mathrm{Rad}(M)$ is semisimple, Since $\mathrm{Rad}(M) \ll M$, $M = K + \mathrm{Rad}(M) + T = K + T$. Moreover, $(K+\mathrm{Rad}(M))/\mathrm{Rad}(M) \cap T/\mathrm{Rad}(M) = [(K + \mathrm{Rad}(M)) \cap T]/\mathrm{Rad}(M) = [(K \cap T) + \mathrm{Rad}(M)]/\mathrm{Rad}(M) = 0$. Hence $(K \cap T) \subseteq \mathrm{Rad}(M) \ll M$. Therefore $K \cap T \ll M$. Let $\pi_1 : K \to K/(K \cap T)$ be the canonical epimorphism. Define a map $g : M \to M/K \oplus K/(K \cap T)$ by $t + k \rightsquigarrow (\pi(t), \pi_1(k))$, for $t \in T$, $k \in K$. Then g is well-defined and a small cover. By hypothesis, there exists a homomorphism $h : T \to M = K + T$ such that $gh = if$, where $i : M/K \to M/K \oplus K/(K \cap T)$ is an inclusion map. Then $\pi h = f$. Thus M is M-projective.
(i)$\Longrightarrow$(iii) is trivial.
(iii)$\Longrightarrow$(i) Since M is finitely generated, $\mathrm{Rad}(M) \ll M$. By assumption, M is Rad-M-projective. □

By Theorem 3.4 and [9, Corollary 3.12], we get the following:

Corollary 3.5. *Let R be a semilocal ring and let M be a finitely generated module satisfying one of the following:*
(*a*) *M is Rad-self-projective,*
(*b*) *M is small self-projective.*
Then $End_R(M)$ is semilocal.

Let L and M be modules. Following [5], L is *im-summand* (*im-co-closed*) *M-projective* if the canonical epimorphism $g : M \to M/K$ such that and any homomorphism $f : L \to M/K$ such that Im f is a direct summand (co-closed) in M/K, there exists a homomorphism $h : L \to M$ such that $gh = f$. Note that im-co-closed M-projective modules are im-summand M-projective. L is *M-epi-projective* if for any epimorphisms $p : M \to N$ and $f : L \to N$, there exists a homomorphism $h : L \to M$ such that $ph = f$. L is *epi-projective* if it is L-epi-projective. L is *im-summand* (*im-co-closed*) *small M-projective* if the canonical epimorphism $g : M \to M/K$ and any homomorphism $f : L \to M/K$ such that Im f is a direct summand (co-closed) in M/K and $K \ll M$, there exists a homomorphism $h : L \to M$ such that $gh = f$. It is easy to see that im-co-closed small M-projective modules are im-summand small M-projective.

Remark 3.6. It is obvious that the following implications hold for a module:

(a) M-projective $\Longrightarrow$ small M-projective.
(b) M-projective $\Longrightarrow$ im-summand M-projective $\Longrightarrow$ im-summand small M-projective.
(c) M-projective $\Longrightarrow$ im-co-closed M-projective $\Longrightarrow$ im-co-closed small M-projective.
(d) projective $\Longrightarrow$ self-projective (or quasi-projective) $\Longrightarrow$ epi-projective.

Lemma 3.7. *Let M be a weakly supplemented module. Then the following conditions are equivalent:*
(i) *M is M-epi-projective (i.e., M is epi-projective);*
(ii) *M is M-projective (i.e., M is self-projective);*
(iii) *M is small M-projective (i.e., M is small self-projective).*

Proof. (ii)$\Longrightarrow$(i) is obvious.
(i)$\Longrightarrow$(ii) Consider a diagram

$$\begin{array}{ccccc} & & M & & \\ & & \big\downarrow{\scriptstyle f} & & \\ M & \xrightarrow{g} & M/K & \longrightarrow & 0, \end{array}$$

where any homomorphism $f : M \to M/K$ and the canonical epimorphism $g : M \to M/K$. Then Im $f \leq M/K$. If Im $f = M/K$, then, by assumption, there exists a homomorphism $h : M \to M$ such that the above diagram commutes. If Im $f \lneq M/K$. Put Im $f = L/K$. Since M/K is weakly supplemented, there exists a submodule T/K of M/K such that $L/K + T/K = M/K$ and $L/K \cap T/K \ll M/K$. Then $M = L + T$ and $g(L) =$ Im f. By hypothesis, there exists a homomorphism $h : M \to M$ such that the above diagram commutes. (ii)$\Longleftrightarrow$(iii) This follows from [6, Lemma 2.1]. □

By Lemma 3.7, "An epi-projective weakly supplemented module is self-projective" (cf., [6, Corollary 3.2]).

Lemma 3.8 (cf., [6, Corollary 2.2])**.** *Let L be a module and let H be a hollow module. Then the following conditions are equivalent:*
(i) *L is im-summand H-projective;*
(ii) *L is im-summand small H-projective.*

Proof. (i)$\Longrightarrow$(ii) is obvious.
(ii)$\Longrightarrow$(i) Consider a diagram

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle f} & & \\ H & \xrightarrow{g} & H/K & \longrightarrow & 0, \end{array}$$

where any homomorphism $f : L \to H/K$ with Im $f \leq_{\oplus} H/K$ and the canonical epimorphism $g : H \to H/K$. Since H is hollow, $K \ll H$. By assumption, there exists a homomorphism $h : L \to H$ such that the above diagram commutes. □

Using a proof similar to that of Lemma 3.8, we get the following two results.

Corollary 3.9 (cf., [6, Corollary 2.2])**.** *Let L be a module and let H be a hollow*

module. Then the following conditions are equivalent:
(i) *L is im-co-closed H-projective;*
(ii) *L is im-co-closed small H-projective.*

Corollary 3.10. *Let L be a module and let H be a hollow module. Then the following conditions are equivalent:*
(i) *L is im-summand H-projective;*
(ii) *L is H-epi-projective.*

Recall that a module M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement P of A such that $P \subseteq B$. It is well-known from [12] that the following implications hold for a module:

"lifting $\Longrightarrow$ amply supplemented $\Longrightarrow$ supplemented $\Longrightarrow$ weakly supplemented $\Longrightarrow$ semilocal"

Lemma 3.11. *Let L be a module and let M be an amply supplemented module. Suppose that every co-closed submodule of a factor module of M is a direct summand. Then the following conditions are equivalent:*
(i) *L is im-summand* (*small*) *M-projective;*
(ii) *L is im-co-closed* (*small*) *M-projective.*

Proof. (ii)$\Longrightarrow$(i) Since every direct summand of a module is a co-closed submodule, L is im-summand M-projective.
(i)$\Longrightarrow$(ii) Consider a diagram

$$\begin{array}{ccccc} & & L & & \\ & & \downarrow f & & \\ M & \xrightarrow{g} & M/K & \longrightarrow & 0, \end{array}$$

where any homomorphism $f : L \to M/K$ with Im f is co-closed in M/K and the canonical epimorphism $g : M \to M/K$. Since M/K is amply supplemented, there exists a co-closure T of Im f in M/K which is a direct summand of M/K. i.e., $T \leq_c$ Im f in M/K such that T is co-closed in M/K. Since Im f is co-closed in M/K, $T =$ Im $f \leq_{\oplus} M/K$. By assumption, there exists a homomorphism $h : L \to M$ such that the above diagram commutes. □

By Lemma 3.8, 3.11, and Corollary 3.9, 3.10, the following holds:

Corollary 3.12. *Let L be a module and let H be a hollow module. Then the following conditions are equivalent:*
(i) *L is im-summand* (*small*) *H-projective;*
(ii) *L is im-co-closed* (*small*) *H-projective;*
(iii) *L is H-epi-projective.*

We recall that a module M is *strongly discrete* if it is self-projective and supplemented. It is well-known from [6] that the following implications hold for a module:

"strongly discrete $\Longrightarrow$ discrete $\Longrightarrow$ quasi-discrete $\Longrightarrow$ lifting".

The converse implications are not true in general.

Example 3.13. (1) Consider the quotient field K of a discrete valuation domain R which is not complete. Then K as an R-module is discrete but is not self-projective (cf., [6, pp. 902-903]).
(2) Let R be a discrete valuation ring with a prime ideal P. Then an injective hull $E(R/P)$ of R/P is quasi-discrete but not discrete.
(3) Put $R = \mathbb{Z}/4\mathbb{Z}$ and $Q_R = R \oplus R$. Then a submodule $M_R = (1,2)R \oplus (1,0)R$ of Q_R is lifting but not quasi-discrete.

Theorem 3.14 (cf., [6, Theorem 3.4])**.** *Let H be a hollow module. Then the following conditions are equivalent:*
(i) *H is strongly discrete;*
(ii) *For every $K \leq H$ ($K \ll H$) and for any homomorphism $f : H \to H/K$ with Im $f = L/K$, where L is co-closed in H, f can be lifted to H;*
(iii) *H is im-summand (small) H-projective;*
(iv) *H is im-co-closed (small) H-projective;*
(v) *H is epi-projective.*

Proof. From the proof of [6, Lemma 2.1] we see that the $\leq$- and $\ll$-versions of condition (ii) are equivalent.
(i)$\Longrightarrow$(ii) Since H is lifting, there exist co-closures of submodules of H. Hence (ii) follows.
(ii)$\Longrightarrow$(iii) Since H is amply supplemented, the proof is similar to (b)$\Longrightarrow$(c) of [6, Theorem 2.4].
(iii)$\Longleftrightarrow$(iv) This follows from Corollary 3.12.
(iv)$\Longrightarrow$(v) By Corollary 3.12, H is epi-projective.
(v)$\Longrightarrow$(i) Since H is lifting and epi-projective, H is strongly discrete by [6, Theorem 3.3]. □

By [6, Lemma 2.3, Theorem 2.4] and Corollary 3.12, we obtain the following:

Theorem 3.15. *Let H be a projective hollow module satisfying one of the following:*
(*a*) *H is im-summand (small) H-projective,*
(*b*) *H is im-co-closed (small) H-projective,*
(*c*) *H is epi-projective.*
Then H is discrete.

4. Characterizations of (semi-)perfect rings

Following [1], a ring R is *right generalized perfect* (or *right G-perfect*) if every right R-module has a flat cover. It is easy to see that right perfect rings are right generalized perfect.

We give characterizations for right perfect rings.

Theorem 4.1. *The following statements are equivalent for a ring R:*
(i) *R is right perfect;*
(ii) *Every flat right R-module is lifting and R is right generalized perfect;*

(iii) *R is semilocal and every non-zero right R-module has a maximal submodule;*
(iv) *Every quasi-projective right R-module is lifting;*
(v) *Every countably generated free right R-module is lifting.*

Proof. (i)$\Longrightarrow$(ii) Let L be a flat right R-module. By Lemma 2.3 and [2, Theorem 28.4], L is lifting. Since every projective module is flat, every right R-module has a flat cover.
(ii)$\Longrightarrow$(i) Consider a diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow f & & \\ L & \xrightarrow{g} & M & \longrightarrow & 0, \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

where any epimorphism $f : P \to M$ with P is projective and $g : L \to M$ is a flat cover. Since P is projective, there exists a homomorphism $h : P \to L$ such that the above diagram commutes. Since Ker $g \ll L$, h is an epimorphism. Then h^{-1}(Ker g) = Ker gh. By assumption, L is lifting. By Lemma 2.11, h^{-1}(Ker g) = Ker gh = Ker $f \ll P$. Hence $P \xrightarrow{f} M$ is a projective cover.
(ii)$\Longrightarrow$(iii) By assumption, R_R is lifting. Let A be a submodule of R_R with $J(R) \subseteq A$. We put $\overline{A} = A/J(R)$ and $\overline{R} = R/J(R)$. We may show $\overline{A} \leq_{\oplus} \overline{R}$. Since R_R is lifting, there exists a decomposition $R_R = A^* \oplus A^{**}$ such that $A^* \leq A$ and $A \cap A^{**} \ll R$. Consider the canonical map $\varphi = \varphi\mid_{J(R)}: R \to \overline{R}$. Then $\overline{R} = \varphi(A) \oplus \varphi(A^{**})$. In fact, $\varphi(A) = \overline{A}$. Hence $\overline{A} \leq_{\oplus} \overline{R}$. Therefore $\overline{R}$ is semisimple. Let M be a right R-module. By assumption, we can consider a diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow f & & \\ L & \xrightarrow{g} & M & \longrightarrow & 0, \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

where any epimorphism $f : P \to M$ with P is projective and $g : L \to M$ is a flat cover. Since P is projective, there exists a homomorphism $h : P \to L$ such that the above diagram commutes. Since Ker $g \ll L$, h is an epimorphism. By hypothesis, P is projective lifting. Then there exists a maximal submodule K of P such that Ker $f \subseteq K$. It is sufficient to show that $P/$Ker f has a maximal submodule. By Lemma 2.5, $P/$Ker f has a projective cover. Say $Q \xrightarrow{q} P/$Ker f. Thus $Q/$Ker q$\simeq P/$Ker f. Since Q is projective, Q has a maximal submodule L. Hence Ker q $\subseteq$ Rad$(Q) \subseteq L$. This implies that $L/$Ker q is a maximal submodule of $Q/$Ker q. Hence $P/$Ker f has a maximal submodule. Therefore M has a maximal submodule.
(iii)$\Longrightarrow$(i) holds by [2, Theorem 28.4].

(i)$\Longrightarrow$(v) This follows from Lemma 2.3.
(v)$\Longrightarrow$(i) By hypothesis, R is semiperfect and $\overline{R}$ is semisimple. Since $R^{(\mathbb{N})}$ is lifting, there exists a decomposition $R^{(\mathbb{N})} = X \oplus Y$ such that $X \leq \mathrm{Rad}(R^{(\mathbb{N})})$ and $\mathrm{Rad}(R^{(\mathbb{N})} \cap Y \ll Y$. Since $\mathrm{Rad}(R^{(\mathbb{N})}) = \mathrm{Rad}(X) \oplus \mathrm{Rad}(Y)$ and $X \leq \mathrm{Rad}(R^{(\mathbb{N})})$, we see $\mathrm{Rad}(X) = X$. Hence, by [2, Lemma 28.3], $J(R)$ is right T-nilpotent. Thus R is right perfect.
(iv)$\Longrightarrow$(iii) By assumption, R_R is lifting. Then $\overline{R}$ is semisimple. Let M be a non-zero right R-module. Then there is an epimorphism $f : P \to M$ with P projective. By hypothesis, P is projective lifting. Then there exists a maximal submodule K such that $\mathrm{Ker}\ f \subseteq K$. Since $(P/\mathrm{Ker}\ f)/(K/\mathrm{Ker}\ f) \simeq P/K$ is simple, $K/\mathrm{Ker}\ f$ is a maximal submodule of $P/\mathrm{Ker}\ f$. Hence M has a maximal submodule.
(i)$\Longrightarrow$(iv) Assume that R is right perfect. Then, by Lemma 2.3, every projective right R-module is lifting. Let Q be a quasi-projective module and let A be a submodule of Q. Consider the canonical epimorphism $f : Q \to Q/A$. We can take a projective module P such that Q is a homomorphic image of P. i.e., we have an epimorphism $g : P \to Q$. Since P is lifting, by [2, Lemma 17.17], there exists a decomposition $P = P_1 \oplus P_2$ such that $P_1 \leq g^{-1}(A)$ and $fg|_{P_2} : P_2 \to Q/A$ is a projective cover. Because Q is a quasi-projective module, the decomposition $P = P_1 \oplus P_2$ induces a direct decomposition $Q = g(P_1) \oplus g(P_2)$ by Theorem 2.1. Then $g(P_1) \leq A$ and $g(P_2) \cap A \ll g(P_2)$ hold. □

We give characterizations for semiperfect rings.

Theorem 4.2. *The following statements are equivalent for a ring R:*
(i) *R is semiperfect;*
(ii) *Every finitely generated flat right R-module is lifting;*
(iii) *R is semilocal and idempotents lift modulo $J(R)$;*
(iv) *R_R is lifting;*
(v) *R_R satisfies the lifting property for simple factor modules;*
(vi) *R is semilocal and every simple right R-module has a flat cover.*

Proof. (i)$\Longrightarrow$(ii) Let L be a finitely generated flat right R-module. By [11, Corollary 2] and Lemma 2.3, L is lifting.
(ii)$\Longrightarrow$(iii) We put $R/J(R) = \overline{R}$. By hypothesis, R_R is lifting. By the proof of Theorem 4.1, $\overline{R}$ is semisimple. Let $\overline{g}$ be an idempotent in $\overline{R}$. Then there exists a decomposition $\overline{R} = \overline{gR} \oplus \overline{(1-g)R}$. Put $\overline{gR} = \overline{g_1 R}$ and $\overline{(1-g)R} = \overline{g_2 R}$. We consider the canonical epimorphism $R \xrightarrow{\varphi} \overline{R}$. Since R_R is lifting, there exists a decomposition $R_R = A_i \oplus A_i^*$ such that $A_i \leq_c \varphi^{-1}(\overline{g_i R})$ in R_R $(i = 1, 2)$. Then $R_R = A_1 + A_2 + \mathrm{Ker}\ \varphi$. Since $\mathrm{Ker}\ \varphi \ll R_R$, $R_R = A_1 + A_2$. Moreover, $A_1 \cap A_2 \ll R_R$. By [12, 41.14], $R_R = A_1 \oplus A_2$. Thus there exists a (necessarily) complete set $\{e_1, e_2\}$ of pairwise orthogonal idempotents in R with $A_i = e_i R$ $(i = 1, 2)$. Then $\overline{1} = \overline{e_1} + \overline{e_2}$, where $\overline{e_i} \in \overline{g_i R}$ $(i = 1, 2)$. On the other hand, $\overline{1} = \overline{g_1} + \overline{g_2}$. By the uniqueness, $\overline{e_i} = \overline{g_i}$ $(i = 1, 2)$.
(iii)$\Longrightarrow$(i) holds by [2, Theorem 27.6].
(iv)$\Longrightarrow$(v) is trivial.
(v)$\Longrightarrow$(iv) By Proposition 2.8, this part is clear.
(iii)$\Longrightarrow$(v) This part is a direct consequence of Proposition 2.7.

(iv)⟹(iii) Assume that R_R is lifting. Then, by the proof of (ii)⟹(iii), $\overline{R}$ is semisimple and idempotents lift modulo $J(R)$.
(iv)⟹(vi) From the proof of Theorem 4.1 we see that R is semilocal. By Lemma 2.5, every factor module of R_R has a projective cover, hence every cyclic right R-module has a projective cover. Therefore every simple right R-module has a flat cover.
(vi)⟹(v) By [9, Theorem 3.8], every simple right R-module has a projective cover. Let K be a maximal submodule of R_R and let $\varphi : R \to R/K$ be the canonical epimorphism. Since R/K has a projective cover, by [2, Lemma 17.17], there exists a decomposition $R_R = eR \oplus (1-e)R$ such that $(\varphi|_{eR}) : eR \to R/K$ is a projective cover and $(1-e)R \leq K$. Hence Ker $(\varphi|_{eR}) = K \cap eR \ll eR$. i.e., $R = eR \oplus (1-e)R$ such that $K \cap eR \ll eR$. Thus R_R satisfies the lifting property for simple factor modules. □

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