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Maximal Hypersurfaces of $(m + 2)$-Dimensional Lorentzian Space Forms

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Abstract. We determine maximal space-like hypersurfaces which are the images of subbundles of the normal bundle of m-dimensional totally geodesic space-like submanifolds of an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$ under the normal exponential map. Then we construct examples of maximal space-like hypersurfaces of $\widetilde{M}_1^{m+2}(c)$.

1. Introduction

A maximal hypersurface in a Lorentz-Minkowski n-space L^n is a space-like hypersurface with zero mean curvature. It is well known that the maximal and constant mean curvature space-like hypersurfaces are important in both mathematics and physics points of view. They play some important roles in general relativity (see for instance [12] and references therein).

One of the most important global results about maximal surfaces in L^3 is Calabi-Bernstein's theorem, which states that the only complete maximal surfaces in the Lorentz-Minkowski space L^3 are the space-like planes. This theorem was first proved by Calabi in [4], and later it was extended to n-dimensional case by Cheng and Yau in [5]. As a generalization of this result, complete space-like hypersurfaces with constant mean curvature in a Lorentz manifold have been investigated in [1], [15], [14], [10], [18], [3].

Recently, maximal space-like surfaces in the Lorentz-Minkowski 3-space L^3 has been studied in [9], [11], [2], [13]. For instance, in [11], maximal surfaces in L^3 which are foliated by pieces of circles were classified; in [9], maximal rotation and ruled surfaces in L^3 were investigated, and also, maximal helicoidal surfaces in L^3 were studied in [13].

In [8], Kimura determined minimal hypersurfaces M foliated by geodesics of a 4-dimensional space forms $\widetilde{M}^4$ that given by $M = \{\exp_p(t\xi) | p \in \Sigma, t \in \mathbb{R}\}$, where Σ is a minimal surface of a 4-dimensional space form $\widetilde{M}^4$ and ξ is a local unit normal vector field on Σ. As a partial generalization of Kimura's work, in

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[7] we constructed the minimal hypersurfaces which are the image of a subbundle, spanned by a nonparallel unit normal vector field ξ, of the normal bundle of an m-dimensional totally geodesic submanifold of an $(m+2)$-dimensional space form under the normal exponential map. These hypersurfaces are foliated by the geodesics of the space forms.

Then, it is natural to construct the similar hypesurfaces of an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. However, in a Lorentzian space vectors with different causal characters usually turn into a wider variety of cases to consider. In this work, for a space-like non-parallel unit normal vector field ξ, we build up space-like hypersurfaces of a Lorentzian space form $\widetilde{M}_1^{m+2}(c)$ under a constrain condition. More precisely, we start with a totally geodesic immersion f from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$ and a non-parallel space-like unit normal vector field ξ to define a map $F : M \times I \to \widetilde{M}_1^{m+2}(c)$ by $F(x,t) = \exp(x, t\xi)$, where $x \in M,\ t \in I$ which is an open subset of $\mathbb{R}$. The image $F(M \times I)$ is a space-like hypersurface of $\widetilde{M}_1^{m+2}(c)$ foliated by the geodesics of $\widetilde{M}_1^{m+2}(c)$ under a constraint condition, which does not appear in the Riemannian space form [7]. We show that F is a maximal immersion under some conditions on the components of the normal connection form of f. We also construct some examples.

2. Preliminaries

Let $\widetilde{M}_q^m$ be an m-dimensional pseudo-Riemannian manifold with pseudo-Riemannian metric tensor $\tilde{g}$ of index q. Denoting by $\langle\, ,\, \rangle$ the associated nondegenerate inner product on $\widetilde{M}_q^m$, a tangent vector X to $\widetilde{M}_q^m$ is said to be *space-like* if $\langle X, X\rangle > 0$ (or $X = 0$), *time-like* if $\langle X, X\rangle < 0$ or *light-like* (*null*) if $\langle X, X\rangle = 0$ and $X \neq 0$.

Let M^m be a submanifold of a pseudo-Riemannian manifold $\widetilde{M}_q^{m+n}$. If the pseudo-Riemannian metric tensor $\tilde{g}$ of $\widetilde{M}_q^{m+n}$ induces a pseudo-Riemannian metric g on M^m, then M^m is called a pseudo-Riemannian submanifold of $\widetilde{M}_q^m$. If the index of g is zero then M is called a space-like submanifold.

Let X and Y be tangent vector fields on M^m and let ξ be a normal vector field on M^m in $\widetilde{M}_q^{m+n}$. Then the *Gauss* formula and the *Weingarten* formula are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi,$$

where $\widetilde{\nabla}$ is the Riemannian connection of $\widetilde{M}$, ∇ and $\nabla^\perp$ are, respectively, the induced Riemannian connection of M and the normal connection of M^m in $\widetilde{M}$, h is the second fundamental form of M in $\widetilde{M}_q^{m+n}$ and A_ξ is the shape operator of M with respect to the normal vector ξ. Also the Gauss and Weingarten formulas yield

$$\langle A_\xi(X), Y\rangle = \langle h(X,Y), \xi\rangle. \tag{2.1}$$

Let M^m be a submanifold of a pseudo-Riemannian manifold $\widetilde{M}_q^{m+n}$. Let $\xi_1, \cdots, \xi_n$ be an orthonormal local basis for $T^{\perp}M$. Then the mean curvature vector is given by

$$H = \frac{1}{m}\sum_{i=1}^{n} \varepsilon_i (\mathrm{trace} A_{\xi_i})\xi_i,$$

where $\varepsilon_i = \langle \xi_i, \xi_i \rangle = \pm 1$. For a space-like submanifold M of $\widetilde{M}_q^{m+n}$, if $H = 0$ on M, then M is called a maximal submanifold of $\widetilde{M}_q^{m+n}$.

Let $\widetilde{M}_q^m(c)$ be an m-dimensional connected pseudo-Riemannian manifold of index q and of constant curvature c, which is called an *indefinite space form*. According to $c > 0, c = 0$ or $c < 0$, it is a pseudo-Riemannian sphere $\mathbb{S}_q^m(c)$, a pseudo-Euclidean space $\mathbb{R}_q^m$ or a pseudo-hyperbolic space $\mathbb{H}_q^m(c)$, respectively. For the index $q = 1$, $\mathbb{S}_1^m(c)$, $\mathbb{R}_1^m$ and $\mathbb{H}_1^m(c)$ are, respectively, called the de Sitter space-time, Minkowski space-time and the anti-de Sitter space-time. Hence the indefinite space form $\widetilde{M}_1^m(c)$ is called a *Lorentzian space form*. If $q = 0$, then $\widetilde{M}_q^m(c)$ is a Riemannian space form. For simplicity, we suppose that the constant curvature c of $\widetilde{M}_1^m(c)$ is equal to $1,\ 0,\ -1$ according to whether $c > 0, c = 0, c < 0$.

Let $\mathbb{R}_q^m$ be an m-dimensional pseudo-Euclidean space with metric tensor given by

$$\tilde{g} = -\sum_{i=1}^{q}(dx_i)^2 + \sum_{i=q+1}^{m}(dx_i)^2,$$

where $(x_1, \cdots, x_m)$ is a rectangular coordinate system of $\mathbb{R}_q^m$. So $(\mathbb{R}_q^m, \tilde{g})$ is a flat pseudo-Riemannian manifold of index q. For the pseudo-Riemannian sphere and pseudo-hyperbolic space, we put

$$\mathbb{S}_q^m(1) = \{x \in \mathbb{R}_q^{m+1} |\ \langle x, x\rangle = 1\} \quad \text{and} \quad \mathbb{H}_q^m(-1) = \{x \in \mathbb{R}_{q+1}^{m+1} |\ \langle x, x\rangle = -1\}.$$

Also the hyperbolic space $\mathbb{H}^m(-1)$ is defined by

$$\mathbb{H}^m(-1) = \{x \in \mathbb{R}_1^{m+1} |\ \langle x, x\rangle = -1 \text{ and } x_1 > 0\},$$

where x_1 is the first coordinate in $\mathbb{R}_1^{m+1}$.

Let $f : M^m \to \widetilde{M}_1^{m+2}(c)$ be a smooth isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. Let $\xi,\ \eta$ be a local orthonormal normal basis of M^m in $\widetilde{M}_1^{m+2}(c)$ with signature $\varepsilon_1 = \langle \xi, \xi\rangle$ and $\varepsilon_2 = \langle \eta, \eta\rangle$. Let $X_1, \cdots, X_m$ be a local tangent basis on M and s be the normal connection form for $\nabla^{\perp}$ defined by $s(X_i) = \langle \nabla^{\perp}_{X_i}\xi, \eta\rangle$. Since $\langle \xi, \eta\rangle = 0$, then we see that $\nabla^{\perp}_{X_i}\xi = \varepsilon_2 s(X_i)\eta$ and $\nabla^{\perp}_{X_i}\eta = -\varepsilon_1 s(X_i)\xi$. Here it is seen that if either ξ or η is parallel in the normal space then the normal connection form for $\nabla^{\perp}$ is zero. We therefore suppose that ξ and η are nonparallel.

Denoting by s_i the components of the connection form s, the covariant derivative of the 1-form s is defined by

$$s_{ij} = (\nabla_{X_j} s)(X_i) = X_j(s_i) - s(\nabla_{X_j} X_i).$$

Then it is easily seen that

$$s_{ij} = \langle \nabla^\perp_{X_j} \nabla^\perp_{X_i} \xi - \nabla^\perp_{\nabla_{X_j} X_i} \xi, \eta \rangle.$$

As the ambient space is a space form, the Ricci equation can be written as

$$\langle R^\perp(X,Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle, \quad ([16, \text{p.}125]),$$

where $R^\perp$ denotes the normal curvature tensor of the normal connection $\nabla^\perp$ and $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$. So we express the Ricci equation as

$$s_{ji} - s_{ij} = \langle R^\perp(X_i, X_j)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X_i, X_j\rangle. \tag{2.2}$$

If the normal curvature tensor $R^\perp$ of the normal connection $\nabla^\perp$ vanishes identically, then the normal connection is said to be flat.

Henceforth, for the sake of simplicity of the computations we take a local isothermal coordinate system $(x_1, \cdots, x_m)$ of M such that $\partial_i = \frac{\partial}{\partial x_i} = \varphi X_i$, $i = 1, \cdots, m$, where $X_1, \cdots, X_m$ form an orthonormal tangent basis on M and φ is a positive function on some open set in M. Thus the components of the first fundamental form g on M are $\langle f_i, f_j\rangle = \varphi^2 \delta_{ij}$, $i, j = 1, \cdots, m$. In terms of the chosen tangent basis it is easily seen that

$$\nabla_{X_j} X_i = \sum_{k=1}^m \gamma_{ij}^k X_k, \quad \gamma_{ij}^k = -\frac{1}{\varphi}(X_j(\varphi)\delta_i^k - \Gamma_{ij}^k), \tag{2.3}$$

where Γ_{ij}^k are the Christoffel symbols of M. Hence $s(\nabla_{X_j} X_i) = \sum_{k=1}^m \gamma_{ij}^k s_k$, and we have

$$X_j(s_i) = s_{ij} + \sum_{k=1}^m \gamma_{ij}^k s_k. \tag{2.4}$$

Let ξ be a unit space-like normal vector field on M^m in $\widetilde{M}_1^{m+2}(c)$. Then η is time-like, $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. The normal exponential mapping of M^m in $\widetilde{M}_1^{m+2}(c)$ in direction ξ is given by

$$\exp(x, t\xi) = a(t)f(x) + b(t)\xi(x),$$

where $x \in M$ and $t \in \mathbb{R}$. The functions $a(t)$ and $b(t)$ are given by $a(t) = 1$, $b(t) = t$ if $c = 0$; $a(t) = \cos t$, $b(t) = \sin t$ if $c = 1$; and $a(t) = \cosh t$, $b(t) = \sinh t$ if $c = -1$.

3. Maximal hypersurfaces of $\widetilde{M}_1^{m+2}(c)$

Let $f : M^m \to \widetilde{M}_1^{m+2}(c)$ be a smooth totally geodesic isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. Let I be an open interval containing zero such that $I \subset (-\pi/2, \pi/2)$ when $c = 1$. We then define a map $F : M \times I \to \widetilde{M}_1^{m+2}(c)$ by

$$F(x,t) = \exp(x, t\xi). \tag{3.1}$$

The hypersurface $F(M \times I)$ is the part of the image of the subbundle, spanned by the space-like nonparallel unit normal vector field ξ, of the normal bundle $T^{\perp}M$ under the normal exponential mapping of M in $\widetilde{M}_1^{m+2}(c)$.

The tangent vectors to the hypersurface at $(x_1, \cdots, x_m, t)$ are expressed as

$$F_i = \frac{\partial F}{\partial x_i} = af_i + b\xi_i, \ \ i = 1, \cdots, m, \text{ and } F_t = \frac{\partial F}{\partial t} = a'f + b'\xi,$$

where $F_i,\ F_t,\ f_i,\ \xi_i, ...$ denote the derivatives of $F,\ f$, and ξ with respect to x_i and t; a' and b' are, respectively, the derivatives of $a(t)$ and $b(t)$. As f is totally geodesic we have $A_\xi \equiv 0$ and $A_\eta \equiv 0$. So,

$$F_i = \varphi(aX_i + bD_{X_i}\xi) = \varphi(aX_i + b\nabla^{\perp}_{X_i}\xi) = \varphi(aX_i - bs_i\eta), \quad i = 1, \cdots, m,$$

where D is the covariant differentiation in $\mathbb{R}_1^{m+2}$ or $\mathbb{R}_d^{m+3},\ d = 1, 2$. Hence

$$\langle F_i, F_j\rangle = \varphi^2(a^2\delta_{ij} - b^2 s_i s_j), \ \ \langle F_i, F_t\rangle = 0, \ \ \langle F_t, F_t\rangle = 1, \tag{3.2}$$

where $i, j = 1, \cdots, m$. The tangent vector F_t is space-like, and the tangent vectors $F_i,\ i = 1, \cdots, m,$ are space-like if $\langle F_i, F_i\rangle = \varphi^2(a^2 - b^2 s_i^2) > 0,\ i = 1, \cdots, m$, that is, the map F is space-like. Therefore we have the metric G on $M \times I$ induced by F as

$$G = \left(\begin{array}{c|c} \varphi^2(a^2\delta_{ij} - b^2 s_i s_j) & 0 \\ \hline 0 & 1 \end{array}\right).$$

Note that when the normal vector ξ is time-like, then the tangent vectors F_i's are space-like without any restriction and the tangent vector F_t is time-like, and hence F is Lorentzian, which was studied in [6].

We need the following Lemma to show that the map F is an immersion.

Lemma 3.1. *Let $E = I + \mu v^T v$ be an $m \times m$ matrix, where I is the $m \times m$ identity matrix and $v = (v_1, \cdots, v_m) \in \mathbb{R}^m$. Then E has two distinct eigenvalues 1 and $1 + \mu\|v\|^2$ with multiplicities $m-1$ and 1, respectively, and further $\det E = 1 + \mu\|v\|^2$ and the matrix $I - \mu\frac{1}{\det E}v^T v$ is the inverse of E, where $\mu = \pm 1$ and $\det E \neq 0$ when $\mu = -1$.*

For $\mu = 1$, the Lemma was proved in [7] and the Lemma can be proved similarly for $\mu = -1$.

Proposition 3.2. *Let $f : M^m \to \widetilde{M}_1^{m+2}(c)$ be a smooth totally geodesic isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)-$dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. If $\alpha^2 = a^2 - b^2(s_1^2 + \cdots + s_m^2) > 0$ on some connected open subset $U \subset M \times I$, then the map $F : U \subset M \times I \to \widetilde{M}_1^{m+2}(c)$ defined by (3.1) is a space-like immersion.*

Proof. As f is totally geodesic, using the Lemma 3.1, the determinant of G is calculated as

$$\begin{aligned}\det G =& \det(\varphi^2(a^2\delta_{ij} - b^2 s_i s_j)) = (a^2\varphi^2)^m \det(\delta_{ij} - \frac{b^2}{a^2} s_i s_j)\\ =&(a^2\varphi^2)^m\{1 - \frac{b^2}{a^2}(s_1^2 + \cdots + s_m^2)\} = a^{2(m-1)}\varphi^{2m}(a^2 - b^2\hat{s}^2),\end{aligned}$$

where $\hat{s}^2 = s_1^2 + \cdots + s_m^2$. Since φ is a positive function, $\alpha^2 = a^2 - b^2(s_1^2 + \cdots + s_m^2) > 0$ on the open subset U and the functions $a(t)$ and $b(t)$ have no zeros simultaneously, then $\det G = 0$ if and only if $a(t) = 0$. Therefore F is an immersion if and only if $a(t) \neq 0$. In fact, for $c = 0$ and $c = -1$, respectively, $a(t) = 1$ and $a(t) = \cosh t$, which have no zeros for all $t \in \mathbb{R}$, and for $c = 1$, $a(t) = \cos t \neq 0$ on $I \subset (-\pi/2, \pi/2)$.

From the condition $a^2 - b^2(s_1^2 + \cdots + s_m^2) > 0$, we have $a^2 - b^2 s_i^2 > 0$, $i = 1, \cdots, m$. Thus the coordinate vectors F_i, $i = 1, \cdots, m$, are all space-like. As F_t is also space-like, then F is a space-like immersion. □

If F is an immersion, then from the Lemma 3.1, the inverse of G is obtained as

$$G^{-1} = \left(\begin{array}{c|c} \frac{1}{\alpha^2\varphi^2 a^2}(\alpha^2\delta_{ij} + b^2 s_i s_j) & 0 \\ \hline 0 & 1 \end{array}\right).$$

By considering (2.3) and (2.4), the second derivatives of F are calculated as

$$\begin{aligned}(3.3)\quad F_{ij} &= \frac{\partial^2 F}{\partial x_i \partial x_j}\\ &= \frac{\partial \varphi}{\partial x_j}(aX_i - bs_i\eta) + \varphi^2(aD_{X_j}X_i - bX_j(s_i)\eta - bs_i\nabla^{\perp}_{X_j}\eta)\\ &= X_j(\varphi)F_i + \varphi^2\{\sum_{k=1}^m \gamma_{ij}^k(aX_k - bs_k\eta) - ac\delta_{ij}f + b(s_i s_j\xi - s_{ij}\eta)\}\\ &= \sum_{k=1}^m (X_j(\varphi)\delta_{ik} + \varphi\gamma_{ij}^k)F_k + b\varphi^2(s_i s_j\xi - s_{ij}\eta) - ac\varphi^2\delta_{ij}f\\ &= \sum_{k=1}^m \Gamma_{ij}^k F_k + b\varphi^2(s_i s_j\xi - s_{ij}\eta) - ac\varphi^2\delta_{ij}f, \quad i, j = 1, \cdots, m,\end{aligned}$$

$$F_{it} = \varphi(a'X_i - b's_i\eta), \quad i = 1, \cdots, m, \quad \text{and} \quad F_{tt} = (a''f + b''\xi) = -cF.$$

Let $\bar{h}^N$ denotes the second fundamental form of F relative to the unit normal vector N to F in $\widetilde{M}_1^{m+2}(c)$. So, for the coordinate vector fields $\partial_1, \ldots, \partial_m, \partial_t,$ if we use the Gauss formula for F, then we have

$$\bar{h}^N(\partial_i, \partial_j) = \langle F_{ij}, N\rangle, \quad \bar{h}^N(\partial_i, \partial_t) = \langle F_{it}, N\rangle, \quad \bar{h}^N(\partial_t, \partial_t) = \langle F_{tt}, N\rangle.$$

□

We prove the following theorem.

Theorem 3.3. *Let $f : M^m \to \widetilde{M}_1^{m+2}(c)$ be a smooth totally geodesic isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. If $\alpha^2 = a^2 - b^2(s_1^2 + \cdots + s_m^2) > 0$ on some connected open subset $U \subset M \times I$, then the immersion $F : U \subset M \times I \to \widetilde{M}_1^{m+2}(c)$ defined by (3.1) is maximal if and only if the components, s_i, of the normal connection form s of f satisfy the following equations*

$$\sum_{i=1}^{m} s_{ii} = 0 \quad \text{and} \quad \sum_{i,j=1}^{m} s_i s_j s_{ji} = 0. \tag{3.4}$$

Proof. Let A_N denote the shape operator of F in $\widetilde{M}_1^{m+2}(c)$. By virtue of (2.1), it is given by $A_N = G^{-1}\bar{h}$. Hence we can write the mean curvature vector H of F in $\widetilde{M}_1^{m+2}(c)$ as

$$\begin{aligned} H &= \frac{1}{(m+1)\alpha^2 a^2 \varphi^2} \sum_{i,j=1}^{m} (\alpha^2 \delta_{ij} + b^2 s_i s_j)\langle F_{ji}, N\rangle N \\ &= \frac{1}{(m+1)\alpha^2 a^2 \varphi^2} \sum_{i,j=1}^{m} (\alpha^2 \delta_{ij} + b^2 s_i s_j)(F_{ji})^{\perp}, \end{aligned}$$

where $(F_{ji})^{\perp}$ denotes the projection of F_{ji} on the normal space of F in $\widetilde{M}_1^{m+2}(c)$. If $c \neq 0$, then F is maximal if and only if

$$f \wedge \xi \wedge F_1 \wedge \cdots \wedge F_m \wedge \sum_{i,j=1}^{m} (\alpha^2 \delta_{ij} + b^2 s_i s_j) F_{ji} = 0, \tag{3.5}$$

as $F \wedge F_t = f \wedge \xi$. If $c = 0$, then $F_t = \xi$ and thus F is maximal if and only if

$$\xi \wedge F_1 \wedge \cdots \wedge F_m \wedge \sum_{i,j=1}^{m} (\alpha^2 \delta_{ij} + b^2 s_i s_j) F_{ji} = 0. \tag{3.6}$$

Note that (3.5) and (3.6) do not depend on the chosen local coordinate system. For $c \neq 0$, using (3.3) we obtain

$$f \wedge \xi \wedge F_1 \wedge \cdots \wedge F_m \wedge \sum_{i,j=1}^{m} (\alpha^2 \delta_{ij} + b^2 s_i s_j)\varphi^2 b^2 s_{ji}\eta = 0$$

$$\Longleftrightarrow \sum_{i,j=1}^{m} (\alpha^2\delta_{ij} + b^2 s_i s_j) s_{ji} = 0 \Longleftrightarrow \sum_{i=1}^{m} s_{ii} = 0 \text{ and } \sum_{i,j=1}^{m} s_i s_j s_{ji} = 0.$$

Since M is totally geodesic, then we have $s_{ij} = s_{ji}$ from the Ricci equation (2.2). Similarly, the conditions (3.4) are valid for $c = 0$. □

Note that the hypersurface $F(U)$, which is the part of the image of the subbundle, spanned by the unit space-like nonparallel normal vector field ξ, of the normal bundle $T^{\perp}M$ under the normal exponential mapping of M in $\widetilde{M}_1^{m+2}(c)$ is equivalent the following two conditions: (1) $F(U)$ is foliated by the geodesic of $\widetilde{M}_1^{m+2}(c)$, (2) m-dimensional distribution on $F(U)$ orthogonal to the geodesics in (1) is locally integrable.

4. Construction of examples

Here we construct some examples of the maximal immersion, defined as in the previous section, into space forms $\widetilde{M}_1^{m+2}(c)$. We consider a totally geodesic isometric immersion $f : M^m(c) \to \widetilde{M}_1^{m+2}(c)$ from an m-dimensional Riemannian space form $M^m(c)$ into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$ defined by

$$f(x_1, \cdots, x_m) = \begin{cases} (0, x_1, \cdots, x_m, 0) & \text{if} \quad c = 0, \\ \frac{1}{r^2}(0, c(r^2-2), 2x_1, \cdots, 2x_m, 0) & \text{if} \quad c = \mp 1, \end{cases}$$

where $x_1, \cdots, x_m \in \mathbb{R}$, $r^2 = 1 + c(x_1^2 + \cdots + x_m^2)$ and for $c = -1$, $x_1^2 + \cdots + x_m^2 < 1$.

We will do all computations for $c = \mp 1$. By a direct computations, the components of the induced first fundamental form on $M^m(c)$ are obtained as $\langle f_i, f_j \rangle = \frac{4}{r^4}\delta_{ij}$, $i, j = 1, \cdots, m$, which means that the chosen coordinate system on M is isothermal and $\varphi = \frac{2}{r^2}$. Thus, $X_i = \frac{r^2}{2}\frac{\partial}{\partial x_i}$, $i = 1, \cdots, m$, form a local orthonormal tangent basis on $M^m(c)$. In terms of this metric the Christoffel symbols are obtained as

$$\Gamma_{ij}^k = -\frac{2c}{r^2}(x_i\delta_{kj} + x_j\delta_{ik} - x_k\delta_{ij}). \tag{4.1}$$

For the normal space of $M^m(c)$ in $\widetilde{M}_1^{m+2}(c)$, an orthonormal local basis can, generally, be chosen as

$$\xi = (\sinh\theta, 0, \cdots, 0, \cosh\theta), \quad \eta = (\cosh\theta, 0, \cdots, 0, \sinh\theta),$$

where $\theta = \theta(x_1, \cdots, x_m)$ is a smooth function on some open subset of M. We will find θ, which determines the unit nonparallel normal vector ξ on $M^m(c)$ such that the immersion F defined in previous section is maximal on some open subset $U \subset M \times I$ under the condition $\alpha^2 = a^2 - b^2(s_1^2 + \cdots + s_m^2) > 0$. Now we will calculate the components s_i of the normal connection s and their covariant derivatives s_{ij}. From the definition of s_i, we have

$$s_i = \langle \nabla_{X_i}^{\perp}\xi, \eta \rangle = \langle D_{X_i}\xi, \eta \rangle = \frac{r^2}{2}\langle \frac{\partial \xi}{\partial x_i}, \eta \rangle = \frac{r^2}{2}\frac{\partial \theta}{\partial x_i}, \tag{4.2}$$

that is, $s_i = -\frac{r^2}{2}\theta_i$, $i = 1, \cdots, m$, and hence

$$X_j(s_i) = \frac{r^2}{2}\frac{\partial}{\partial x_j}(\frac{r^2}{2}\theta_i) = -\frac{r^2}{2}(cx_j\theta_i + \frac{r^2}{2}\theta_{ij}).$$

Using the equations (2.3) and (4.1), we have $\gamma_{ij}^k = -c(x_i\delta_{kj} - x_k\delta_{ij})$. Therefore, by considering (2.4), we obtain

$$s_{ij} = -\frac{r^2}{2}[cx_j\theta_i + \frac{r^2}{2}\theta_{ij} + c\sum_{k=1}^{m}(x_i\delta_{kj} - x_k\delta_{ij})\theta_k] = \frac{r^4}{4}(-\theta_{ij} + \sum_{k=1}^{m}\Gamma_{ij}^k\theta_k). \tag{4.3}$$

Here it is clear that $s_{ij} = s_{ji}$ if and only if $\theta_{ij} = \theta_{ji}$. Thus, by using (4.2) and (4.3), the first equation of (3.4) turns out to be

$$2c(2-m)\tilde{\theta} + r^2\sum_{i=1}^{m}\theta_{ii} = 0, \tag{4.4}$$

and the second equation of (3.4) becomes

$$2c\tilde{\theta}\hat{\theta}^2 + r^2\sum_{i,j=1}^{m}\theta_i\theta_j\theta_{ji} = 0, \tag{4.5}$$

where $\tilde{\theta} = \sum_{i=1}^m x_i\theta_i$ and $\hat{\theta}^2 = \sum_{i=1}^m \theta_i^2$.

For $c = 0$, by a straightforward computation we can see that the equations (4.4) and (4.5) are valid.

The solutions of the partial differential equations (4.4) and (4.5), which were studied for $m = 2$ in [8] give us the function θ, that is, the normal vector fields on $M^m(c)$ which are not parallel in the normal space of f in $\widetilde{M}_1^{m+2}(c)$ unless θ is a constant function on $M^m(c)$. For $m > 2$, the following some special solutions of the equations (4.4) and (4.5) were studied in [7]. In the domains of these solutions, the map F may not be a space-like immersion. For some solutions, we will obtain some connected open subset of the domain of f to find an interval for t with the restriction $\alpha^2 = a^2 - b^2\hat{s}^2 > 0$ such that F is a smooth maximal immersion.

Example 4.1. For $c = 0$, we have a linear solution of the equations (4.4) and (4.5), which is

$$\theta(x_1, \cdots, x_m) = C_1x_1 + \cdots + C_mx_m, \tag{4.6}$$

and hence

$$F(x_1, \cdots, x_m, t) = \big(t\sinh\theta, x_1, \cdots, x_m, t\cosh\theta\big),$$

is a smooth maximal immersion for $(x_1, \cdots, x_m) \in \mathbb{R}^m$ and $|t| < \frac{2}{\sqrt{C_1^2+\cdots+C_m^2}}$, which comes from the condition $\alpha^2 = a^2 - b^2\hat{s}^2 > 0$.

Let ℓ be a positive integer such that $\ell \leq m/2, \quad m \geq 2$. For $c = -1, 0, 1$, the function

$$\theta(x_1, \cdots, x_m) = \sum_{i=1}^{\ell} C_i \arctan \frac{x_{2i}}{x_{2i-1}}, \tag{4.7}$$

is a solution of the equations (4.4) and (4.5) in the open domain $D_0 = \{(x_1, \cdots, x_m) \in \mathbb{R}^m : \ x_{2i-1} \neq 0, \ i = 1, \cdots, \ell\}$, where $C_1, \cdots C_\ell \in \mathbb{R}$. However, the map F may not be a space-like immersion on D_0. For the following examples, we will consider some connected open subset of D_0 to find an interval for the parameter t such that F is a smooth maximal immersion. Moreover, we will show that F is a ruled immersion. Let $c_1 \neq 0$. Then $x_1 = \theta - \bar{C}_2 x_2 - \cdots - \bar{C}_m x_m, \ \bar{C}_i = C_i / C_1, \ i = 2, \cdots, m$, and

$$F(\theta, x_2 \cdots, x_m, t) = \big(t \sinh \theta, \ \theta - \bar{C}_2 x_2 - \cdots - \bar{C}_m x_m, x_2, \cdots, x_m, t \cosh \theta\big).$$

This expression means that F is a ruled maximal immersion in $\mathbb{R}^{m+2}_1$ because $x_2 \cdots, x_m, t$ are linear parameters, which span totally geodesic m-planes $\mathbb{R}^m$.

Example 4.2. For $c = 0$, a linear combinations of the solutions (4.6) and (4.7) gives

$$\theta(x_1, \cdots, x_{2\ell}, x_{2\ell+1}, \cdots, x_m) = \sum_{i=1}^{\ell} C_i \arctan \frac{x_{2i}}{x_{2i-1}} + \sum_{j=2\ell+1}^{m} C_i x_i, \tag{4.8}$$

which is also a solution of (4.4) and (4.5) in the open domain D_0, (see [7]). Now let us consider the open set $W(\rho_0) = \{(x_1, \cdots, x_m) \in \mathbb{R}^m : \ x_1^2 + \cdots + x_{2\ell}^2 > \rho_0^2\}$ for some constant ρ_0. Using (4.2), when we calculate $\hat{s}^2$ for the function (4.8) we obtain

$$\hat{s}^2 = \frac{1}{4}\left(\theta_1^2 + \cdots + \theta_m^2\right) = \frac{1}{4}\left(\frac{C_1^2}{x_1^2 + x_2^2} + \cdots + \frac{C_\ell^2}{x_{2\ell-1}^2 + x_{2\ell}^2} + C_{2\ell+1}^2 + \cdots + C_m^2\right).$$

On the set $W(\rho_0)$, $x_{2i-1}^2 + x_{2i}^2 > \rho_0^2$ for $i = 1, 2, \cdots, \ell$. Hence

$$\hat{s}^2 < \frac{1}{4}\left(\frac{C_1^2 + \cdots + C_\ell^2}{\rho_0^2} + C_{2\ell+1}^2 + \cdots + C_m^2\right).$$

Therefore we have

$$\alpha^2 = 1 - t^2 \hat{s}^2 > 1 - \frac{t^2}{4}\left(\frac{C_1^2 + \cdots + C_\ell^2}{\rho_0^2} + C_{2\ell+1}^2 + \cdots + C_m^2\right) > 0,$$

as $a(t) = 1$ and $b(t) = t$. This implies that

$$|t| < \frac{2\rho_0}{\sqrt{C_1^2 + \cdots + C_\ell^2 + \rho_0^2(C_{2\ell+1}^2 + \cdots + C_m^2)}} = t_0.$$

As a result, the map F given by

$$F(x_1,\cdots,x_m,t)=\Big(t\sinh(\sum_{i=1}^{\ell}C_i\arctan\frac{x_{2i}}{x_{2i-1}}+\sum_{i=2\ell+1}^{m}C_ix_i),x_1,\cdots,x_m,$$
$$t\cosh(\sum_{i=1}^{\ell}C_i\arctan\frac{x_{2i}}{x_{2i-1}}+\sum_{i=2\ell+1}^{m}C_ix_i)\Big),$$

is a smooth maximal immersion on each connected component of the open set $U=(D_0\cap W(\rho_0))\times(-t_0,t_0)$.

For $m=2$ ($\ell=1$), if we put $u=\sqrt{x_1^2+x_2^2}$, then we can write the map F as

$$F(\theta,u,t)=(t\sinh\theta,u\cos\frac{\theta}{C_1},u\sin\frac{\theta}{C_1},t\cosh\theta),$$

which shows that F is a 2-ruled maximal immersion in $\mathbb{R}^4_1$ as u and t are linear parameters that span totally geodesic planes $\mathbb{R}^2$.

Example 4.3. For $c=1$, we consider the solution (4.7) of the equations (4.4) and (4.5). Let $W(\rho_0\mu_0)=\{(x_1,\cdots,x_m)\in\mathbb{R}^m:\ \rho_0^2<x_1^2+\cdots+x_m^2<\mu_0^2\}$ for some constants $\rho_0<1$ and $\mu_0>1$. Since $r^2=1+x_1^2+\cdots+x_m^2$, when we calculate $\hat{s}^2$ we get

$$\hat{s}^2=\frac{r^2}{4}\left(\theta_1^2+\cdots+\theta_m^2\right)=\frac{(1+x_1^2+\cdots+x_m^2)^2}{4}\sum_{i=1}^{\ell}\frac{C_i^2}{x_{2i-1}^2+x_{2i}^2}.$$

On the set $W(\rho_0\mu_0)$ we can write $x_{2i-1}^2+x_{2i}^2>\rho_0^2$ for $i=1,\cdots,\ell$. Hence, on the set $D_0\cap W(\rho_0\mu_0)$

$$\hat{s}^2<\frac{(1+\mu_0^2)^2}{4}\sum_{i=1}^{\ell}\frac{C_i^2}{x_{2i-1}^2+x_{2i}^2}<\frac{(1+\mu_0^2)^2}{4}\sum_{i=1}^{\ell}\frac{C_i^2}{\rho_0^2}=\frac{(1+\mu_0^2)^2}{4\rho_0^2}\sum_{i=1}^{\ell}C_i^2.$$

Therefore, using $a(t)=\cos t,\quad b(t)=\sin t$, we have

$$\alpha^2=\cos^2t-\sin^2t\,\hat{s}^2>\cos^2t-\sin^2t\,\frac{(1+\mu_0^2)^2}{4\rho_0^2}\sum_{i=1}^{\ell}C_i^2>0,$$

which gives us $|t|<\arctan\left(\dfrac{2\rho_0}{(1+\mu_0^2)\sqrt{\sum_{i=1}^{\ell}C_i^2}}\right)=\tilde{t}_0$. Thus the map F

$$F(x_1,\cdots,x_m,t)=\Big(\sin t\sinh(\sum_{i=1}^{\ell}C_i\arctan\frac{x_{2i}}{x_{2i-1}}),\frac{(r^2-2)\cos t}{r^2},$$
$$\frac{2x_1\cos t}{r^2},\cdots,\frac{2x_m\cos t}{r^2},\sin t\cosh(\sum_{i=1}^{\ell}C_i\arctan\frac{x_{2i}}{x_{2i-1}})\Big),$$

is a smooth maximal immersion on each connected component of the open set $U = (D_0 \cap W(\rho_0\mu_0)) \times (-\tilde{t}_0, \tilde{t}_0)$, where $r^2 = 1 + x_1^2 + \cdots + x_m^2$.

Example 4.4. For $c = -1$, we reconsider the solution (4.7) of the equations (4.4) and (4.5). Let $W(\rho_0 1) = \{(x_1, \cdots, x_m) \in \mathbb{R}^m : \ \rho_0^2 < x_1^2 + \cdots + x_m^2 < 1\}$ for some constants $\rho_0 < 1$. Since $r^2 = 1 - x_1^2 - \cdots - x_m^2$, when we calculate $\hat{s}^2$ we get

$$\hat{s}^2 = \frac{r^2}{4}\left(\theta_1^2 + \cdots + \theta_m^2\right) = \frac{(1 - x_1^2 - \cdots - x_m^2)^2}{4} \sum_{i=1}^{\ell} \frac{C_i^2}{x_{2i-1}^2 + x_{2i}^2}.$$

On the set $W(\rho_0 1)$ we can write $x_{2i-1}^2 + x_{2i}^2 > \rho_0^2$ for $i = 1, \cdots, \ell$, and thus

$$\hat{s}^2 < \frac{(1-\rho_0^2)^2}{4} \sum_{i=1}^{\ell} \frac{C_i^2}{x_{2i-1}^2 + x_{2i}^2} < \frac{(1-\rho_0^2)^2}{4} \sum_{i=1}^{\ell} \frac{C_i^2}{\rho_0^2} = \frac{(1-\rho_0^2)^2}{4\rho_0^2} \sum_{i=1}^{\ell} C_i^2.$$

Therefore, using $a(t) = \cosh t, \ b(t) = \sinh t$, we have

$$\alpha^2 = \cosh^2 t - \sinh^2 t\, \hat{s}^2 > \cosh^2 t - \sinh^2 t\, \frac{(1-\rho_0^2)^2}{4\rho_0^2} \sum_{i=1}^{\ell} C_i^2 > 0,$$

which gives us $|t| < \tanh^{-1}\left(\dfrac{2\rho_0}{(1-\rho_0^2)\sqrt{\sum_{i=1}^{\ell} C_i^2}}\right) = \hat{t}_0$, and the map F becomes

$$F(x_1, \cdots, x_m, t) = \Bigg(\sinh t \sinh\Big(\sum_{i=1}^{\ell} C_i \arctan\frac{x_{2i}}{x_{2i-1}}\Big), \frac{(2 - r^2)\cosh t}{r^2},$$
$$\frac{2x_1 \cosh t}{r^2}, \cdots, \frac{2x_m \cosh t}{r^2}, \sinh t \cosh\Big(\sum_{i=1}^{\ell} C_i \arctan\frac{x_{2i}}{x_{2i-1}}\Big)\Bigg),$$

which is a smooth maximal immersion on each connected component of the open set $U = (D_0 \cap W(\rho_0 1)) \times (-\hat{t}_0, \hat{t}_0)$, where $r^2 = 1 - x_1^2 - \cdots - x_m^2$.

For $c = -1, 0, 1$ and $m > 2$, we have another solution of the differential equations (4.4) and (4.5) from [7] as

$$\theta(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m) = \arctan\left(\frac{C_1 x_1 + \cdots + C_n x_n}{C_{n+1}x_{n+1} + \cdots + C_m x_m}\right), \tag{4.9}$$

when $\sum_{i=1}^{n} C_i^2 = \sum_{i=n+1}^{m} C_i^2$ in the open domain $D = \{(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m) \in \mathbb{R}^m : \sum_{i=n+1}^{m} C_i x_i \neq 0,$ and if $c = -1, x_1^2 + \cdots + x_m^2 < 1\}$, where $C_1, \cdots C_m \in \mathbb{R}$. As in the above examples, it can be shown that for some connected open subset of D, there is an open interval for t for which F is a smooth maximal immersion.

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