KYUNGPOOK Math. J. 48(2008), 101-108

Global Small Solutions of the Cauchy Problem for Non-isotropic Schrödinger Equations

Xiangqing Zhao
Department of Mathematics, Zhejiang Ocean University, Zhoushan, Zhejiang 316000, P. R. China
e-mail : zhao-xiangqing@163.com

Shangbin Cui
Department of Mathematics, Sun Yat-Sen University, Guangzhou, Guangdong 510275, P. R. China
e-mail : cuisb3@yahoo.com.cn

Abstract. In this paper we study the existence of global small solutions of the Cauchy problem for the non-isotropically perturbed nonlinear Schrödinger equation: $iu_t + \Delta u + |u|^\alpha u + a\sum_i^d u_{x_ix_ix_ix_i} = 0$, where a is real constant, $1 \leq d < n$ is a integer, α is a positive constant, and $x = (x_1, x_2, \cdots, x_n) \in R^n$. For some admissible α we show the existence of global(almost global) solutions and we calculate the regularity of those solutions.

1. Introduction

This paper is concerned with the Cauchy problem of the following fourth-order nonlinear dispersive equation in $R^n \times R$:

$$(1)\qquad \begin{cases} iu_t + \Delta u + |u|^\alpha u + a\sum_i^d u_{x_ix_ix_ix_i} = 0, & x \in R^n,\ t \in R. \\ u(x,0) = \varphi(x), & x \in R^n. \end{cases}$$

where d is an integer, $1 \leq d < n$, a is a nonzero real constants, and α is a positive constant. This equation is a modified version of the semi-discrete nonlinear Schrodinger equation (see [1]), or a non-isotropic higher-order perturbation of the second-order nonlinear Schrodinger equation:

$$(2)\qquad iu_t + \Delta u + |u|^\alpha u = 0,\ x \in R^n,\ t \in R.$$

Clearly, the following equations are special cases of (1)

$$iu_t(x,y,z,t) + \Delta u + |u|^\alpha u + au_{xxxx} = 0,$$

Received August 28, 2006.
2000 Mathematics Subject Classification: 35Q53, 35Q35.
Key words and phrases: non-isotropic Schrödinger equations, global solutions, almost global solutions, regularity.

$$iu_t(x,y,t) + \Delta u + |u|^\alpha u + au_{xxxx} = 0.$$

The first equation arose in the study of solitary wave propagation in a bulk medium, and the second equation arose in a similar problem in a planar waveguide geometric medium (see [1]-[2]). We also refer the reader to see Fibich, Ilan and Schochet [3] and Fibich and Papanicolaou [4]-[5] for more details of the physical background of these equations.

Ribaud and Youssfi in [6] studied self-similar solution of the classical Schrödinger equation (2). To establish self-similar solutions, they investigate global small solutions first. By using the general existence result to a special class of initial datum they obtained the existence of self-similar solutions. As an byproduct, they also obtain regularity of the solutions. Because of the lack of scaling invariant for the non-isotropically perturbed nonlinear Schrödinger equation (1), we will only study its global small solutions and calculate the regularity of those solutions.

Before stating our main results precisely, we first introduce some notations. As usual $S(R^n)$ denotes the Schwartz's space of test functions, $S'(R^n)$ is its dual. Let p and s be reals such that $1 < p < \infty$, $0 < s < \frac{n}{p}$. Let ∇^s be the pseudo-differential operator with symbol $|\xi|^s$, then $\dot{H}^s_p(R^n)$ is the set of all $f \in L^p(R^n)$ such that $\nabla^s f \in L^p(R^n)$. So $\dot{H}^s_p(R^n)$ is a Banach space of tempered distributions equipped with the norm $\|f\|_{\dot{H}^s_p(R^n)} = \|\nabla^s f\|_{L^p(R^n)}$. We take the solution spaces E_s and $E^s_{T_0}$ to be the spaces of all Bochner measurable functions $u : (0,\infty) \to \dot{H}^s_p(R^n)$ such that

$$\|u\|_{E^s} \equiv \sup_{t>0} t^\theta \|u\|_{\dot{H}^s_p(R^n)} < \infty,$$

and

$$\|u\|_{E^s_{T_0}} \equiv \sup_{0<t\le T_0} t^\theta \|u\|_{\dot{H}^s_p(R^n)} < \infty,$$

for any T_0 satisfying $0 < T_0 < \infty$.

Where θ will be determined in the proof of the main theorem later.

As in [6], we define the admissible α and the range of regularity for the solutions: I_α.

Definition 1.

1) If $\alpha < 1$, $I_\alpha = \{0\} \cap (\frac{n}{2} - \frac{2n(\alpha+2)}{(2n-d)\alpha}, \frac{n}{2} - \frac{2n(\alpha+2)}{(2n-d)\alpha(\alpha+1)})$.

2) If $\alpha \ge 1$ and $\alpha \notin 2N$, $I_\alpha = (\frac{n}{2} - \frac{2n(\alpha+2)}{(2n-d)\alpha}, \frac{n}{2} - \frac{2n(\alpha+2)}{(2n-d)\alpha(\alpha+1)}) \cap [0,\alpha)$.

3) If $\alpha \ge 1$ and $\alpha \in 2N$, $I_\alpha = (\frac{n}{2} - \frac{2n(\alpha+2)}{(2n-d)\alpha}, \frac{n}{2} - \frac{2n(\alpha+2)}{(2n-d)\alpha(\alpha+1)}) \cap [0,\infty)$.

Definition 2. We will say that α is admissible if I_α is not empty.

The main result is

Theorem. *Let α be admissible, $s \in I_\alpha$, and set $u_0(t,x) = [S(t)\phi](x)$.*

1) *(global solution) Let* $a < 0$, *if there exists* $0 < \varepsilon \ll 1$ *such that*

$$\|u_0\|_{E^s} \leq \varepsilon,$$

then there exists a unique solution $u(t,x) \in E^s$ *of system (1) with* $\|u\|_{E^s} \leq 2\varepsilon$.

2) *(almost global solution) Let* $a > 0$, *if there exists* $0 < \varepsilon \ll 1$ *such that*

$$\|u_0\|_{E^s_{T_0}} \leq \varepsilon,$$

then there exists a unique solution $u(t,x) \in E^s_{T_0}$ *of system (1) with* $\|u\|_{E^s_{T_0}} \leq 2\varepsilon$. *Here* T_0 *is an arbitrary positive number satisfying* $0 < T_0 < \infty$.

In the sequel, C will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters and for $p \geq 1$ we set $p' = \frac{p}{p-1}$.

2. Preliminary lemmas

The solutions of the free equation

$$\begin{cases} iu_t + \Delta u + a\sum_i^d u_{x_ix_ix_ix_i} = 0, & x \in R^n,\ t \in R. \\ u(x,0) = \varphi(x), & x \in R^n. \end{cases}$$

is

$$u(x,t) = S(t)\varphi = (I * \varphi)(x,t),$$

where

$$I(x,t) = (2\pi)^{-n} \int_{R^n} e^{2\pi x\cdot\xi + it((-|\xi|^2 + a\sum_{k=1}^d \xi_k^4)d\xi}.$$

Lemma 1. *We have the* $L^{p'} - L^p$ *estimates for the linear operator* $S(t)$ *as follows:*

1) *let* $a > 0$, *then* $\forall\phi \in L^{p'}$, $0 < T_0 < \infty$, *we have*

$$\|S(t)\phi\|_{L^p} \leq C|t|^{-\frac{2n-d}{4}(1-\frac{2}{p})}\|\phi\|_{L^{p'}}, \quad 0 < |t| \leq T_0. \tag{3}$$

Furthermore,

$$\|S(t)\phi\|_{\dot{H}^s_p} \leq C|t|^{-\frac{2n-d}{4}(1-\frac{2}{p})}\|\phi\|_{\dot{H}^s_{p'}}, \quad 0 < |t| \leq T_0,$$

where $s \in R$ *and* $2 \leq p \leq \infty$.

2) *let* $a < 0$, *then* $\forall\phi \in L^{p'}$, $|t| \neq 0$ *we have*

$$\|S(t)\phi\|_{L^p} \leq C|t|^{-\frac{2n-d}{4}(1-\frac{2}{p})}\|\phi\|_{L^{p'}}.$$

Furthermore,

$$\|S(t)\phi\|_{\dot{H}^s_p} \leq C|t|^{-\frac{2n-d}{4}(1-\frac{2}{p})}\|\phi\|_{\dot{H}^s_{p'}},$$

where $s \in R$ and $2 \le p \le \infty$.

Proof. Letting

$$I_j(x,t) = \begin{cases} (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i[x_j\xi_j - t(\xi_j^2 - a\xi_j^4)]} d\xi_j & \text{for } 1 \le j \le d, \\ (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i(x_j\xi_j - t\xi_j^2)} d\xi_j & \text{for } d+1 \le j \le n, \end{cases}$$

we see that

$$I(x,t) = \prod_{j=1}^{n} I_j(x,t).$$

By Lemma 2.1 of [9] (see also Theorem 1 of [10]) we have

$$I_j(x,t) = \begin{cases} C|t|^{-\frac{1}{4}} & \text{for } 1 \le j \le d, \\ C|t|^{-\frac{1}{2}} & \text{for } d+1 \le j \le n, \end{cases} \tag{4}$$

for $0 < |t| \le T_0$.
Thus

$$I(x,t) \le C|t|^{-\frac{2n-d}{4}}, \quad 0 < |t| \le T_0.$$

By Young's inequality, this implies that

$$\|S(t)\phi\|_{L^\infty} \le C|t|^{-\frac{2n-d}{4}} \|\phi\|_{L^1},\ 0 < |t| \le T_0. \tag{5}$$

Besides, since the polynomial $P(\xi)$ is real, it is clear that

$$\|S(t)\phi\|_{L^2} = \|\phi\|_{L^2}. \tag{6}$$

Hence, by interpolation between (5) and (6) we immediately obtain (3) for $0 < |t| \le T_0$.

If $a < 0$ then the polynomial $P_1(\xi) = -\xi_j^2 + a\xi_j^4$ satisfies

$$P_1''(\xi) \le 12|a|\xi_j^2, \text{ for any } \xi_j \in R.$$

It follows from the proof of Lemma 2.1 of [9] (taking δ there to be zero) that the inequalities in (4) on the part $1 \le j \le d$ hold for all $t \in R \setminus \{0\}$. Since it is well-known that the inequalities in (4) on the part $d+1 \le j \le n$ also hold for all $t \in R \setminus \{0\}$, we conclude that (5) holds for all $t \in R \setminus \{0\}$. Thus (3) holds for all $t \in R \setminus \{0\}$.

To prove the second inequality in 1), 2), the following facts should be noticed.

$\forall \omega \in S'(R^n)$, we have

$$\begin{aligned} &\mathcal{F}^{-1}(\omega \widehat{S(t)\varphi}) = \mathcal{F}^{-1}(\omega e^{it(-|\xi|^2 + a\sum_{k=1}^d \xi_k^4)} \hat{\varphi}) \\ &= \mathcal{F}^{-1}(e^{it(-|\xi|^2 + a\sum_{k=1}^d \xi_k^4)} \mathcal{F}\mathcal{F}^{-1}(\omega\hat{\varphi})) = S(t)(\mathcal{F}^{-1}(\omega\hat{\varphi})). \end{aligned}$$

In particular, by the first inequality, we obtain

$$\|\mathcal{F}^{-1}(\omega S(\hat{t})\varphi)\|_{L^p} \le C|t|^{-\frac{2n-d}{4}(1-\frac{2}{p})} \|\mathcal{F}^{-1}(\omega\phi)\|_{L^{p'}}.$$

Taking $\omega = |\xi|^s$ in the above inequality, we obtain the desired results. $\square$

Remark 1. Based on lemma 1, the so called Strichartz estimates follows easily. The Strichartz estimates and the wellposedness of (1) followed from them will appear in [7].

Lemma 2 (see [6]). *Let $\alpha > 0$, $s \in I_\alpha$ and let*

$$p = \frac{n(\alpha+2)}{\alpha s + n}$$

It holds

$$\| |f|^\alpha f \|_{\dot H^s_{p'}} \le C \|f\|^{\alpha+1}_{\dot H^s_p}, \tag{7}$$

and

$$\| |f|^\alpha f - |g|^\alpha g \|_{\dot H^s_{p'}} \le C \|f-g\|_{\dot H^s_p} \Big[\|f\|^\alpha_{\dot H^s_p} + \|g\|^\alpha_{\dot H^s_p} \Big].$$

Lemma 2 was proved in [6] with the help of a series of lemmas. Here we put great emphasis on the following one of those lemmas because I_α is determined partly by it.

Lemma 3 (see[8]). *Let $\alpha > 0$, $s > 0$ and $1 < r < \infty$ such that*

$$s < \min(\frac{n}{r}, \alpha+1), \quad (\alpha+1)(\frac{n}{r} - s) \le n. \tag{8}$$

Define t by

$$t = \frac{n}{s + (\alpha+1)(\frac{n}{r} - s)}$$

a) Then, for all $f \in \dot H^s_r$ we have:

$$\| |f|^\alpha f \|_{\dot H^s_t} \le C \|f\|^{\alpha+1}_{\dot H^s_r} \tag{9}$$

and

$$\| |f|^{\alpha+1} \|_{\dot H^s_t} \le C \|f\|^{\alpha+1}_{\dot H^s_r}. \tag{10}$$

b) Furthermore, if α is an even integer (respectively an odd integer) then (9) *(respectively* (10)*) holds without the restriction $s < \alpha + 1$.*

Remark 2. We point out that we get (7) if we take $r = p$, $t = p'$ in (9).

3. Proof of the main result

We shall make use of the fixed point theorem to solve the equivalent integral equation

$$u(t) = S(t)\phi - i \int_0^t S(t-\tau)|u|^\alpha u(\tau))d\tau.$$

Since the proof of 1) and 2) is all the same, we will prove 1) only.

Define

$$\mathcal{T}(u) = S(t)\phi - i\int_0^t S(t-\tau)|u|^\alpha u(\tau))d\tau$$

Let α be admissible and $s \in I_\alpha$. First by virtue of Lemma 1 we have

$$\begin{aligned} t^\theta\|\mathcal{T}(u)\|_{\dot{H}^s_p} &\leq t^\theta\|S(t)\phi\|_{\dot{H}^s_p} + t^\theta\int_0^t \|S(t-\tau)|u|^\alpha u(\tau))\|_{\dot{H}^s_p}d\tau \\ &\leq \|u_0\|_{E^s} + Ct^\theta\int_0^t |t-\tau|^{-\frac{2n-d}{4}(1-\frac{2}{p})}\||u|^\alpha u(\tau)\|_{\dot{H}^s_{p'}}d\tau. \end{aligned}$$

Hence from the first inequality of the proposition it follows that

$$\begin{aligned} & t^\theta\|\mathcal{T}(u)\|_{\dot{H}^s_p} \\ \leq\ & \varepsilon + Ct^\theta\int_0^t |t-\tau|^{-\frac{2n-d}{4}(1-\frac{2}{p})}\|u\|^{\alpha+1}_{\dot{H}^s_p}d\tau \\ \leq\ & \varepsilon + C\|u\|^{\alpha+1}_{E^s}t^\theta\int_0^t |t-\tau|^{-\frac{2n-d}{4}(1-\frac{2}{p})}|\tau|^{-\theta(\alpha+1)}d\tau \\ \leq\ & \varepsilon + C\|u\|^{\alpha+1}_{E^s}t^{-\frac{2n-d}{4}(1-\frac{2}{p})-\theta\alpha}\int_0^t |1-\frac{\tau}{t}|^{-\frac{2n-d}{4}(1-\frac{2}{p})}|\frac{\tau}{t}|^{-\theta(\alpha+1)}d\tau \\ \leq\ & \varepsilon + C\|u\|^{\alpha+1}_{E^s}t^{1-\frac{2n-d}{4}(1-\frac{2}{p})-\theta\alpha}\int_0^1 |1-\tau|^{-\frac{2n-d}{4}(1-\frac{2}{p})}|\tau|^{-\theta(\alpha+1)}d\tau \\ \leq\ & \varepsilon + CB(1-\frac{2n-d}{4}(1-\frac{2}{p}), 1-\theta(\alpha+1))\|u\|^{\alpha+1}_{E^s}t^{1-\frac{2n-d}{4}(1-\frac{2}{p})-\theta\alpha}. \end{aligned}$$

where $B(m,n)$ $(m,n>0)$ is the well-known Beta function.

Taking $\theta = \frac{1}{\alpha} - \frac{2n-d}{4\alpha}(1-\frac{2}{p})$, we have $1 - \frac{2n-d}{4}(1-\frac{2}{p}) - \theta\alpha = 0$ and

$$\frac{2n-d}{4}(1-\frac{2}{p}) < 1, \quad \theta(\alpha+1) < 1. \tag{11}$$

It follows that

$$t^\theta\|F(u)\|_{\dot{H}^s_p} \leq \varepsilon + C\|u\|^{\alpha+1}_{E^s}.$$

Thus we have

$$\|\mathcal{T}(u)\|_{E^s} \leq \varepsilon + C\|u\|^{\alpha+1}_{E^s}.$$

Choose $\varepsilon \leq (\frac{1}{C2^{\alpha+1}})^{\frac{1}{\alpha}}$ and let $B_\varepsilon = \{u \in E^s, \|u\|_{E^s} \leq 2\varepsilon\}$, then $L(B_\varepsilon) \subseteq B_\varepsilon$.

Next as above, by using the second inequality of the proposition we derive

$$\||u|^\alpha u - |v|^\alpha v\|_{\dot{H}^s_{p'}} \leq C\|u(\tau,\cdot) - v(\tau,\cdot)\|_{\dot{H}^s_p}[\|u(\tau,\cdot)\|^\alpha_{\dot{H}^s_p} + \|v(\tau,\cdot)\|^\alpha_{\dot{H}^s_p}],$$

so that

$$\||u|^\alpha u - |v|^\alpha v\|_{\dot{H}^s_{p'}} \leq C\tau^{-\theta(\alpha+1)}\|u-v\|_{E^s}[\|u\|^\alpha_{E^s} + \|v\|^\alpha_{E^s}].$$

Assuming that $\|u\|_{E^s} < 2\varepsilon$ and $\|v\|_{E^s} < 2\varepsilon$, the same reasoning as above gives that

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{E^s} \leq C\varepsilon^{\alpha+1}\|u-v\|_{E^s} \leq \frac{1}{2}\|u-v\|_{E^s}$$

for ε small enough. This implies that $\mathcal{T}$ is a contraction map from B_ε into B_ε. Thus, by the Banach's fixed point theorem, for all admissible α and for all $s \in I_\alpha$, there exists a unique solution $u \in E^s$ of (1) with $\|u\|_{E^s} < 2\varepsilon$. □

Remark 3.
(1) Till now, we find that I_α is determined by (11) and lemma 3, especially (8).
(2) We recall that if $a = 0$, Authors in [6] obtained global solutions with initial data ϕ in $H^{\tilde{s}}(R^n)$. In fact, assume the admissible α satisfies the following additional condition:

$$\alpha \geq \alpha_c = \frac{4}{n-2}.$$

(α_c is the so called H^1 crtical value.) and let $\phi \in H^{\tilde{s}}(R^n)$ where

$$\tilde{s} > s_c = \frac{n}{2} - \frac{2}{\alpha},$$

then, thanks to Sobolev embedding, one can check that there always exists $s \in I_\alpha$ such that $H^{\tilde{s}} \subset \dot{H}^s_p$. So, for $0 < t < T$ we have

$$t^\theta\|S(t)\phi\|_{\dot{H}^s_p} \leq t^\theta\|S(t)\phi\|_{H^{\tilde{s}}} \leq t^\theta\|\phi\|_{H^{\tilde{s}}} \leq T^\theta\|\phi\|_{H^{\tilde{s}}}.$$

Choosing T small enough, it follows that $\|S(t)\phi\|_{E_s} \leq \varepsilon$ and from the local version of the theorem there exists a unique solution $u \in E^s$ with $\phi \in H^{\tilde{s}}(R^n)$ as initial data.

As for general a, the critical indices $s_c = \frac{n}{2}(1 - \frac{8}{(2n-d)\alpha})$, but it is not easy to determine $\tilde{s}$, we'll have to leave it over.

References

[1] G. Fibich and B. Ilan, *Discretization effects in the nonlinear Schrödinger equation*, Appl. Numer. Math., **44**(2002), 63-75.

[2] S. Wen and D. Fan, *Spatiotemporal instabilities in nonlinear kerr media in the presence of arbitrary higher order dispersions*, J. Opt. Soc. Am. B, **19**(2002), 1653-1659.

[3] G. Fibich and B. Ilan and S. Schochet, *Critical exponents and collapse of nonlinear Schr Odinger equations with anisotropic fourth-order dispersion*, Nonliearity, **16**(2003), 1809-1821.

[4] G. Fibich and G. C. Papanicolaou, *A modulation method for self-focusing in the perturbed critical nonlinear Schrödinger equation*, Phys. Lett. A, **239**(1998), 167-173.

[5] G. Fibich and G. C. Papanicolaou, *Self-focusing in the presence of small time dispersion and nonparaxiality*, Opti. Lett., **22**(1997), 1379-1381.

[6] F. Ribaud and A. Youssfi, *Regular and self-similar solutions of nonlinear Schrödinger equation*, J. Math. Pures Appl., **77**(1998), 1065-1079.

[7] Cuihua Guo and Shangbin Cui, *Solvability of the Cauchy problem of nonisotropic Schrödinger equation in Sobolev spaces*, submitted.

[8] T. Runst and W. Sickel, Sobolev spaces of fractional order, Nemytskii operators and nonlinear partial differential equations, de Gruyter, Berlin, 1996.

[9] S. Cui and S. Tao, *Strichartz estimates for dispersive equations and solvability of the Kawahara equation*, J. Math. Anal. Appl., **304**(2005), 683-702.

[10] S. Cui, *Pointwise estimates for a class of oscillatory integrals and and related $L^p - L^q$ estimates*, J. Fourier Anal. Appl., **11**(2005), 441-457.