KYUNGPOOK Math. J. 48(2008), 81-92

On the Trajectory Null Scrolls in 3-Dimensional Minkowski Space-Time E_1^3

Soley Ersoy and Murat Tosun
Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, 54187 Sakarya, Turkey
e-mail : `sersoy@sakarya.edu.tr` *and* `tosun@sakarya.edu.tr`

Abstract. In this paper, the trajectory null scroll in 3-dimensional Minkowski space-time E_1^3 is given by a firmly connected null oriented line moving with Cartan frame along null curve. Some theorems and results between curvatures of base curve and distribution parameter of this surface are obtained. Moreover, some theorems and results related to being developable and minimal of this surface are given. And also, some relationships among geodesic curvature, geodesic torsion and the curvatures of null base curve of trajectory null scroll are found.

1. Introduction

In literature there are many studies related to ruled surfaces and their invariants (distribution parameters, Blaschke invariants, sectional curvature, apex angles, etc) in 3-dimensional Euclidean space E^3, [1], [2]. In a spatial motion, the trajectories of oriented lines embedded in a moving space (or in a moving rigid body) are generally trajectory ruled surfaces (or ruled surfaces). Therefore the geometry of trajectory ruled surfaces is important in the study of space kinematics or spatial mechanisms. And also, the developable of the trajectory ruled surfaces have a number of applications in geometric modeling and model-based manufacturing of mechanical products, [3], [4], [5]. Lorentz metric in 3-dimensional Minkowski space-time E_1^3 is indefinite. In theory of relativity, geometry of indefinite metric is very crucial. Hence, the theory of ruled surfaces in Minkowski space-time which has the metric $ds^2 = -dx^2 + dy^2 + dz^2$ attracted much attention. The situation is much more complicated than the Euclidean case, since the ruled surfaces may have a definite metric (space-like surfaces), Lorentz metric (time-like surfaces) or mixed metric. Recently, the time-like or space-like ruled surfaces in have been studied systematically, [6], [7], [8], [9]. From the differential geometric point of view, the study of null curves has its own geometric interest. Many of the classical results from Riemannian geometry have Lorentz counterparts. In fact, space-like curves or time-like

Received September 1, 2006.
2000 Mathematics Subject Classification: 53B30.
Key words and phrases: Null scroll, distribution parameter, mean curvature, Gaussian curvature.

curves can be studied by a similar approach to that positive definite Riemannian geometry. However, null curves have many properties very different from space-like and time-like curves. In the other words, null curve theory has many results which have no Riemannian analogues. For general theory of parametrized null curves we refer to, [10]. In geometry of null curves difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. A method of proceeding is to introduce a new parameter called pseudo-arc which normalizes the derivative of the tangent vector. Many authors generalize the results of Bonnor since for a null curve in an n-dimensional Lorentzian space form they introduce a Frenet frame with the minimum number of curvature functions (which call the Cartan frame), and they study the null helices in those spaces, that is, null curves with constant curvatures, [11], [12], [13]. In the first time, [14] introduced the notion of B-scrolls as based on a null curve and a null line in the 3-dimensional Minkowski space-time E_1^3. The null scrolls in 3-dimensional Minkowski space-time E_1^3 have been studied systematically, [15], [16].

2. Preliminaries

Let E_1^3 denote the 3-dimensional Minkowski space-time, i.e. the Euclidean space E^3 with standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is rectangular coordinate system of E_1^3 since g is indefinite metric, recall that a vector v in E_1^3 can have one of three casual characters: it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \mp 1$. Furthermore, vectors v and w are said to be orthogonal if $g(v,w) = 0$, [17]. For any vectors $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3) \in E_1^3$, the Lorentzian product $v \wedge w$ of v and w is defined as [18]

$$v \wedge w = (v_3 w_2 - v_2 w_3, v_1 w_3 - v_3 w_1, v_1 w_2 - v_2 w_1).$$

An arbitrary curve $\alpha : I \to E_1^3$ in the space E_1^3 can locally be null iff the velocity vector $\alpha'(t)$ is null. Furthermore, α is a unit speed curve if $g(\alpha'(s), \alpha'(s)) = \mp 1$, [19].

A surface in the 3-dimensional Minkowski space-time E_1^3 is called a time-like surface if induced metric on the surface is a Lorentzian metric i.e. the normal on the surface is a space-like vector. Let M be a three dimensional Lorentzian manifold and α be a null curve in M. A null frame of E_1^3 is a positively oriented triple (λ, N, W) of vector satisfying

$$\begin{aligned} g(\lambda,\lambda) = g(N,N) = 0, \qquad & g(\lambda, N) = 1 \\ g(\lambda, W) = g(N, W) = 0, \qquad & g(W,W) = 1. \end{aligned} \tag{1}$$

A null frame of a null curve $\alpha(s)$ is a frame field $F(s) = (\lambda(s), N(s), W(s))$ such that $\dfrac{d\alpha}{ds}$ is a positive scalar multiple of λ, [10]. In such a case, α is said to be framed by $F(s)$. Frames of null curves are not unique. Moreover frames are changed under parametrizations of a curve. Therefore, the curve and a frame must be given together. Now suppose that α is framed by $F = (\lambda, N, W)$ with $\lambda = \dfrac{d\alpha}{ds}$. Then the vector fields N and W define line bundles $ntr(\alpha)$ and $S\left(T\alpha^{\perp}\right)$ over α, respectively. The line bundle $S\left(T\alpha^{\perp}\right)$ is called the screen vector bundle and $ntr(\alpha)$ the null transversal vector bundle of with respect to $S\left(T\alpha^{\perp}\right)$, respectively. The Frenet formula of α with respect to the frame F is given by [10]

$$\begin{aligned}\frac{d\lambda}{ds} &= h\lambda + k_1 W\\ \frac{dN}{ds} &= -hN + k_2 W\\ \frac{dW}{ds} &= -k_2\lambda - k_1 N.\end{aligned} \tag{2}$$

The functions h, k_1 and k_2 are called the curvature functions of α. There always exists a parameter s of α such that $h = 0$ in (1). This parameter is called a distinguished parameter of α, [10]. The distinguished parameter is uniquely determined for prescribed screen vector bundle up to affine transformation. In case s is a distinguished parameter of a null curve α, then we put

$$\xi(s) = \frac{d\alpha}{dt}(s), \qquad n(s) = -N(s), \qquad u(s) = W(s). \tag{3}$$

Thus the Frenet formula of α with respect to $F = (\xi, n, u)$ become

$$\begin{aligned}\xi' &= k_1 u\\ n' &= -k_2 u\\ u' &= -k_2\xi + k_1 n.\end{aligned} \tag{4}$$

Here the prime $'$ denotes differentiation with respect to the distinguished parameter s. The null frame F is called the Cartan frame of $\alpha(s)$. A parametrized null curve parametrized by the distinguished parameter s together with its Cartan frame is called a Cartan framed null curve, [10].
A ruled surface is a surface swept out by a straight line Y moving along a curve α. The various positions of the generating line Y are called the rulings of the surface. Such a surface has a parametrization in ruled form as follows

$$\varphi(s, v) = \alpha(s) + vY(s).$$

We call α to be base curve and Y to be director line. Alternatively, we may visualize Y as a vector field on α. Frequently, it is necessary to restrict v to some interval, so the rulings may not be entire straight lines. If the tangent plane is constant along a fixed ruling, then the ruled surface is developable surface. The remaining ruled surfaces are called skew surfaces. If there exists a common perpendicular to two

preceding rulings in the skew surface, then the foot of the common perpendicular on the main ruling is called a central point. The locus of the central points is called the curve of striction, [15].
In the three-dimensional Minkowski space-time E_1^3, a surface is called as a null scroll, if it produced by the movement of a null curve α of a null line N with direction of a unit vector $e(s)$ and it is denoted by M. It is easy to check that M is a time-like surface, [15]. We give the parametrization of this null scroll in the following form

$$\varphi(s,v)=\alpha(s)+vN(s).$$

Throughout this paper N, denotes a unit vector which is a null vector, α is a null curve. Here we call the curve α as the base curve.
Distribution parameter, mean and Gaussian curvature of a null scroll M are as follows, respectively

$$\delta=-\frac{\det\left(\alpha'(s),N(s),N'(s)\right)}{\left\|\vec{N'}(s)\right\|^2}, \tag{5}$$

$$H=\frac{eG-2fF+gE}{2\left(EG-F^2\right)} \tag{6}$$

and

$$K=\frac{eg-f^2}{EG-F^2} \tag{7}$$

where E, F and G are the coefficients of the first fundamental form, whereas e, f and g are the coefficients of the second fundamental form, [17]. Striction curve of null scroll, which is non-developable in E_1^3, is given by [15]

$$\beta(s)=\alpha(s)-\frac{g\left(\alpha'(s),N'(s)\right)}{\left\|N'(s)\right\|^2}N(s). \tag{8}$$

3. Some theorems and results on the trajectory null scrolls in E_1^3

Let $\alpha=\alpha(s)$ be a null curve given by Cartan frame $\{\xi,n,u\}$ defined by equation (1) and (3). We also consider that a null oriented line X in E_1^3 such that it is firmly connected to the Cartan frame of the null curve α is represented, uniquely with respect to this frame, in the form

$$\begin{aligned}&X(s)=x_1(s)\,\xi(s)+x_2(s)\,n(s)+x_3(s)\,u(s),\langle X(s),X(s)\rangle=0\\&\left(x_3(s)\right)^2-2x_1(s)\,x_2(s)=0\end{aligned} \tag{9}$$

where the components $x_i\,(s)$ $(1 \leq i \leq 3)$ are the scalar functions of the distinguished parameter of the Cartan framed base curve α. The trajectory null scrolls generated by ξ, n and X are

$$M_1 : \quad \varphi_1\,(s,v) = \alpha\,(s) + v\xi\,(s) \tag{10}$$

$$M_2 : \quad \varphi_2\,(s,z) = \alpha\,(s) + zn\,(s) \tag{11}$$

$$M_3 : \quad \varphi_3\,(s,w) = \alpha\,(s) + wX\,(s) \tag{12}$$

respectively.
Now we consider the surfaces M_1, M_2, M_3 and give the theorem related to these trajectory null scrolls (or M_i-surface, $1 \leq i \leq 3$). First of all we take trajectory null scroll M_1. If one considers equations (5), (6) and (7) with the equation (10), the following theorem can be given related to the distribution parameter, mean and Gaussian curvature of the trajectory null scroll M_1.

Theorem 1. *The distribution parameter of trajectory null scroll M_1 is equal to zero. Furthermore the mean and Gaussian curvature of the trajectory null scroll M_1 are undefined.*

As $\delta_{M_1} = 0$, the following result can be given.

Result 1. Trajectory null scroll M_1 is developable.

Now we take into consideration trajectory null scroll M_2. Again equations (5), (6) and (7) with the equation (11) give us the following theorem.

Theorem 2. *The distribution parameter, the mean and Gaussian curvature of the trajectory null scroll M_2 are*

$$\delta_{M_2} = \frac{1}{k_2} \quad , \quad H_{M_2} = k_2 \quad , \quad K_{M_2} = (k_2)^2$$

respectively.

By taking into consideration the last theorem, the distribution parameter of M_2-surface is not zero, so we give the following result.

Result 2. (i) M_2-surface is non-developable.
(ii) Non-developable M_2-surface is not minimal.
(iii) There is a relation

$$H_{M_2} = \sqrt{K_{M_2}}$$

between mean and Gaussian curvature of M_2-surface.
(iv) The relation

$$\delta_{M_2} = \frac{1}{\sqrt{K_{M_2}}}$$

holds between distribution parameter and Gaussian curvature of M_2-surface.

If equation (8) is taken into consideration, then the striction curve of non-developable M_2-surface becomes

$$\beta(s) = \alpha(s) - \frac{g(\alpha'(s), n'(s))}{\|n'(s)\|^2} n(s).$$

With the last equation together equations (3) and (4) one reaches the following result.

Result 3. The striction curve of M_2-surface generated by n in E_1^3 is base curve.

Finally we consider trajectory null scroll M_3. From equations (5), (6) and (7) we give the following theorem.

Theorem 3. *The distribution parameter, the mean and Gaussian curvature of M_3-surface are*

$$\delta_{M_3} = \frac{\left(x_3^2 - x_1x_2\right)k_1 + x_2^2k_2 - x_2'x_3 + x_2x_3'}{\left(x_1k_1 - x_2k_2 + x_3'\right)^2 - 2\left(x_1' - x_3k_2\right)\left(x_2' + x_3k_1\right)}, \tag{13}$$

$$H_{M_3} = -\left(\frac{x_3}{x_2}\right)' + \frac{k_1x_1}{x_2} + k_2 \tag{14}$$

and

$$K_{M_3} = \left[-\left(\frac{x_3}{x_2}\right)' + \frac{k_1x_1}{x_2} + k_2\right]^2 \tag{15}$$

respectively.

From equation (13) the following result is reached.

Result 4. (i) If the oriented null line X is in nu-plane (i.e., $x_1 = 0$) and M_3-surface is developable then

$$k_2 = \left(\frac{x_3}{x_2}\right)' - \left(\frac{x_3}{x_2}\right)^2 k_1\ , \quad x_2 \neq 0.$$

(ii) If the oriented null line X is in ξu-plane (i.e., $x_2 = 0$) then M_3-surface is developable.
(iii) If the oriented null line X is in ξn-plane (i.e., $x_3 = 0$) and M_3-surface is developable then

$$x_2 = 0 \text{ or } \frac{k_2}{k_1} = \frac{x_1}{x_2}\ , \quad x_2 \neq 0.$$

Let us consider equation (9) and the equations (i.) and (iii.) of the result 4. If the oriented null line X is in both nu-plane ($x_1 = 0$) or ξn-plane ($x_3 = 0$) and

M_3-surface is developable, then curve α is a planar curve, where $x_2 \neq 0$. Taking into consideration equation (14) gives the following result.

Result 5. i. If the oriented null line X is in nu-plane (i.e., $x_1 = 0$) and M_3-surface is minimal then

$$k_2 = -\left(\frac{x_3}{x_2}\right)' = 0.$$

This means that the null curve α is a planar curve.
ii. If the oriented null line X is in ξu-plane (i.e., $x_2 = 0$) then mean and Gaussian curvature of M_3-surface is undefined.
iii. If the oriented null line X is in ξn-plane (i.e., $x_3 = 0$) and M_3-surface surface is minimal then

$$\frac{k_2}{k_1} = \frac{x_1}{x_2}\ , \quad x_2 \neq 0.$$

Therefore the null curve α is a planar curve.

Considering equations (14) and (15) gives us the following result.

Result 6. The relation

$$H_{M_3} = \sqrt{K_{M_3}}$$

holds between the mean and Gaussian curvature of trajectory null scroll generated by the oriented null line X in E_1^3.

If the oriented null line X is constant i.e. $dX = 0$ and M_3-surface is developable then

$$x_2 = \frac{x_3^2 k_1}{x_1 k_1 - x_2 k_2} \quad , \quad \frac{k_2}{k_1} \neq \frac{x_1}{x_2}. \tag{16}$$

Therefore the following theorem can be given.

Theorem 4. *Let α be a null curve and the oriented null line X be a fixed line that is firmly connected to the Cartan frame of α in E_1^3. If the trajectory null scroll generated by X is developable then equation (16) exists.*

If $x_1 k_1 - x_2 k_2 = c = \textit{constant}$ in equation (16), then α is a Bertrand offset. So, we can give the following result.

Result 7. If α is a Bertrand offset, $k_1 = c\frac{x_2}{x_3^2}$.

From equation (8), the striction curve of non-developable trajectory null scroll generated by the oriented null line X in E_1^3 parametrically is in the following form

$$\beta(s) = \alpha(s) + \frac{x_2'(s) + x_3(s)\, k_1(s)}{\|X'(s)\|^2} X(s)\,.$$

From the last equation we give the following theorem.

Theorem 5. *The striction curve of non-developable trajectory null scroll generated by X in E_1^3 is the base curve if and only if*

$$x_2 = -\int x_3 k_1 ds + C.$$

Unit normal vector $\eta(s,w)$ of the trajectory null scroll M_3 is given by

$$\eta(s,w) = \frac{\varphi_{3s} \wedge \varphi_{3w}}{||\varphi_{3s} \wedge \varphi_{3w}||} = \frac{\alpha'(s) \wedge X(s) + wX'(s) \wedge X(s)}{||\varphi_{3s} \wedge \varphi_{3w}||}. \tag{17}$$

Therefore, from equations (1) and (17) unit normal vector of M_3-surface at point $(s,0)$ is

$$\eta\,(s,0) = \frac{-x_3 n + x_2 u}{|x_2|}\ , \quad x_2 \neq 0. \tag{18}$$

So, the following theorem can be given.

Theorem 6. *If a null curve on trajectory null scroll M_3 in E_1^3 is a geodesic curve then $x_3 = 0$ and the base curve of null scroll is a striction curve.*

If we consider equations (1), (4), and (18) we find that the geodesic and normal curvature and geodesic torsion are given by the following equation

$$k_g = g\,(\eta \wedge \xi, \xi') = k_1 \frac{x_3}{|x_2|}, \tag{19}$$

$$k_n = g\,(\alpha'', \eta) = k_1 \frac{x_2}{|x_2|} = \mp k_1 \tag{20}$$

and

$$\tau_g = g\,(\eta \wedge \eta', \xi) = -k_1 k_2 \frac{x_3}{|x_2|} \tag{21}$$

respectively. These three equations imply the following theorem.

Theorem 7. *The relation*

$$\tau_g = -k_2 k_g$$

holds between the torsion and the geodesic torsion of the base curve of trajectory null scroll M_3 in E_1^3.

The last theorem gives the following result.

Result 8. If the base curve of M_3-surface is a planar or geodesic curve then $\tau_g = 0$.

Example 1. Let a null curve of E_1^3 be

$$\alpha : IR \to E_1^3 \qquad (22)$$
$$t \to \alpha(t) = (\sinh t, t, \cosh t)$$

such that α has a Cartan frame $F = (\xi, n, u)$ defined as follows

$$\xi(t) = \frac{d\alpha}{dt} = (\cosh t, 1, \sinh t) \qquad (23)$$

$$n(t) = \frac{1}{2}(\cosh t, -1, \sinh t) \qquad (24)$$

$$u(t) = (\sinh t, 0, \cosh t) \qquad (25)$$

Let oriented null line X in E_1^3 be firmly connected to the Cartan frame of the null curve $\alpha(t)$ and represented, uniquely with respect to Cartan frame, in the form

$$X(t) = x_1(t)\,\xi(t) + x_2(t)\,n(t) + x_3(t)\,u(t) \qquad (26)$$

$$\left\langle \overrightarrow{X}(t), \overrightarrow{X}(t) \right\rangle = 0, \quad (i.e., (x_3(t))^2 - 2x_1(t)\,x_2(t) = 0) \qquad (27)$$

where the components $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the scalar functions of the distinguished parameter of the Cartan framed base curve α. Therefore the trajectory null scrolls generated by ξ, n and X are M_1, M_2 and M_3, respectively. Now, we first consider M_1-surface. From equation (22) and (23), M_1-surface is defined parametrically as (see Figure I)

$$M_1 : \varphi_1(t, v) = (\sinh t, t, \cosh t) + v(\cosh t, 1, \sinh t). \qquad (28)$$

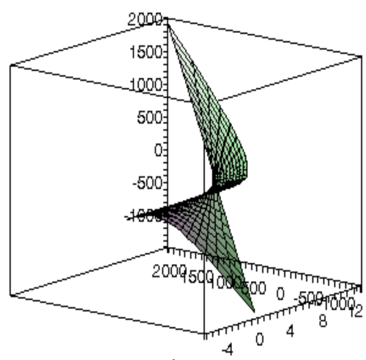

Figure I

Considering equation (28) with equations (5), (6) and (7) gives us the distribution parameter, mean and Gaussian curvature of M_1-surface as

$$\delta_{M_1} = 0\ ,\ H_{M_1} = \frac{0}{0}\ ,\ K_{M_1} = \frac{0}{0},$$

respectively.
We now take the M_2-surface and write parametrically from equations (22) and (24) the following relation (see Figure II)

$$M_2 : \varphi_2(t,z) = (\sinh t, t, \cosh t) + \frac{z}{2}(\cosh t, -1, \sinh t)\,. \tag{29}$$

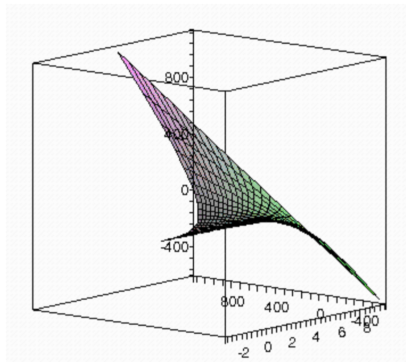

Figure II

As in M_1-surface, we find the following results for the distribution parameter, mean and Gaussian curvature of M_2-surface

$$\delta_{M_2} = 2\ ,\ H_{M_2} = \frac{1}{2}\ ,\ K_{M_2} = \frac{1}{4},$$

respectively.
Lastly, we take M_3-surface into consideration. From equations (22)-(26) we define M_3-surface parametrically as

$$\begin{aligned} & M_3 : \varphi_3(t,w) \\ = \ & (\sinh t,\ t,\ \cosh t) + w\left(x_1(t)\cosh t + \frac{x_2(t)}{2}\cosh t + x_3(t)\sinh t,\right. \\ & \left. x_1(t) - \frac{x_2(t)}{2},\ x_1(t)\sinh t + \frac{x_2(t)}{2}\sinh t + x_3(t)\cosh t\right) \end{aligned} \tag{30}$$

Considering equation (30) and (5) gives the distribution parameter of M_3-surface as

$$\delta_{M_3} = \frac{2\left(x_2^2 - 2x_3^2 + 2x_1x_2 - 2x_2'x_3 + 2x_2x_3'\right)}{\left(4x_1^2 + x_2^2 - 4x_3^2 + 4\left(x_3'\right)^2 + 4x_1x_2 - 8x_1'x_3 + 8x_1x_3' + 4x_2x_3' - 4x_2'x_3 - 8x_1'x_2'\right)}.$$

From equation (6) the mean curvature of M_3-surface becomes

$$H_{M_3} = -\left(\frac{x_3}{x_2}\right)' + \frac{x_1}{x_2} - \frac{1}{2}.$$

Equation (7) gives the Gaussian curvature of M_3-surface to be

$$K_{M_3} = \left(-\left(\frac{x_3}{x_2}\right)' + \frac{x_1}{x_2} - \frac{1}{2} \right)^2 .$$

Special Case: Let us choose the components in oriented null line X given by equation (26) as

$$\begin{aligned} x_1(t) &= \cosh t, \\ x_2(t) &= \frac{1}{2}\cosh t, \\ x_3(t) &= \cosh t. \end{aligned}$$

So the trajectory null scroll generated by X (i.e., M_3-surface) can be written parametrically as (see Figure III)

$$\begin{aligned} (31) \quad \varphi_3(t,w) \quad = \quad & \big(\sinh t + w(5/4\cosh^2 t + \cosh t \sinh t), \\ & t + 3/4w\cosh t, \;\; \cosh t + w(5/4\cosh t \sinh t + \cosh^2 t)\big) \end{aligned}$$

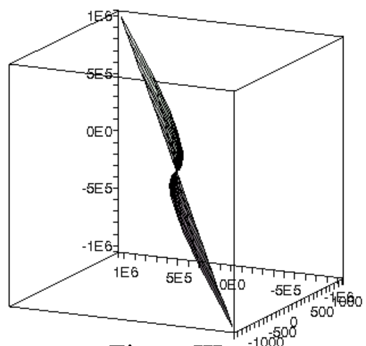

Figure III

If we consider equation (5), the distribution parameter of the trajectory null scroll equation (31) becomes

$$\delta = \frac{2}{3}.$$

From equation (6) the mean curvature this trajectory null scroll is found to be

$$H = \frac{3}{2}.$$

Furthermore, the Gaussian curvature of this trajectory null scroll is reached to be

$$K = \frac{9}{4}.$$

References

[1] H. Frank and O. Giering, *Verallgemeinerte Regelflachen*, Math. Z., **150**(1976), 261-271.

[2] H. H. Hacisalihoglu, *Diferensiyel Geometri*, Inonu Uni. Fen-Edeb. Fak., 1983

[3] R. T. Farouki, *The approximation off non-degenerate offset surfaces*, Comp. Aided Geom. Design, **3**(1986), 15-43.

[4] B. Ravani and T. S. Ku, *Bertrand offset and developable surfaces*, Comp. Aided Geom. Design, **23(2)**(1991), 145-152.

[5] A. T. Yang, Y. Kirson and B. Roth, *On a kinematics theory for ruled surface*, Proceedings of Fourth World Congress on the theory of Machines and Mechanisms, Newcastle Upon Tyne, England, September, pp. 737-742, 1975.

[6] Y. H. Kim and D. W. Yoon, *Classification of ruled surfaces in Minkowski 3-spaces*, Journal of Geometry and Physics, **49(1)**(2004), 89-100.

[7] A. Turgut and H. H. Hacisalihoglu, *Time-like ruled surfaces in the Minkowski 3-space*, Tr. J. of Math., **22**(1998), 33-46, Tubitak.

[8] V. D. I. Woestijine, *Minimal surfaces of 3-dimensional Minkowski Space*, word Scientific Publishing, Singapore, 344-369, 1990.

[9] A. Kucuk, *On the developable time-like trajectory ruled surfaces in a Lorentz 3-space* R_1^3, Appl. Math. Comput., **157**(2004), 483-489.

[10] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi Riemannian manifolds and applications*, Kluwer Academic Publishers, Dorecht,1996.

[11] A. Ferrandez,A. Gimenez and P. Lucas, *Null helices in Lorentzian space forms*, Int. J. Mod. Phys., A **16**(2001), 48 45-63.

[12] B. Sahin, E. Kilic, and R. Gunes, *Null Helices in* IR_1^3, differential geometry dynamical systems, **3(2)**(2001), 31-36.

[13] A. F. Yaliniz and H. H. Hacisalihoglu, *Null generalized Helices in* L^3 *and* L^4*, 3 and 4 dimensional Lorentzian space*, Math. And Comp. Appl. **10(1)**(2005),105-111.

[14] L. K. Graves, *Codimension one isometric immersions between Lorentz space*, Trans. Amer. Math. Soc., **252**(1979), 367-392.

[15] H. Balgetir. M. Bektas. and M. Ergut, *Null scroll in the 3-dimensional Lorentzian space*, Appl. Sciences, **5(1)**(2003), 1-5.

[16] W. B. Bonnor, *Null curves in a Minkowski space-time*, Tensor N. S., **20**(1969), 229-242.

[17] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[18] K. Akutagawa and S. Nshkawa, *The Gauss map and space-like surfaces with prescribed mean curvature in Minkowski 3-space*, Thoku Math. J., **42**(1990), 67-82.

[19] T. Ikaw, *On curves and submanifolds indefinite Riemannian manifold*, Tsukuba J. Math., **9(2)**(1985), 353-371.