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$\overline{\mathcal{WT}}$-Classes of Differential Forms on Riemannian Manifolds

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Abstract. The purpose of this paper is to study the relations between quasilinear elliptic equations on Riemannian manifolds and differential forms. Two classes of differential forms are introduced and it is shown that some differential expressions are connected in a natural way to quasilinear elliptic equations.

1. Introduction

The theory of $\mathcal{A}$-harmonic forms plays a crucial role in many fields, such as potential theory, partial differential equations and quasiconformal analysis. At the same time, they are extensions of harmonic functions and p-harmonic functions, $p > 1$. In recent years, there have been remarkable advances made in the field of $\mathcal{A}$-harmonic forms. Many interesting results about them and their applications in fields such as potential theory, quasiregular analysis and the theory of elasticity have been found; see [4], [5] and [1]. There are also some other interesting results, such as the relations between quasiregular mappings on Riemannian manifolds and differential forms, see [3], [10] and [11]. In this paper, we introduce two classes of differential forms and show that some differential expressions are connected in a natural way to quasilinear elliptic equations.

We next introduce some notations and symbols used in this paper. Most of

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them can be found in [3]. We list them all here for the sake of completeness.

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold with boundary or without boundary. Throughout this paper we assume that the manifold $\mathcal{M}$ is orientable and of class C^3. By $T(\mathcal{M})$ we denote the tangent bundle and by $T_m(\mathcal{M})$ the tangent space at the point $m \in \mathcal{M}$. For each pair of vectors $x, y \in T_m(\mathcal{M})$ the symbol $\langle x, y\rangle$ denotes their scalar product.

Below we shall use standard notation for function classes on manifolds. Thus, for example, the symbol $L^p_{loc}(D)$ stands for the set of all Lebesgue measurable functions on an open set $D \subset \mathcal{M}$, locally integrable to the power p, $1 \leq p \leq \infty$, on D. The symbol $W^{1,p}_{loc}(D)$ stands for the set of functions that have generalized partial derivatives in the sense of Sobolev of class $L^p_{loc}(D)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of class $C^k, k \geq 3$, and $F : D \to \mathcal{N}, D \subset \mathcal{M}$, a mapping. We shall say that $F \in L^p_{loc}(D)$ if for an arbitrary function $\phi \in C^0(\mathcal{N})$ we have $\phi \circ F \in L^p_{loc}(D)$. The mapping F is in the class $W^{1,p}_{loc}(D)$, if $\phi \circ F \in W^{1,p}_{loc}(D)$ for every $\phi \in C^1(\mathcal{N})$.

Let $V(\mathcal{M})$ be a vector bundle on $\mathcal{M}$. Let in the elements of this bundle be given an Euclidean scalar product and let the linear connection on $V(\mathcal{M})$ preserve this scalar product. In this case we may say that $V(\mathcal{M})$ is a Riemannian vector bundle over $\mathcal{M}$.

By $\bigwedge^k(\mathcal{M})$ and $\bigwedge_k(\mathcal{M})$ we denote Riemannian vector bundles $\bigwedge^k(T_m(\mathcal{M}))$ and $\bigwedge_k(T_m(\mathcal{M}))$. The sections of these bundles are the fields of k-covectors (k-forms) and k-vectors.

Let $x^1, x^2, \cdots, x^n$ be local coordinates in the neighborhood of of point $m \in \mathcal{M}$. The square of a line element on $\mathcal{M}$ has the following expression in terms of the local coordinates $x^1, x^2, \cdots, x^n$

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j.$$

By the symbol g^{ij} we shall denote the contravariant tensor defined by the equality

$$(g^{ik})(g_{kj}) = (\delta^i_j), \quad i, j = 1, \cdots, n,$$

where δ^i_j is the Kronecker symbol.

Each section α of the bundle $\bigwedge^k(\mathcal{M})$ (that is a differential form) can be written in terms of the local coordinates $x^1, x^2, \cdots, x^n$ as the linear combination

$$\alpha = \sum_{I \in \wedge(k,n)} \alpha_I dx_I = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where we have denoted by $\bigwedge(k, n)$ the set of all ordered multi-indices $I = (i_1, i_2, \cdots, i_k)$ of integers $1 \leq i_1 < \cdots < i_k \leq n$.

Let α be a differential form defined on an open set $D \subset \mathcal{M}$. If $\mathcal{F}(D)$ is a class of functions defined on D then we say that the differential from α is in this class provided that $\alpha_I \in \mathcal{F}(D)$. For instance, the differential form α is in the class $L^p(D)$ if all its coefficients are in this class.

The operator $\star : \bigwedge^k(\mathcal{M}) \to \bigwedge^{n-k}(\mathcal{M})$, called Hodge star operator has the following properties:

If $\alpha, \beta \in \bigwedge^k(\mathcal{M})$, and $a, b \in \mathrm{R}$, then

$$\star(a\alpha + b\beta) = a \star \alpha + b \star \beta.$$

For every w with deg$w = k$, we have

$$\star(\star w) = (-1)^{k(n-k)} w.$$

We introduce the following notation. Let w be a differential form of degree k, we set

$$\star^{-1} w = (-1)^{k(n-k)} \star w.$$

The operator $\star^{-1}$ is an inverse to $\star$ in the sense that $\star^{-1}(\star w) = \star(\star^{-1} w) = w$.

The inner or scalar product has the usual properties of the scalar product. We set

$$\langle \alpha, \beta \rangle = \star^{-1} \langle \alpha, \star\beta \rangle = \star(\alpha \wedge \star\beta).$$

We denote by dv the volume element on $\mathcal{M}$.

If α, $\deg \alpha = k, 0 \leq k \leq n$, is a differential from whose coefficients are in $C^1(\mathcal{M})$ then $d\alpha$, $\deg(d\alpha) = k + 1$, denotes its differential defined by

$$d\alpha = \sum_{I \in \wedge(k,n)} d\alpha_I \wedge dx_I.$$

The differentiation is a linear operation for which the following properties hold:

If α and β are arbitrary differential forms that are differentiable in a domain $U \subset \mathcal{M}$ then

(i) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$,

(ii) $d(d\alpha) = 0$,

where $k = \deg(w)$ is the degree of the differential form α.

The operator $\star$ and the exterior differentiation d define the codifferential operator d^* by the formula

$$d^* \alpha = (-1)^k \star^{-1} d \star \alpha$$

for a differential form α of degree k. Clearly, $d^*\alpha$ is a differential form of degree $k - 1$.

2. Differential forms on Riemannian manifolds

The following definition comes from [3], see also [5].

Definition 2.1. A differential form α of degree k on the manifold $\mathcal{M}$ with coefficients $\alpha_{i_1 \cdots i_k} \in L^p_{loc}(\mathcal{M})$ is called weakly closed if for each differential form β, deg$\beta = k + 1$, with

$$\mathrm{supp}\beta \bigcap \partial\mathcal{M} = \phi, \quad \mathrm{supp}\beta = \overline{\{m \in \mathcal{M} : \beta \neq 0\}} \subset \mathcal{M}$$

and with coefficients in the class $W^{1,q}_{loc}(\mathcal{M})$, $\frac{1}{p}+\frac{1}{q}=1, 1\le p,q\le\infty$, we have

$$\int_{\mathcal{M}} \langle \alpha, d^*\beta\rangle dv = 0. \tag{2.1}$$

We next introduce two classes of differential forms with generalized derivatives. These classes can be used to study the associated classes of quasilinear elliptic partial differential equations.

Definition 2.2. A weakly closed differential form

$$w \in L^p_{loc}(\mathcal{M}), \deg w = k, 0\le k\le n, p>1 \tag{2.2}$$

is said to be of the class $\overline{\mathcal{WT}}_1$ on $\mathcal{M}$ if there exists a weakly closed differential form

$$\theta \in L^q_{loc}(\mathcal{M}), \deg\theta = n-k, \frac{1}{p}+\frac{1}{q}=1 \tag{2.3}$$

such that almost everywhere on $\mathcal{M}$ we have

$$\langle w, \star\theta\rangle \le \nu_1|\theta|^q, \tag{2.4}$$

where ν_1 is a constant.

Definition 2.3. A pair of weakly closed differential form

$$(w_1,w_2)\in L^p_{loc}(\mathcal{M})\times L^p_{loc}(\mathcal{M}), \deg w_1 = \deg w_2 = k, 0\le k\le n, p>1 \tag{2.5}$$

is said to be of the class $\overline{\mathcal{WT}}_2$ on $\mathcal{M}$ if there exists a weakly closed differential form θ (2.3) such that almost everywhere on $\mathcal{M}$ the conditions

$$\nu_2|w_1-w_2|^2(|w_1|+|w_2|)^{p-2} \le \langle w_1-w_2, \star\theta\rangle \tag{2.6}$$

and

$$|\theta| \le \nu_3|w_1-w_2|(|w_1|+|w_2|)^{p-2} \tag{2.7}$$

are satisfied, with constants $\nu_2,\nu_3>0$.

The following definition comes form [3].

Definition 2.4. The differential form (2.2) is said to be of the class $\mathcal{WT}_2$ on $\mathcal{M}$ if there exists a differential form (2.3) such that almost everywhere on $\mathcal{M}$ the conditions

$$\nu_2|w|^p \le \langle w, *\theta\rangle$$

and

$$|\theta| \le \nu_3|w|^{p-1}$$

are satisfied, with constants $\nu_2, \nu_3 > 0$.

3. Quasilinear elliptic equations

Let $\mathcal{M}$ be a Riemannian manifold and let

$$\mathcal{A} : \bigwedge^k(T(\mathcal{M})) \to \bigwedge^k(T(\mathcal{M}))$$

be a mapping defined almost everywhere on the k-vector tangent bundle $\bigwedge^k(T(\mathcal{M}))$. We assume that for almost every $m \in \mathcal{M}$ the mapping $\mathcal{A}$ is defined on the k-vector tangent space $\bigwedge^k(T(\mathcal{M}))$, that is for almost every $m \in \mathcal{M}$ the mapping

$$\mathcal{A}(m, \cdot) : \xi \in \bigwedge^k(T(\mathcal{M})) \to \bigwedge^k(T(\mathcal{M}))$$

is defined and continuous. We assume that the mapping $m \mapsto \mathcal{A}_m(m, \xi)$ is measurable for all measurable k-vector fields ξ. Suppose that for almost every $m \in \mathcal{M}$ and for all $\xi \in \bigwedge^k(T(\mathcal{M}))$ we have

$$\langle \xi, \mathcal{A}(m, \xi)\rangle \le \nu_1 |\mathcal{A}(m, \xi)|^q \tag{3.1}$$

with the constants $q > 1$ and $\nu_1 > 0$.

Definition 3.1. A differential from $w \in W^{1,p}_{loc}(\mathcal{M})$ is said to be $\mathcal{A}$-harmonic if it is a solution of the $\mathcal{A}$-harmonic equation

$$d^* \mathcal{A}(m, dw) = 0 \tag{3.2}$$

understood in the weak sense, that is

$$\int_{\mathcal{M}} \langle d\varphi, \mathcal{A}(m, dw)\rangle dv = 0 \tag{3.3}$$

for all differential forms $\varphi \in W^{1,p}_{loc}(\mathcal{M})$, $\frac{1}{p} + \frac{1}{q} = 1$, with supp $\varphi \bigcap \partial\mathcal{M} = \phi$.

Theorem 3.1. *If the differential form $w \in W^{1,p}_{loc}(\mathcal{M})$ is $\mathcal{A}$-harmonic with the property* (3.1)*, then the differential form dw is in the class $\overline{\mathcal{WT}}_1$ on $\mathcal{M}$.*

Proof. Let w, $\deg w = k - 1$ be a solution of (3.2) understood in the weak sense. Let the differential form $\alpha(m)$ be associated with the vector field $\mathcal{A}(m, dw)$ at the point m and set $\theta = *^{-1}\alpha$. The differential form dw is weakly closed because of Poincaré's Lemma and the weak closedness of θ follows from (3.3), that is

$$(-1)^k \int_{\mathcal{M}} \langle \theta, d^\star \psi\rangle dv = \int_{\mathcal{M}} \langle \star^{-1}\alpha, \star^{-1} d \star \psi\rangle dv$$

$$= \int_{\mathcal{M}} \langle \alpha, d \star \psi\rangle dv = \int_{\mathcal{M}} \langle \mathcal{A}(m, dw), d\varphi\rangle dv = 0$$

for all $\psi = \star^{-1}\varphi \in W^{1,p}(\mathcal{M})$ with supp$\psi \bigcap \partial\mathcal{M} = \phi$. Further, by (3.1) we get

$$\langle dw, \star\theta\rangle = \langle dw, \mathcal{A}(m, dw)\rangle \leq \nu_1|\mathcal{A}(m, dw)|^q = \nu_1|\theta|^q,$$

which guarantees (2.4). □

Form now on we assume that the vector field $\mathcal{A}(m,\xi)$ satisfies the conditions

$$\nu_2|\xi_1 - \xi_2|^2(|\xi_1| + |\xi_2|)^{p-2} \leq \langle \xi_1 - \xi_2, \mathcal{A}(m,\xi_1) - \mathcal{A}(m,\xi_2)\rangle \tag{3.4}$$

$$|\mathcal{A}(m,\xi_1) - \mathcal{A}(m,\xi_2)| \leq \nu_3|\xi_1 - \xi_2|(|\xi_1| + |\xi_2|)^{p-2} \tag{3.5}$$

for almost every $m \in \mathcal{M}$ and all $\xi_1, \xi_2 \in \bigwedge^k(T(\mathcal{M}))$, where ν_2, ν_3 are some constants.

Theorem 3.2. *If the differential forms $w_1, w_2 \in W^{1,p}_{loc}(\mathcal{M})$ are $\mathcal{A}$-harmonic with the properties* (3.4) *and* (3.5)*, then (dw_1, dw_2) is in the class $\overline{\mathcal{WT}}_2$ on $\mathcal{M}$.*

Proof. Let the differential form $\alpha(m)$ be associated with the vector field $\mathcal{A}(m, dw_1) - \mathcal{A}(m, dw_2)$ at the point m and set $\theta = *^{-1}\alpha$. The weak closedness of dw_1, dw_2 and θ follow as in the proof of Theorem 3.1. From (3.4), one sees that

$$\begin{aligned} & \nu_2|dw_1 - dw_2|^2(|dw_1| + |dw_2|)^{p-2} \\ \leq \; & \langle dw_1 - dw_2, \mathcal{A}(m, dw_1) - \mathcal{A}(m, dw_2)\rangle = \langle dw_1 - dw_2, \star\theta\rangle \end{aligned}$$

and from (3.5)

$$|\theta| = |\star\theta| = |\mathcal{A}(m, dw_1) - \mathcal{A}(m, dw_2)| \leq \nu_3|dw_1 - dw_2|(|dw_1| + |dw_2|)^{p-2}.$$

This completes the proof of Theorem 3.2. □

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