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# On the Generalized Hyers-Ulam-Rassias Stability for a Functional Equation of Two Types in $p$-Banach Spaces

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Abstract. We investigate the generalized Hyers-Ulam-Rassias stability in $p$-Banach spaces for the following functional equation which is two types, that is, either cubic or quadratic:

$$2f(x+3y)+6f(x-y)+12f(2y)=2f(x-3y)+6f(x+y)+3f(4y).$$

The concept of Hyers-Ulam-Rassias stability originated essentially with the Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72 (1978), 297-300.

## 1. Introduction

Under what condition does there is a homomorphism near an approximately homomorphism between a group and a metric group? This is called the stability problem of functional equations which was first raised by S. M. Ulam [37] in 1940. In next year, D. H. Hyers [11] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [25]. The terminology Hyers-Ulam-Rassias stability originates from this historical background. Since then, a great deal of work has been done by a number of authors (for instances, [2], [4], [6], [7], [8], [10], [12], [13], [16], [20], [21], [23], [24], [26], [27], [28], [29], [30], [31], [32], [33]). In particular, one of the important functional equations studied is the following functional equation [1], [5], [15], [17], [19] :

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

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The quadratic mapping $f(x) = qx^2$ is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic. A Hyers-Ulam stability problem for the quadratic functional equation was first proved by F. Skof [35] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ a Banach space. S. Czerwik [5] generalized the Hyers-Ulam stability of the quadratic functional equation. The cubic mapping $f(x) = cx^3$ satisfies the functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \tag{1.1}$$

The equation (1.1) was solved by K.-W. Jun and H.-M. Kim [14] (see also [22]).

In this note we promise that the equation (1.1) is called a cubic functional equation and every solution of the cubic functional equation (1.1) is said to be a cubic mapping. Now, let us introduce the following functional equation:

$$2f(x + 3y) + 6f(x - y) + 12f(2y) = 2f(x - 3y) + 6f(x + y) + 3f(4y). \tag{1.2}$$

It is easy to see that all the real-valued mappings $f : \mathbb{R} \to \mathbb{R}$ of the two types, i.e., either $f(x) = cx^3$ or $f(x) = qx^2$ satisfy the functional equation (1.2).

Our main goal in this note is to investigate the generalized Hyers-Ulam-Rassias stability problem (or the stability in the sense of Găvruta [10]) for the equation (1.2) in quasi-Banach spaces.

We first recall some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 1.1**([3], [34]). Let $X$ be a linear space. A *quasi-norm* $\|\cdot\|$ is a real-valued function on $X$ satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all scalar $\lambda$ and all $x \in X$.

(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

A quasi-normed space is a linear space together with a specified quasi-norm.

A *quasi-Banach space* means a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if the inequality

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

holds for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space*. Clearly, $p$-norms are continuous, and in fact, if $\|\cdot\|$ is a $p$-norm on $X$, then the formula $d(x, y) := \|x - y\|^p$ defines a translation invariant metric for $X$ and $\|\cdot\|^p$ is a $p$-homogeneous $F$-norm. The Aoki–Rolewicz theorem [34] (see also [3, 18]) yields that each quasi-norm is equivalent to some $p$-norm, for some $0 < p \leq 1$. Since it is much easier to work with $p$-norms than quasi-norms, henceforth we restrict our attention mainly to $p$-norms. In [36], J. Tabor has investigated a version of the Hyers-Rassias-Gajda theorem (see [9]) in quasi-Banach spaces. In this paper, we will prove the Hyers–Ulam–Rassias stability of mappings satisfying approximately the equations (1.2)in $p$-Banach spaces.

## 2. Solutions of equation (1.2)

Let $X$ and $Y$ be linear spaces. In this section we will find out the general solution of (1.2).

**Lemma 2.1.** *A mapping $f : X \to Y$ is cubic if and only if $f$ is odd and satisfies the functional equation $f(x+3y)+3f(x-y) = f(x-3y)+3f(x+y)+48f(y)$ for all $x,\ y \in X$.*

*Proof.* ($\Rightarrow$) Suppose that $f$ is cubic, that is, the functional equation

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x) \tag{2.1}$$

holds for all $x,\ y \in X$. By putting $x = y = 0$ in (2.1), we see that $f(0) = 0$, and setting $x = 0$ in (2.1) yields the fact that $f$ is odd. If we interchange $x$ and $y$ in (2.1), we have

$$f(x+2y)-f(x-2y) = 2f(x+y)-2f(x-y)+12f(y). \tag{2.2}$$

Let $x := x+y$ and $x := x-y$, respectively, in (2.2). Then we obtain

$$f(x+3y)-f(x-y) = 2f(x+2y)-2f(x)+12f(y)$$

and

$$f(x+y)-f(x-3y) = 2f(x)-2f(x-2y)+12f(y).$$

Comparing the above two results, we get

$$f(x+3y)-f(x-3y)-f(x-y)+f(x+y) = 2f(x+2y)-2f(x-2y)+24f(y),$$

which, by (2.2), gives

$$f(x+3y)+3f(x-y) = f(x-3y)+3f(x+y)+48f(y).$$

($\Leftarrow$) Assume that $f$ is odd and satisfies the functional equation

$$f(x+3y)+3f(x-y) = f(x-3y)+3f(x+y)+48f(y) \tag{2.3}$$

for all $x,\ y \in X$. By interchanging $x$ and $y$ in (2.3), we obtain

$$f(3x+y)+f(3x-y) = 3f(x+y)+3f(x-y)+48f(x). \tag{2.4}$$

We substitute $x = 0 = y$ in (2.4) and then $y = 0$ in (2.4) to obtain $f(0) = 0$ and

$$f(3x) = 27f(x). \tag{2.5}$$

Putting $y = x$ in (2.4), we get

$$f(4x) = 2f(2x) + 48f(x), \tag{2.6}$$

and replacing $y$ by $3x$ in (2.4) and employing (2.5), we obtain

$$10f(2x) = f(4x) + 16f(x). \tag{2.7}$$

Now it follows from (2.6) and (2.7) that

$$f(2x) = 8f(x). \tag{2.8}$$

If we set $y := -x + y$ and $y := -x - y$ in (2.4), respectively and then compare the results, then we obtain

$$f(4x + y) + f(4x - y) = 2f(2x + y) + 2f(2x - y) + 96f(x). \tag{2.9}$$

Finally, replacing $y$ by $2y$ in (2.9) and using (2.8), we get the functional equation (1.1), that is, $f$ is cubic. □

**Lemma 2.2.** *A mapping $f : X \to Y$ is quadratic if and only if $f(0) = 0$, $f$ is even and satisfies the functional equation $f(x+3y)+3f(x-y) = f(x-3y)+3f(x+y)$ for all $x,\ y \in X$.*

*Proof.* ($\Rightarrow$) Suppose that $f$ is quadratic, that is, the functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.10}$$

holds for all $x,\ y \in X$. Putting $x = y = 0$ in (2.10) gives $f(0) = 0$, and setting $x = 0$ in (2.10) leads to the fact that $f$ is even. We also obtain $f(2x) = 4f(x)$ by letting $y := x$ in (2.10). From the substitutions $x := x + 2y$ and $y := x - 2y$ in (2.10), it follows that

$$2f(x) + 8f(y) = f(x + 2y) + f(x - 2y). \tag{2.11}$$

Putting $x := x + y$ and $x := x - y$ in (2.11), respectively, we obtain

$$2f(x + y) + 8f(y) = f(x + 3y) + f(x - y) \tag{2.12}$$

and

$$2f(x - y) + 8f(y) = f(x + y) + f(x - 3y). \tag{2.13}$$

If we subtract (2.13) from (2.12), we get

$$f(x + 3y) + 3f(x - y) = f(x - 3y) + 3f(x + y).$$

($\Leftarrow$) Assume that $f(0) = 0$, $f$ is even and satisfies the functional equation

$$f(x + 3y) + 3f(x - y) = f(x - 3y) + 3f(x + y) \tag{2.14}$$

for all $x,\ y \in X$. Let us replace $x$ by $y$ in (2.14) and then put $y := \frac{y}{2}$. Then we get $f(2y) = 4f(y)$. If we set $x := 3y$ in (2.14) and use $f(2y) = 4f(y)$, then we have $f(3y) = 9f(y)$. Substituting $x := x - y$ and $x := x + y$ in (2.14), respectively and then comparing the results, we obtain

$$f(x + 4y) + 2f(x - 2y) = f(x - 4y) + 2f(x + 2y). \tag{2.15}$$

Replacing $x$ by $2x$ in (2.15) and using $f(2y) = 4f(y)$, we have

$$f(x + 2y) + 2f(x - y) = f(x - 2y) + 2f(x + y). \tag{2.16}$$

From the substitutions $x := x + y$ and $y := x - y$ in (2.16), we deduce

$$f(3x - y) + 8f(y) = f(x - 3y) + 8f(x),$$

and replacing $y$ by $-y$ gives

$$f(3x + y) + 8f(y) = f(x + 3y) + 8f(x),$$

that is,

$$f(3x + y) - f(x + 3y) = 8f(x) - 8f(y), \tag{2.17}$$

Setting $x + y$ instead of $x$ in (2.16), we get

$$f(x + 3y) + 2f(x) = 2f(x + 2y) + 2f(x - y), \tag{2.18}$$

and interchanging $x$ and $y$ in (2.18) yields

$$f(3x + y) + 2f(y) = 2f(2x + y) + 2f(x - y). \tag{2.19}$$

If we subtract (2.19) from (2.18) and use (2.17), we obtain

$$f(x + 2y) + 3f(x) = f(2x + y) + 3f(y), \tag{2.20}$$

which, by putting $y := 2y$ in (2.20) and using $f(2y) = 4f(y)$, leads to

$$f(x + 4y) + 3f(x) = 4f(x + y) + 12f(y). \tag{2.21}$$

Interchanging $x$ with $y$ in (2.21) gives

$$f(4x + y) + 3f(y) = 4f(x + y) + 12f(x), \tag{2.22}$$

and by replacing $y$ by $-y$ in (2.22), we arrive at

$$f(4x - y) + 3f(y) = 4f(x - y) + 12f(x). \tag{2.23}$$

Comparing (2.22) with (2.23), we have

$$f(4x + y) + f(4x - y) + 6f(y) = 4f(x + y) + 4f(x - y) + 24f(x). \tag{2.24}$$

Now utilizing the substitutions $x := x + y$ and $y := x - \frac{y}{2}$ in (2.20), we obtain

$$f(3x) + 3f(x + y) = f\Big(3\Big(x + \frac{y}{2}\Big)\Big) + 3f\Big(x - \frac{y}{2}\Big),$$

and letting $y := -y$ in this relation yields

$$f(3x) + 3f(x - y) = f\Big(3\Big(x - \frac{y}{2}\Big)\Big) + 3f\Big(x + \frac{y}{2}\Big).$$

Since $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$, we add the above two relations to obtain

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x). \tag{2.25}$$

Replacing $x$ by $2x$ in (2.25), we get

$$f(4x + y) + f(4x - y) = f(2x + y) + f(2x - y) + 24f(x),$$

which, by (2.25), gives

$$f(4x + y) + f(4x - y) = f(x + y) + f(x - y) + 30f(x). \tag{2.26}$$

By comparing (2.24) with (2.26), we conclude that

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

which implies that $f$ is quadratic. □

Our main result in this section is

**Theorem 2.3.** *A mapping $f : X \to Y$ satisfies the equation* (1.2) *for all $x,\ y \in X$ if and only if there exist a cubic mapping $C : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = C(x) + Q(x)$ for all $x \in X$.*

*Proof.* (⇒) Define the mappings $C,\ Q : X \to Y$ by $C(x) = \frac{1}{2}[f(x) - f(-x)]$ and $Q(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$, respectively. Then we have $C(0) = 0,\ C(-x) = -C(x),\ Q(-x) = Q(x)$,

$$2C(x + 3y) + 6C(x - y) + 12C(2y) = 2C(x - 3y) + 6C(x + y) + 3C(4y) \tag{2.27}$$

and

$$2Q(x + 3y) + 6Q(x - y) + 12Q(2y) = 2Q(x - 3y) + 6Q(x + y) + 3Q(4y) \tag{2.28}$$

for all $x,\ y \in X$.

First, we claim that $C$ is cubic. If we let $x := y$ in (2.27), we get

$$8C(2y) = C(4y), \tag{2.29}$$

and replacing $y$ by $\frac{y}{2}$ in (2.29) gives

$$C(2y) = 8C(y).$$

Therefore the equation (2.27) is reduced to the form

$$C(x+3y) + 3C(x-y) = C(x-3y) + 3C(x+y) + 48C(y)$$

for all $x,\ y \in X$ and Lemma 2.1 guarantees that $C$ is cubic.

Secondly, we claim that $Q$ is quadratic. By letting $x = y = 0$ in (2.28), we get $Q(0) = 0$. If we put $x = 0$ in (2.28) and then replace $y$ by $\frac{y}{2}$, we have

$$Q(2y) = 4Q(y).$$

Hence (2.28) can be written in the form

$$Q(x+3y) + 3Q(x-y) = Q(x-3y) + 3Q(x+y),$$

which shows that $Q$ is quadratic according to Lemma 2.2.

That is, if $f : X \to Y$ satisfies the equation (1.2), then we have $f(x) = C(x) + Q(x)$ for all $x \in X$.

($\Leftarrow$) Suppose that there exist a cubic mapping $C : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = C(x) + Q(x)$ for all $x \in X$.

Since $C(2x) = 8C(x)$ and $Q(2x) = 4Q(x)$ for all $x \in X$, it follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned}
&2f(x+3y) + 6f(x-y) + 12f(2y) - 2f(x-3y) - 6f(x+y) - 3f(4y)\\
&= 2C(x+3y) + 6C(x-y) + 12C(2y) - 2C(x-3y) - 6C(x+y) - 3C(4y)\\
&\quad +2Q(x+3y) + 6Q(x-y) + 12Q(2y) - 2Q(x-3y) - 6Q(x+y) - 3Q(4y)\\
&= 2[C(x+3y) + 3C(x-y) - C(x-3y) - 3C(x+y) - 48C(y)]\\
&\quad +2[Q(x+3y) + 3Q(x-y) - Q(x-3y) - 3Q(x+y)] = 0
\end{aligned}$$

for all $x,\ y \in X$. □

## 3. Stability of equation (1.2) in $p$-Banach spaces

In this section $X$ and $Y$ will be a quasi-normed space and a $p$-Banach space, respectively. Given a mapping $f : X \to Y$, we set

$$Df(x,y) := 2f(x+3y) + 6f(x-y) + 12f(2y) - 2f(x-3y) - 6f(x+y) - 3f(4y)$$

for all $x,\ y \in X$. Let $\phi : X \times X \to [0,\infty)$ be a mapping satisfying one of the conditions (3.1) and (3.2), and one of the conditions (3.3) and (3.4) below:

$$\text{(3.1)} \qquad \varepsilon_1(x) := \frac{1}{16^p}\sum_{i=0}^{\infty}\frac{1}{8^{pi}}\alpha(2^i x)^p < \infty, \quad \frac{\phi(2^n x, 2^n y)}{8^n} \to 0 \quad \text{as} \quad n \to \infty,$$

$$(3.2)\quad \varepsilon_2(x) := \frac{1}{2^p}\sum_{i=0}^{\infty} 8^{pi}\alpha(2^{-(i+1)}x)^p < \infty, \quad 8^n\phi(2^{-n}x, 2^{-n}y) \to 0 \text{ as } n \to \infty,$$

where $\alpha(x) := \phi(\frac{x}{2}, \frac{x}{2}) + \phi(-\frac{x}{2}, -\frac{x}{2})$ for all $x,\ y \in X$, and

$$(3.3)\qquad \varepsilon_3(x) := \frac{1}{24^p}\sum_{i=0}^{\infty} \frac{1}{4^{pi}}\beta(2^i x)^p < \infty, \quad \frac{\phi(2^n x, 2^n y)}{4^n} \to 0 \text{ as } n \to \infty,$$

$$(3.4)\quad \varepsilon_4(x) := \frac{1}{6^p}\sum_{i=0}^{\infty} 4^{pi}\beta(2^{-(i+1)}x)^p < \infty, \quad 4^n\phi(2^{-n}x, 2^{-n}y) \to 0 \text{ as } n \to \infty,$$

where $\beta(x) := \phi(0, \frac{x}{2}) + \phi(0, -\frac{x}{2})$ for all $x,\ y \in X$.

**Theorem 3.1.** *If the mapping $f : X \to Y$ satisfies the inequality*

$$(3.5)\qquad \|Df(x, y)\| \le \phi(x, y)$$

*for all $x,\ y \in X$, then there exist a unique cubic mapping $C : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$(3.6)\qquad \|f(x) - (C(x) + Q(x))\| \le [\varepsilon_k(x) + \varepsilon_j(x)]^{\frac{1}{p}},$$

$$(3.7)\qquad \left\|\frac{f(x) - f(-x)}{2} - C(x)\right\| \le \varepsilon_k(x)^{\frac{1}{p}},$$

*and*

$$(3.8)\qquad \left\|\frac{f(x) + f(-x)}{2} - Q(x)\right\| \le \varepsilon_j(x)^{\frac{1}{p}}$$

*for all $x \in X$, where $k = 1$ or $2$ and $j = 3$ or $4$.*

*The mappings $C$ and $Q$ are given by*

$$C(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(2^n x) - f(-2^n x)}{2 \cdot 8^n} & \text{if } \phi \text{ satisfies (3.3)} \\ \lim_{n\to\infty} 8^n \cdot \dfrac{1}{2}\left[f(\dfrac{x}{2^n}) - f(-\dfrac{x}{2^n})\right] & \text{if } \phi \text{ satisfies (3.4)} \end{cases}$$

$$Q(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} & \text{if } \phi \text{ satisfies (3.3)} \\ \lim_{n\to\infty} 4^n \cdot \dfrac{1}{2}\left[f(\dfrac{x}{2^n}) + f(-\dfrac{x}{2^n})\right] & \text{if } \phi \text{ satisfies (3.4)} \end{cases}$$

*for all $x \in X$.*

*Proof.* Let $g : X \to Y$ be the mapping defined by $g(x) = \frac{1}{2}[f(x) - f(-x)]$ for all $x \in X$. Then we have $g(0) = 0$, $g(-x) = -g(x)$ and

$$\begin{aligned} \|Dg(x,y)\| &= \|2g(x+3y) + 6g(x-y) + 12g(2y) \\ &\quad -2g(x-3y) - 6g(x+y) - 3g(4y)\| \\ &\leq \frac{1}{2}[\phi(x,y) + \phi(-x,-y)] \end{aligned} \tag{3.9}$$

for all $x,\ y \in X$. Putting $y := x$ in (3.9) yields

$$\|8g(2x) - g(4x)\|^p \leq \frac{1}{2^p}[\phi(x,x) + \phi(-x,-x)]^p, \tag{3.10}$$

which, by setting $x := \frac{x}{2}$ and dividing by $8^p$ in (3.10), gives

$$\left\|g(x) - \frac{g(2x)}{8}\right\|^p \leq \frac{1}{16^p}\alpha(x)^p \tag{3.11}$$

for all $x \in X$.

Assume that $\phi$ satisfies the condition (3.1). Substituting $2x$ for $x$ in (3.11) and dividing by $8^p$, we get

$$\left\|\frac{g(2x)}{8} - \frac{g(2^2x)}{8^2}\right\|^p \leq \frac{1}{16^p}\frac{1}{8^p}\alpha(2x)^p$$

for all $x \in X$. An induction argument now implies that

$$\left\|g(x) - \frac{g(2^nx)}{8^n}\right\|^p \leq \frac{1}{16^p}\sum_{i=0}^{n-1}\frac{1}{8^{pi}}\alpha(2^ix)^p \tag{3.12}$$

for all $x \in X$. We claim that $\{8^{-n}g(2^nx)\}$ is a Cauchy sequence in $Y$. For $m < n$,

$$\begin{aligned} &\|8^{-n}g(2^nx) - 8^{-m}g(2^mx)\|^p \\ \leq\ & \sum_{i=m}^{n-1}\|8^{-i}g(2^ix) - 8^{-(i+1)}g(2^{i+1}x)\|^p \\ \leq\ & \frac{1}{16^p}\sum_{i=m}^{n-1}\frac{1}{8^{pi}}\alpha(2^ix)^p \end{aligned} \tag{3.13}$$

for all $x \in X$. Taking the limit as $m \to \infty$, we get

$$\lim_{m\to\infty}\|8^{-n}g(2^nx) - 8^{-m}g(2^mx)\|^p = 0$$

for all $x \in X$. Since $Y$ is a Banach space, it follows that the sequence $\{8^{-n}g(2^n)\}$ converges. We define a mapping $C : X \to Y$ by

$$C(x) = \lim_{n\to\infty} 8^{-n}g(2^nx) \tag{3.14}$$

for all $x \in X$. It is clear that $C(-x) = -C(x)$ for all $x \in X$, and it follows from (3.14) that

$$\begin{aligned}\|DC(x,y)\|^p &= \lim_{n\to\infty} 8^{-pn}\|Dg(2^n x, 2^n y)\|^p \\ &\le \frac{1}{2}\lim_{n\to\infty} 8^{-pn}[\phi(2^n x, 2^n y)^p + \phi(-2^n x, -2^n y)^p] = 0\end{aligned}$$

for all $x, y \in X$. Hence we see that $C$ is cubic as in the proof of Theorem 2.3. To prove the inequality (3.7), taking the limit in (3.12) as $n \to \infty$, we have

$$\|g(x) - C(x)\| \le \varepsilon_1(x)^{\frac{1}{p}} \tag{3.15}$$

for all $x \in X$. Now it remains to show that $C$ is unique. Suppose that $\widetilde{C} : X \to Y$ is another cubic mapping satisfying (3.15). Then it is obvious that $C(2x) = 8C(x)$ for all $x \in X$, and so it follows from (3.15) that

$$\begin{aligned}\|\widetilde{C}(x) - C(x)\|^p &= 8^{-pn}\|\widetilde{C}(2^n x) - C(2^n x)\|^p \\ &\le 8^{-pn}(\|\widetilde{C}(2^n x) - g(2^n x)\|^p + \|g(2^n x) - C(2^n x)\|^p) \\ &\le 2\cdot 8^{-pn}\varepsilon_1(2^n x)\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in this inequality, we have $\widetilde{C}(x) = C(x)$ for all $x \in X$.

If $\phi$ satisfies the condition (3.2), then we replace $x$ by $\dfrac{x}{4}$ in (3.10) to obtain

$$\left\|g(x) - 8g(2^{-1}x)\right\|^p \le \frac{1}{2^p}\alpha(2^{-1}x)^p$$

for all $x \in X$. By following the corresponding part of the proof of the case (3.1), we see that the inequality

$$\left\|g(x) - 8^n g(2^{-n}x)\right\|^p \le \frac{1}{2^p}\sum_{i=0}^{n-1} 8^{pi}\alpha(2^{-(i+1)}x)^p$$

holds for all $x \in X$ and $\{8^n g(2^{-n}x)\}$ is a Cauchy sequence in $Y$, from which the mapping $C : X \to Y$ defined by

$$C(x) = \lim_{n\to\infty} 8^n g(2^{-n}x)$$

for all $x \in X$ is cubic and unique such that

$$\|g(x) - C(x)\| \le \varepsilon_2(x)^{\frac{1}{p}}$$

for all $x \in X$.

Now let $h : X \to Y$ be the mapping defined by $h(x) = \frac{1}{2}[f(x) + f(-x)]$ for all $x \in X$. Then we have $h(-x) = h(x)$ and

$$\begin{aligned} \|Dh(x,y)\| &= \|2h(x+3y) + 6h(x-y) + 12h(2y) \\ &\quad -2h(x-3y) - 6h(x+y) - 3h(4y)\| \\ &\leq \frac{1}{2}[\phi(x,y) + \phi(-x,-y)] \end{aligned} \tag{3.16}$$

for all $x,\ y \in X$. By setting $x := 0$ in (3.16) and then letting $y := x$, we get

$$\|12h(2x) - 3h(4x)\|^p \leq \frac{1}{2^p}[\phi(0,x) + \phi(0,-x)]^p. \tag{3.17}$$

Replacing $x$ by $\frac{x}{2}$ in (3.17) and then dividing by $12^p$, we obtain

$$\left\|h(x) - \frac{h(2x)}{4}\right\|^p \leq \frac{1}{24^p}\beta(x)^p \tag{3.18}$$

for all $x \in X$.

Assume that $\phi$ satisfies the condition (3.3). Substituting $2x$ for $x$ in (3.18) and dividing by $4^p$, we get

$$\left\|\frac{h(2x)}{4} - \frac{h(2^2x)}{4^2}\right\|^p \leq \frac{1}{24^p}\cdot\frac{1}{4^p}\beta(2x)^p$$

for all $x \in X$. By induction we see that

$$\left\|h(x) - \frac{h(2^nx)}{4^n}\right\|^p \leq \frac{1}{24^p}\sum_{i=0}^{n-1}\frac{1}{4^{pi}}\beta(2^ix)^p \tag{3.19}$$

for all $x \in X$. We claim that $\{4^{-n}h(2^n)\}$ is a Cauchy sequence in $Y$. For $m < n$,

$$\begin{aligned} &\|4^{-n}h(2^nx) - 4^{-m}h(2^mx)\|^p \\ \leq\ &\sum_{i=m}^{n-1}\|4^{-i}h(2^ix) - 4^{-(i+1)}h(2^{i+1}x)\|^p \\ \leq\ &\frac{1}{24^p}\sum_{i=m}^{n-1}\frac{1}{4^{pi}}\beta(2^ix)^p \end{aligned}$$

for all $x \in X$. Taking the limit as $m \to \infty$, we get

$$\lim_{m\to\infty}\|4^{-n}h(2^nx) - 4^{-m}h(2^mx)\|^p = 0$$

for all $x \in X$. Since $Y$ is a Banach space, it follows that the sequence $\{4^{-n}h(2^nx)\}$ converges. We define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}4^{-n}h(2^nx) \tag{3.20}$$

for all $x \in X$. It is clear that $Q(-x) = Q(x)$ for all $x \in X$, and it follows from (3.20) that

$$\begin{aligned}\|DQ(x,y)\|^p &= \lim_{n\to\infty} 4^{-pn}\|Dh(2^n x, 2^n y)\|^p \\ &\leq \frac{1}{2}\lim_{n\to\infty} 4^{-pn}[\phi(2^n x, 2^n y)^p + \phi(-2^n x, -2^n y)^p] = 0\end{aligned}$$

for all $x, y \in X$. Thus we see that $Q$ is quadratic as in the proof of Theorem 2.3. By taking the limit in (3.19) as $n \to \infty$ to prove the inequality (3.8), we obtain

$$\|h(x) - Q(x)\| \leq \varepsilon_3(x)^{\frac{1}{p}} \tag{3.21}$$

for all $x \in X$. To show that $Q$ is unique, let us assume that $\widetilde{Q} : X \to Y$ is another quadratic mapping satisfying (3.21). Then it is obvious that $Q(2x) = 4Q(x)$ for all $x \in X$, and so it follows from (3.21) that

$$\begin{aligned}\|\widetilde{Q}(x) - Q(x)\|^p &= 4^{-pn}\|\widetilde{Q}(2^n x) - Q(2^n x)\|^p \\ &\leq 4^{-pn}(\|\widetilde{Q}(2^n x) - h(2^n x)\|^p + \|h(2^n x) - Q(2^n x)\|^p) \\ &\leq 2 \cdot 4^{-pn}\varepsilon_3(2^n x)\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in this inequality, we have $\widetilde{Q}(x) = Q(x)$ for all $x \in X$. If $\phi$ satisfies the condition (3.4), then we replace $x$ by $\frac{x}{4}$ in (3.17) and divide by $3^p$ to obtain

$$\left\|h(x) - 4h(2^{-1}x)\right\|^p \leq \frac{1}{6^p}\beta(2^{-1}x)^p$$

for all $x \in X$. The rest of the proof goes through the corresponding part of the proof of the case (3.3), that is, the inequality

$$\left\|h(x) - 4^n h(2^{-n}x)\right\|^p \leq \frac{1}{6^p}\sum_{i=0}^{n-1} 4^{pi}\beta(2^{-(i+1)}x)^p$$

holds for all $x \in X$ and $\{4^n h(2^{-n})\}$ is a Cauchy sequence in $Y$, whence we obtain the unique quadratic mapping $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty} 4^n h(2^{-n}x)$$

for all $x \in X$ such that

$$\|h(x) - Q(x)\| \leq \varepsilon_4(x)^{\frac{1}{p}}$$

for all $x \in X$. Since we have $f(x) = g(x) + h(x)$ for all $x \in X$, we see that

$$\begin{aligned}&\|f(x) - (C(x) + Q(x))\|^p \\ \leq\ &\|g(x) - C(x)\|^p + \|h(x) - Q(x)\|^p \\ \leq\ &\varepsilon_k(x) + \varepsilon_j(x)\end{aligned}$$

for all $x \in X$, where $k = 1$ or 2 and $j = 3$ or 4. We complete the proof of the theorem. $\square$

From Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [25] of the functional equation (1.2).

Let $q \neq 2, 3$ be any real number. For the sake of convenience, let

$$\lambda_1(q) := \frac{1}{2^{(q+2)p}(1 - 2^{(q-3)p})}, \quad \lambda_2(q) := \frac{1}{2^{(2q-1)p}(1 - 2^{(3-q)p})},$$

and

$$\lambda_3(q) := \frac{1}{3^p \cdot 2^{(q+2)p}(1 - 2^{(q-2)p})}, \quad \lambda_4(q) := \frac{1}{3^p \cdot 4^{qp}(1 - 2^{(2-q)p})}.$$

**Corollary 3.2.** *Let $q \neq 2, 3$ and $\theta > 0$ be real numbers. If the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \leq \theta(\|x\|^q + \|y\|^q)$$

*for all $x,\ y \in X$, then there exist a unique cubic mapping $C : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - (C(x) + Q(x))\| \leq \lambda(q)^{\frac{1}{p}} \theta \|x\|^q,$$

$$\left\| \frac{f(x) - f(-x)}{2} - C(x) \right\| \leq \lambda_k(q)^{\frac{1}{p}} \theta \|x\|^q \quad (k = 1 \quad or \quad 2),$$

*and*

$$\left\| \frac{f(x) + f(-x)}{2} - Q(x) \right\| \leq \lambda_j(q)^{\frac{1}{p}} \theta \|x\|^q \quad (j = 3 \quad or \quad 4)$$

*for all $x \in X$, where*

$$\lambda(q) = \begin{cases} \lambda_2(q) + \lambda_4(q) & \text{if } q > 3 \\ \lambda_1(q) + \lambda_4(q) & \text{if } 2 < q < 3 \\ \lambda_1(q) + \lambda_3(q) & \text{if } q < 2. \end{cases}$$

*The mappings $C$ and $Q$ are given by*

$$C(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(2^n x) - f(-2^n x)}{2 \cdot 8^n} & \text{if } q < 3 \\ \lim_{n\to\infty} 8^n \cdot \dfrac{1}{2}\left[f(\dfrac{x}{2^n}) - f(-\dfrac{x}{2^n})\right] & \text{if } q > 3, \end{cases}$$

$$Q(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} & \text{if } q < 2 \\ \lim_{n\to\infty} 4^n \cdot \dfrac{1}{2}\left[f(\dfrac{x}{2^n}) + f(-\dfrac{x}{2^n})\right] & \text{if } q > 2 \end{cases}$$

*for all $x \in X$.*

*Proof.* Let $\phi(x, y) := \theta(\|x\|^q + \|y\|^q)$ for all $x \in X$. If $q < 3$, then a simple calculation gives $\alpha(2^i x) = 4 \cdot 2^{(i-1)q}\theta\|x\|^q$, and so we have

$$\varepsilon_1(x) = \frac{1}{16^p}\sum_{i=0}^{\infty}\frac{1}{8^{pi}}\alpha(2^i x)^p = \lambda_1(q)\theta^p\|x\|^{qp}$$

for all $x \in X$. If $q > 3$, then, by considering $\alpha(2^{-(i+1)}x) = 4 \cdot 2^{-(i+2)q}\theta\|x\|^q$, we obtain

$$\varepsilon_2(x) = \frac{1}{2^p}\sum_{i=0}^{\infty}8^{pi}\alpha(2^{-(i+1)}x)^p = \lambda_2(q)\theta^p\|x\|^{qp}$$

for all $x \in X$. On the other hand, suppose that $q < 2$. Since $\beta(2^i x) = 2 \cdot 2^{(i-1)q}\theta\|x\|^q$, we see that

$$\varepsilon_3(x) = \frac{1}{24^p}\sum_{i=0}^{\infty}\frac{1}{4^{pi}}\beta(2^i x)^p = \lambda_3(q)\theta^p\|x\|^{qp}$$

for all $x \in X$. Finally, if $q > 2$, then we know that

$$\varepsilon_4(x) = \frac{1}{6^p}\sum_{i=0}^{\infty}4^{pi}\beta(2^{-(i+2)}x)^p = \lambda_4(q)\theta^p\|x\|^{qp}$$

because of $\beta(2^{-(i+1)}x) = 2 \cdot 2^{-(i+2)q}\theta\|x\|^q$ for all $x \in X$. Therefore, we deduce that

$$\varepsilon_k(x) + \varepsilon_j(x) := \lambda(q)\theta^p\|x\|^q = \begin{cases} (\lambda_2(q) + \lambda_4(q))\theta^p\|x\|^{qp} & \text{if } q > 3 \\ (\lambda_1(q) + \lambda_4(q))\theta^p\|x\|^{qp} & \text{if } 2 < q < 3 \\ (\lambda_1(q) + \lambda_3(q))\theta^p\|x\|^{qp} & \text{if } q < 2 \end{cases}$$

for all $x \in X$. □

The following corollary is the Hyers-Ulam stability of the equation (1.2) which is an immediate consequence of Corollary 3.2 by setting $\theta := \frac{1}{2}\theta$ and $q = 0$.

**Corollary 3.3.** *Let $\theta > 0$ be a real number. If the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \le \theta$$

*for all $x,\ y \in X$, then there exist a unique cubic mapping $C : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - (C(x) + Q(x))\| \le \left[\frac{1}{(8^p - 1)^{1/p}} + \frac{1}{6(4^p - 1)^{1/p}}\right]\theta,$$

$$\left\|\frac{f(x) - f(-x)}{2} - C(x)\right\| \le \frac{1}{(8^p - 1)^{1/p}}\,\theta,$$

*and*

$$\left\| \frac{f(x) + f(-x)}{2} - Q(x) \right\| \leq \frac{1}{6(4^p - 1)^{1/p}}\,\theta$$

*for all* $x \in X$.

*The mappings* $C$ *and* $Q$ *are given by*

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2 \cdot 8^n}$$

*and*

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n}$$

*for all* $x \in X$.

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