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Uniqueness Theorems For Differential Polynomials Concerning Fixed-Points

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Abstract. In this article, we deal with the uniqueness problems on meromorphic functions concerning differential polynomials that share fixed-points. Moreover we extend former results of W. C. Lin and H. X. Yi.

1. Introduction, definitions and results

In this paper, the term meromorphic will always mean meromorphic in the complex plane C. Let a be a complex number and $\alpha(z)$ be a meromorphic function such that $T(r,\alpha) = o(T(r,f))$, we say f and g share the value a CM, if $f-a$ and $g-a$ assume the same zeros with the same multiplicity ; if $f(z)-\alpha(z)$ and $g(z)-\alpha(z)$ assume the same zeros with the same multiplicities, then we say $f(z)$ and $g(z)$ share $\alpha(z)$ CM, especially we say that $f(z)$ and $g(z)$ have the same fixed-points when $\alpha(z) = z$. It is assumed that the reader is familiar with the notations of the Nevanlinna theory, that can be found, for instance, in [4]. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_2\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_k\left(r, \frac{1}{f-a}\right).$$

It is well known that if f and g share four distinct values CM, then f is a fractional transformation of g. In 1997, corresponding to one famous question of Hayman, C. C. Yang and X. H. Hua showed the similar conclusions hold for certain types of differential polynomials when they share only one value. They proved the following result.

Theorem A([7]). *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ an integer and $a \in C - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some (n+1)th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.*

In 2001, M. L. Fang and W. Hong obtained the following result.

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Theorem B([2]). *Let f and g be two transcendental entire functions, $n \geq 11$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.*

Recently, W. C. Lin and H. X. Yi extended the above theorem in view of the fixed-point. They proved the following results.

Theorem C([5]). *Let f and g be two transcendental meromorphic functions, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z CM, then either $f(z) \equiv g(z)$ or*

$$f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$$

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$$

where h is a nonconstant meromorphic function.

Theorem D([5]). *Let f and g be two transcendental meromorphic functions, $n \geq 13$ an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share z CM, then $f \equiv g$.*

Now one may ask the following question which is the motivation of the paper: Can the nature of fixed-point z be relaxed to IM in the above theorems? In the paper, we investigate the solution of the above question. We now state the following two theorems which answer the above question.

Theorem 1. *Let f and g be two transcendental meromorphic functions, $n \geq 27$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share z IM, then either $f(z) \equiv g(z)$ or*

$$f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}.$$

Theorem 2. *Let f and g be two transcendental meromorphic functions, $n \geq 28$ an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share z IM, then $f \equiv g$.*

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote H the following function :

$$H = \left(\frac{F''}{F'} - 2\frac{F'}{F-1}\right) - \left(\frac{G''}{G'} - 2\frac{G'}{G-1}\right).$$

Lemma 1([6]). *Let f be a nonconstant meromorphic function, and let $a_1, a_2, \cdots, a_n$ be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_nf^n + \cdots + a_2f^2 + a_1f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2([1]). *Let f, g be two nonconstant meromorphic functions such that they share 1 IM and $H \not\equiv 0$, then*

$$\begin{aligned} T(r,f) \leq\ & N_2\left(r,\frac{1}{f}\right)+N_2(r,f)+N_2\left(r,\frac{1}{g}\right)+N_2(r,g)+2\overline{N}\left(r,\frac{1}{f}\right)+2\overline{N}(r,f) \\ & +\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,g)+S(r,f)+S(r,g). \end{aligned}$$

Lemma 3(e[3]). Let

$$Q(\omega)=(n-1)^2(\omega^n-1)(\omega^{n-2}-1)-n(n-2)(\omega^{n-1}-1)^2$$

then

$$Q(\omega)=(\omega-1)^4(\omega-\beta_1)(\omega-\beta_2)\cdots(\omega-\beta_{2n-6})$$

where $\beta_j \in C-\{0,1\}$ $(j=1,2,\cdots,2n-6)$, which are distinct respectively.

Lemma 4([9]). *Let f be a nonconstant meromorphic function, k be a positive integer, then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f)$$

where $N_p\left(r,\frac{1}{f^{(k)}}\right)$ denotes the counting function of the zeros of $\frac{1}{f^{(k)}}$ where a zero of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Clearly $\overline{N}\left(r,\frac{1}{f^{(k)}}\right)=N_1\left(r,\frac{1}{f^{(k)}}\right)$.

Lemma 5. *Let f and g be two nonconstant meromorphic functions, $m < 7$, $n > 7-m$ positive integers, $\alpha(z)$ denotes as in Section 1 and $\alpha \not\equiv 0, \infty$, and let*

$$F=f^n(f-1)^mf', G=g^n(g-1)^mg'$$

If F and G share $\alpha(z)$ IM, then $S(r,f)=S(r,g)$.

Proof. By Lemma 1, we have

$$(n+m)T(r,f)=T(r,f^n(f-1)^m)+S(r,f)\leq T(r,F)+T(r,f')+S(r,f)$$

Therefore

$$T(r,F)\geq (n+m-2)T(r,f)+S(r,f)$$

By the second fundamental theorem and Lemma 4, we have

$$\begin{aligned} & T(r,F) \\ \leq\ & \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\alpha}\right)+S(r,f) \\ \leq\ & \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{G-\alpha}\right)+S(r,f) \\ \leq\ & 5T(r,f)+T(r,G)+S(r,f) \end{aligned}$$

Note that $T(r,G)\leq T(r,g^n(g-1)^m)+T(r,g')\leq (n+m+2)T(r,g)+S(r,g)$, we deduce that

$$(n+m-7)T(r,f)\leq (n+m+2)T(r,g)+S(r,g)+S(r,f)$$

It follows that the conclusion holds. □

Lemma 6([8]). *Let H be defined as above. If $H\equiv 0$ and*

$$\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)}{T(r)}<1,\ \ r\in I$$

where $T(r)=\max\{T(r,F),T(r,G)\}$ and I is a set with infinite linear measure, then $F\equiv G$ or $FG\equiv 1$.

3. Proof of the main theorems

Proof of Theorem 1. Lemma 4 implies that $S(r,f)=S(r,g)$. Let

$$F=\frac{f^n(f-1)f'}{z} \tag{3.1}$$

$$G=\frac{g^n(g-1)g'}{z} \tag{3.2}$$

and

$$F^*=\frac{1}{n+2}f^{n+2}-\frac{1}{n+1}f^{n+1} \tag{3.3}$$

$$G^*=\frac{1}{n+2}g^{n+2}-\frac{1}{n+1}g^{n+1} \tag{3.4}$$

Thus we obtain that F and G share 1 IM. Moreover, by Lemma 1, we have

$$T(r,F^*)=(n+2)T(r,f)+S(r,f), \tag{3.5}$$

$$T(r,G^*) = (n+2)T(r,g) + S(r,g). \tag{3.6}$$

Since $(F^*)' = Fz$, we deduce

$$m\left(r,\frac{1}{F^*}\right) \le m\left(r,\frac{1}{zF}\right) + S(r,f) \le m\left(r,\frac{1}{F}\right) + \log r + S(r,f) \tag{3.7}$$

and by the first fundamental theorem

$$T(r,F^*) \le T(r,F) + N\left(r,\frac{1}{F^*}\right) - N\left(r,\frac{1}{F}\right) + \log r + S(r,f). \tag{3.8}$$

Note that

$$N\left(r,\frac{1}{F^*}\right) = (n+1)N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-\frac{n+2}{n+1}}\right), \tag{3.9}$$

$$N\left(r,\frac{1}{F}\right) = nN\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{f'}\right). \tag{3.10}$$

So we have

$$\begin{aligned} T(r,F^*) \le\ & T(r,F) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-\frac{n+2}{n+1}}\right) + \log r \\ & -N\left(r,\frac{1}{f-1}\right) - N\left(r,\frac{1}{f'}\right) + S(r,f). \end{aligned} \tag{3.11}$$

Similarly we have

$$\begin{aligned} T(r,G^*) \le\ & T(r,G) + N\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g-\frac{n+2}{n+1}}\right) + \log r \\ & -N\left(r,\frac{1}{g-1}\right) - N\left(r,\frac{1}{g'}\right) + S(r,g). \end{aligned} \tag{3.12}$$

Since F and G share 1 IM, by Lemma 2 we have

$$\begin{aligned} & T(r,F) + T(r,G) \\ \le\ & 2\left(N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)\right) + 3\overline{N}\left(r,\frac{1}{F}\right) \\ & +3\overline{N}(r,F) + 3\overline{N}\left(r,\frac{1}{G}\right) + 3\overline{N}(r,G) + S(r,F) + S(r,G). \end{aligned} \tag{3.13}$$

Obviously we have

$$\begin{aligned} & N_2(r,F) + N_2\left(r,\frac{1}{F}\right) \\ \le\ & 2\overline{N}(r,f) + 2N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f-1}\right) + N\left(r,\frac{1}{f'}\right) + \log r, \end{aligned} \tag{3.14}$$

$$N_2(r,G)+N_2\left(r,\frac{1}{G}\right) \le 2\overline{N}(r,g)+2N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+\log r. \tag{3.15}$$

So from $(3.11),(3.12)$ we have

$$\begin{aligned} &T(r,F^*)+T(r,G^*)\\ \le\ & T(r,F)+T(r,G)+T(r,F)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-\frac{n+2}{n+1}}\right)+\log r\\ &-N\left(r,\frac{1}{f-1}\right)-N\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-\frac{n+2}{n+1}}\right)+\log r\\ &-N\left(r,\frac{1}{g-1}\right)-N\left(r,\frac{1}{g'}\right)+S(r,f). \end{aligned} \tag{3.16}$$

By $(3.13),(3.14),(3.15),(3.16)$ we get

$$(n-26)T(r,f)+(n-26)T(r,g)\le 6\log r+S(r,f)+S(r,g). \tag{3.17}$$

We obtain that $n\le 26$ which contradicts $n\ge 27$. Therefore $H\equiv 0$. That is

$$\frac{F''}{F'}-2\frac{F'}{F-1}\equiv\frac{G''}{G'}-2\frac{G'}{G-1}. \tag{3.18}$$

By integration, we have

$$\frac{A}{F-1}+B=\frac{1}{G-1} \tag{3.19}$$

where $A(\neq 0)$ and B are constants. Thus

$$T(r,F)=T(r,G)+S(r,f). \tag{3.20}$$

Since

$$\overline{N}\left(r,\frac{1}{f'}\right)\le T(r,f')-m\left(r,\frac{1}{f'}\right)\le 2T(r,f)-m\left(r,\frac{1}{f'}\right)+S(r,f), \tag{3.21}$$

we note that

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+\overline{N}(r,G)\\ \le\ & \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+\overline{N}(r,g)\\ &+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &+2\log r+S(r,f) \end{aligned} \tag{3.22}$$

and

$$T(r,F)+m\left(r,\frac{1}{f'}\right) = T\left(r,\frac{f^n(f-1)f'}{z}\right)+m\left(r,\frac{1}{f'}\right) \geq T(r,f^n(f-1))-\log r. \tag{3.23}$$

Similarly we have

$$T(r,G)+m\left(r,\frac{1}{g'}\right) \geq T(r,f^n(f-1))-\log r. \tag{3.24}$$

From $(3.21),(3.22),(3.23),(3.24)$ and we apply Lemma 6, we get $F\equiv G$ or $FG\equiv 1$. We discuss the following two cases.

Case 1. Suppose that $FG\equiv 1$. That is

$$f^n(f-1)f'g^n(g-1)g' \equiv z^2. \tag{3.25}$$

Let $z_0(\neq 0,\infty)$ be a zero of f of order p, so z_0 is a pole of g. Suppose that z_0 is pole of g of order q. From above we obtain

$$np+p-1=nq+2q+1 \tag{3.26}$$

that is, $(n+1)(p-q)=q+2$, which implies that $p\geq q+1$ and $q+2\geq n+1$. Hence $p\geq n$. Let $z_1(\neq 0,\infty)$ be a zero of $f-1$ of order p_1, then we can also deduce that $p_1\geq \frac{n}{2}+2$. Let $z_2(\neq 0,\infty)$ be a zero of f' of order p_2 that is not a zero of $f(f-1)$, we similarly have $p_2\geq n+3$. Moreover, in the same manner as above, we have the similar results for the zeros of $g^n(g-1)g'$. On the other hand, suppose that $z_3(\neq 0,\infty)$ is a pole of f, we get that z_3 is the zero of $g^n(g-1)g'$, thus

$$\begin{aligned}\overline{N}(r,f) &\leq \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\leq \frac{1}{27}N\left(r,\frac{1}{g}\right)+\frac{2}{31}N\left(r,\frac{1}{g-1}\right)+\frac{1}{30}N\left(r,\frac{1}{g'}\right)\\ &< \frac{2}{3}T(r,g)+S(r,g).\end{aligned} \tag{3.27}$$

By the second fundamental theorem, we have

$$\begin{aligned}T(r,f) &\leq \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+S(r,f)\\ &< \frac{5}{24}T(r,f)+\frac{2}{3}T(r,g)+2\log r+S(r,g).\end{aligned} \tag{3.28}$$

Similarly we have

$$T(r,g)<\frac{5}{24}T(r,g)+\frac{2}{3}T(r,f)+2\log r+S(r,f). \tag{3.29}$$

From (3.28), (3.29) we deduce a contradiction.

Case 2: If $F \equiv G$. That is

$$F^* \equiv G^* + c, \tag{3.30}$$

where c is a constant. It follows that

$$T(r,f) = T(r,g) + S(r,f). \tag{3.31}$$

Suppose that $c \neq 0$. By the second fundamental theorem, we have

$$\begin{aligned} &(n+2)T(r,g) = T(r,G^*) \\ < \ & \overline{N}\left(r,\frac{1}{G^*}\right) + \overline{N}\left(r,\frac{1}{G^*+c}\right) + \overline{N}(r,G^*) + S(r,g) \\ \leq \ & \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-\frac{n+2}{n+1}}\right) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{f}\right) \\ & +\overline{N}\left(r,\frac{1}{f-\frac{n+2}{n+1}}\right) + S(r,f) \\ \leq \ & 5T(r,f) + S(r,f), \end{aligned} \tag{3.32}$$

which contradicts the assumption. Therefore $F^* \equiv G^*$, that is

$$f^{n+1}\left(f - \frac{n+2}{n+1}\right) = g^{n+1}\left(g - \frac{n+2}{n+1}\right). \tag{3.33}$$

Let $h = f/g$. If $h \equiv 1$, that is $f \equiv g$. If $h \not\equiv 1$, we deduce that

$$f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},\ g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \tag{3.34}$$

where h is a nonconstant meromorphic function. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let

$$F = \frac{f^n(f-1)^2 f'}{z},\ G = \frac{g^n(g-1)^2 g'}{z} \tag{3.35}$$

and

$$F^* = \frac{1}{n+3}f^{n+3} - \frac{2}{n+2}f^{n+2} + \frac{1}{n+1}f^{n+1} \tag{3.36}$$

$$G^* = \frac{1}{n+3}g^{n+3} - \frac{2}{n+2}g^{n+2} + \frac{1}{n+1}g^{n+1} \tag{3.37}$$

Thus we obtain F and G share 1 IM. Moreover by Lemma 1, we have

$$T(r,F^*)=(n+3)T(r,f)+S(r,f),T(r,G^*)=(n+3)T(r,g)+S(r,g). \tag{3.38}$$

Let H be defined as in Section 2. Suppose that $H\not\equiv 0$. Proceeding as in the proof of Theorem 1, we have

$$T(r,F^*)\le T(r,F)+N\left(r,\frac{1}{F^*}\right)-N\left(r,\frac{1}{F}\right)+\log r+S(r,f). \tag{3.39}$$

Note that

$$N\left(r,\frac{1}{F^*}\right)=(n+1)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right) \tag{3.40}$$

where a_1,a_2 are distinct roots of the algebraic equation $\frac{1}{n+3}z^2-\frac{2}{n+2}z+\frac{1}{n+1}=0$ and

$$N\left(r,\frac{1}{F}\right)=nN\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right). \tag{3.41}$$

Since F and G share 1 IM, by Lemma 2 we have

$$\begin{aligned}&T(r,F)+T(r,G)\\ \le\ &2(N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G))+3\overline{N}\left(r,\frac{1}{F}\right)\\ &+3\overline{N}(r,F)+3\overline{N}\left(r,\frac{1}{G}\right)+3\overline{N}(r,G)+S(r,F)+S(r,G)\end{aligned} \tag{3.42}$$

Obviously we have

$$\begin{aligned}&N_2(r,F)+N_2\left(r,\frac{1}{F}\right)\\ \le\ &2\overline{N}(r,f)+2N\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)+\log r,\end{aligned} \tag{3.43}$$

$$\begin{aligned}&N_2(r,G)+N_2\left(r,\frac{1}{G}\right)\\ \le\ &2\overline{N}(r,g)+2N\left(r,\frac{1}{g}\right)+2N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+\log r.\end{aligned} \tag{3.44}$$

So we have

$$
\begin{aligned}
(3.45)\qquad & T(r,F^*)+T(r,G^*)\\
\le\ & T(r,F)+T(r,G)+N\left(r,\frac{1}{F^*}\right)+N\left(r,\frac{1}{G^*}\right)\\
& -N\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{G}\right)+2\log r+S(r,f)+S(r,g)\\
\le\ & 2\left(N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right)+3\overline{N}\left(r,\frac{1}{F}\right)\\
& +3\overline{N}(r,F)+3\overline{N}\left(r,\frac{1}{G}\right)+3\overline{N}(r,G)+N\left(r,\frac{1}{F^*}\right)+N\left(r,\frac{1}{G^*}\right)\\
& -N\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{G}\right)+2\log r+S(r,f)+S(r,g).
\end{aligned}
$$

From (3.43), (3.44), (3.45) we have

$$(3.46)\qquad (n-27)T(r,f)+(n-27)T(r,g)\le 6\log r+S(r,f)+S(r,g),$$

we obtain a contradiction. Therefore $H\equiv 0$. Note that

$$
\begin{aligned}
(3.47)\qquad T(r,F)+m\left(r,\frac{1}{f'}\right) &= T(r,f^n(f-1)^2f')+m\left(r,\frac{1}{f'}\right)\\
&\ge T(r,f^n(f-1)^2)-\log r.
\end{aligned}
$$

From (3.21), (3.22), (3.47) we apply Lemma 6, we get $F\equiv G$ or $FG\equiv 1$. We discuss the following two cases.

Case 1: $FG\equiv 1$. In the same manner in the proof of Theorem 1, we again deduce a contradiction.

Case 2: $F\equiv G$. Thus $F^*\equiv G^*$, that is

$$
\begin{aligned}
(3.48)\qquad & \frac{1}{n+3}f^{n+3}-\frac{2}{n+2}f^{n+2}+\frac{1}{n+1}f^{n+1}\\
=\ & \frac{1}{n+3}g^{n+3}-\frac{2}{n+2}g^{n+2}+\frac{1}{n+1}g^{n+1}.
\end{aligned}
$$

Set $h=f/g$, we substitute $f=hg$ in the above, it follows that

$$
\begin{aligned}
(3.49)\qquad & (n+2)(n+1)g^2(h^{n+3}-1)-2(n+3)(n+1)g(h^{n+2}-1)\\
& +(n+2)(n+3)(h^{n+1}-1)=0.
\end{aligned}
$$

If h is not a constant, using Lemma 3, we can conclude that

$$(3.50)\quad \{(n+1)(n+2)(h^{n+3}-1)g-(n+3)(n+1)g(h^{n+2}-1)\}^2=-(n+3)(n+1)Q(h),$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{2n})$, $\beta_j \in C-\{0,1\}(j=1,2,\cdots,2n)$, which are pairwise distinct. This implies that every zero of $h-\beta_j(j=1,2,\cdots,2n)$ has a multiplicity of at least 2. By the second fundamental theorem we obtain that $n \leq 2$, which is again a contradiction. Therefore h is a constant. We have that $h^{n+1}-1=0$ and $h^{n+2}-1=0$, which imply $h=1$, and hence $f \equiv g$. This completes the proof of Theorem 2. □

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