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Reliability in Two Independent Uniform and Power Function-Half Normal Distribution

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Abstract

We consider estimation of reliability $P(Y < X)$ and distribution of the ratio when X and Y are independent uniform random variable and power function random variable, respectively and also consider the estimation problem when X and Y are independent uniform random variable and a half-normal random variable, respectively.

Keywords: Half-normal distribution; power function distribution; reliability.

1. Introduction

The power function distribution was introduced in Johnson *et al.* (1994) and Ali and Woo (2005a) studied inference on reliability in two independent power function distributions each having different parameter. And as a special case of power function distribution, it becomes a uniform distribution. Woo (2007a) studied reliability in two independent half-normal distributions each having different scale parameter. Ali and Woo (2005c) studied inference on reliability in a p-dimensional Rayleigh distribution.

Many authors have considered various aspects of a uniform distributions (see, Johnson, *et al.*, 1994). McCool (1991) considered inference on reliability $P(X < Y)$ in the Weibull case. Ali and Woo (2005b) considered inference on reliability $P(Y < X)$ when X and Y are the same Levy distributions. Woo (2006) presented reliability $P(Y < X)$, ratio $X/(X+Y)$ and a skewed-symmetric distribution of two independent random variables, Woo (2007b) studied reliability in a half-triangle distribution and a skew-symmetric distribution.

In this paper, we consider point and interval estimations of the reliability $P(Y < X)$ when X and Y are two independent uniform random variable and power function random variable, respectively, and in the second case when X and Y are independent uniform random variable and a half-normal random variable, respectively. Especially we consider distribution of the ratio when X and Y are two independent uniform random variable and power function random variable, respectively.

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2. Uniform and Power Function

In this section, we consider the case for the estimation of $P(Y < X)$ when (X, Y) is a pair of uniform and power function random variables, respectively. As an application of this case X, representing time to sustain temperature, is a uniform random variable and Y, representing time to sustain air pressure of a tube, is a power function random variable.

2.1. Reliability

Let X and Y be two independent a uniform random variable and a power function random variable (see, Johnson *et al.*, 1994) with the following densities:

$$f_X(x) = \frac{1}{\theta_1}, \qquad 0 < x < \theta_1,\ \theta_1 > 0,$$
$$f_Y(y) = \alpha\theta_2^{-\alpha}y^{\alpha-1}, \quad 0 < y < \theta_2,\ \theta_2 > 0,\ \alpha > 0. \tag{2.1}$$

Especially if $\alpha = 1$ in (2.1), then Y has a uniform distribution over $(0, \theta_2)$.

From the densities (2.1), we obtain the reliability $P(X < Y)$:

Fact 1. When X and Y are independent uniform random variable and power function random variable having the densities (2.1) with known $\alpha > 0$, then, for $\rho \equiv \theta_1/\theta_2$,

$$R \equiv P(X < Y) = \begin{cases} \dfrac{\alpha}{\alpha+1}\rho^{-1}, & \text{if } \theta_2 < \theta_1, \\ 1 - \dfrac{1}{1+\alpha}\rho^{\alpha}, & \text{if } \theta_2 \ge \theta_1, \end{cases}$$

which the reliability $R = P(X < Y)$ is a monotone function of $\rho \equiv \theta_1/\theta_2$.

We now consider estimation on the reliability $P(X < Y)$ when X and Y are independent uniform random variable and power function random variable, respectively having the densities (2.1) with known $\alpha > 0$. Because $R = P(X < Y)$ is a monotone function of ρ in Fact 1, inference on the reliability is equivalent to inference on ρ (see, McCool, 1991). Hence we only consider estimation on $\rho \equiv \theta_1/\theta_2$ instead of estimating $R = P(X < Y)$.

Assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent samples from X and Y with the densities in (2.1) with known $\alpha > 0$, respectively, then we have the followings:

The MLE $\hat{\theta}_i$ of θ_i, $i = 1, 2$ are given by:

$$\hat{\theta}_1 = X_{(m)} = \max(X_1, X_2, \ldots, X_m),$$
$$\hat{\theta}_2 = Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n).$$

The densities of $\hat{\theta}_1$ and $\hat{\theta}_2$ are given by

$$f_{X_{(m)}}(x) = \frac{m}{\theta_1^m}x^{m-1}, \quad 0 < x < \theta_1,$$
$$f_{Y_{(n)}}(x) = \frac{\alpha n}{\theta_2^{\alpha n}}x^{\alpha n-1}, \quad 0 < x < \theta_2, \quad \text{respectively.} \tag{2.2}$$

Table 2.1: MSE of the MLE and an unbiased estimator (units: ρ^2)

m	n	$\alpha=1/2$		$\alpha=1/2$		$\alpha=1$		$\alpha=2$		$\alpha=4$	
		$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$	$\hat{\rho}$	$\tilde{\rho}$
10	10	2.13636	0.81500	0.11616	0.07556	0.02146	0.02094	0.01205	0.01113	0.01239	0.00899
	20	0.11616	0.07556	0.02146	0.02094	0.02105	0.01113	0.01239	0.00897	0.01350	0.00849
	30	0.03846	0.03278	0.01349	0.01348	0.01198	0.00953	0.01307	0.00862	0.01339	0.00840
20	10	2.37085	0.80409	0.13420	0.06909	0.01996	0.01480	0.00509	0.00506	0.00334	0.00293
	20	0.13420	0.06909	0.01996	0.01480	0.00509	0.00506	0.00334	0.00293	0.00353	0.00243
	30	0.04187	0.02657	0.00813	0.00741	0.00358	0.00347	0.00339	0.00256	0.00373	0.00234
30	10	2.46169	0.80188	0.14315	0.06778	0.02134	0.01356	0.00432	0.00382	0.00173	0.00170
	20	0.14315	0.06778	0.02134	0.01356	0.00432	0.00382	0.00173	0.00170	0.00155	0.00120
	30	0.04516	0.02531	0.00799	0.00617	0.00224	0.00223	0.00154	0.00133	0.00144	0.00111

From the densities (2.2), we obtain the k-moments of $X_{(m)}$ and $1/Y_{(n)}$:

$$E\left(X_{(m)}^k\right) = \frac{m}{m+k}\theta_1^k \quad \text{and} \quad E\left(\frac{1}{Y_{(n)}^k}\right) = \frac{\alpha n}{\alpha n - k}\theta_2^{-k}, \quad \text{if } \alpha n > k. \tag{2.3}$$

The MLE of ρ is given by:

$$\hat{\rho} = \frac{\hat{\theta}_1}{\hat{\theta}_2} = \frac{X_{(m)}}{Y_{(n)}}. \tag{2.4}$$

From the results (2.3) and (2.4), we obtain expectation and variance of the MLE $\hat{\rho}$:

$$\begin{aligned} E(\hat{\rho}) &= \frac{\alpha mn}{(m+1)(\alpha n - 1)}\rho, \\ \mathrm{Var}(\hat{\rho}) &= \left\{\frac{\alpha mn}{(m+2)(\alpha n - 2)} - \frac{\alpha^2 m^2 n^2}{(m+1)^2(\alpha n - 1)^2}\right\}\rho^2. \end{aligned} \tag{2.5}$$

From expectation of the MLE in (2.5), we define an unbiased estimator of ρ by:

$$\tilde{\rho} = \frac{(m+1)(\alpha n - 1)}{\alpha mn} \cdot \frac{X_{(m)}}{Y_{(n)}},$$

which has variance as:

$$\mathrm{Var}(\tilde{\rho}) = \left\{\frac{(m+1)^2(\alpha n - 1)^2}{\alpha mn(m+2)(\alpha n - 2)} - 1\right\}\rho^2. \tag{2.6}$$

From the results (2.5) and (2.6), Table 2.1 shows numerical values of mean squared errors(MSE) of the MLE and an unbiased estimator.

From Table 2.1 we observe the following:

Fact 2. The unbiased estimator performs better than the MLE in a sense of MSE.

Next we consider a confidence interval for ρ . First thing the quantity $Q = (1/\rho)(X_{(m)} / Y_{(n)})$ is a pivot quantity having the following density which doesn't involve ρ:

From Theorem 7(21) in Rohatgi (1976, p.141), the density of Q is obtained as:

$$f_Q(x) = \begin{cases} \dfrac{\alpha mn}{m+\alpha n} x^{m-1}, & \text{if } 0 < x < 1, \\ \dfrac{\alpha mn}{m+\alpha n} x^{-\alpha n-1}, & \text{if } x \geq 1. \end{cases} \tag{2.7}$$

From the density (2.7) of Q, we can obtain an confidence interval for ρ:

For $0 < 1-p_1-p_2 < 1$, $p_i > 0$ an $(1-p_1-p_2)100\%$ confidence interval for ρ is given as:

$$\left(\left(\frac{m+\alpha n}{\alpha n} \cdot p_1 \right)^{\frac{1}{m}}, \left(\frac{m+\alpha n}{m} \cdot p_2 \right)^{-\frac{1}{\alpha n}} \right),$$

where $p_1(p_2)$ is a lower (upper) probability of the density of Q and any reall number $p_i's$ are satisfying $0 < 1-p_1-p_2 < 1$, $p_i > 0$.

2.2. Ratio of random variables

Let X and Y be two independent uniform random variable and power function random variable with the densities (2.1). Then, from Theorem 7(21) in Rohatgi and Rahtigi (1976, p.141), the density of $W = Y/X$ is given by:

$$f_W(w) = \begin{cases} \dfrac{\alpha}{\alpha+1} \cdot \rho^\alpha w^{\alpha-1}, & \text{if } 0 < w < \dfrac{1}{\rho}, \\ \dfrac{\alpha}{(\alpha+1)\rho} \cdot w^{-}2, & \text{if } w \geq \dfrac{1}{\rho}. \end{cases} \tag{2.8}$$

Let $R = X/(X+Y)$ be a ratio of X-variates among $(X+Y)$-variates. Then $R = 1/(1+W)$ and hence, we obtain the density of ratio R by the density (2.8):

$$f_R(r) = \begin{cases} \dfrac{\alpha}{(\alpha+1)\rho} \cdot \dfrac{1}{(1-r)^2}, & \text{if } 0 < r \leq \dfrac{\rho}{1+\rho}, \\ \dfrac{\alpha}{\alpha+1} \cdot \dfrac{\rho^\alpha (1-r)^{\alpha-1}}{r^{\alpha+1}}, & \text{if } \dfrac{\rho}{1+\rho} < r < 1. \end{cases} \tag{2.9}$$

From the density (2.9) of the ratio R and the formula 6.6.1 in Abramowitz and Stegun (1970, p.263), the k^{th} moment of the ratio $R = X/(X+Y)$ is obtained by:

$$E(R^k) = \frac{\alpha}{(1+\alpha)\rho} \left\{ B_{\frac{\rho}{1+\rho}}(k+1, -1) + \rho^{\alpha+1} \cdot B_{1-\frac{\rho}{1+\rho}}(\alpha, k-\alpha) \right\}, \tag{2.10}$$

where $B_x(a,b)$ is the incomplete beta function of $0 < x < 1$.

To evaluate numerical mean and variance of the ratio R, we simplify the following incomplete beta function from the recursion formulas 15.1.3 & 15.1.8, 15.2.18 & 15.2.20 in Abramowitz and Stegun (1970, pp. 556–558).

Table 2.2: Numerical values of mean and variance of the ratio $R = X/(X+Y)$

α	$\rho=1/4$		$\rho=1/2$		$\rho=2$		$\rho=4$	
	mean	variance	mean	variance	mean	variance	mean	variance
1/4	0.54066	0.12787	0.61570	0.11548	0.74494	0.09598	0.78697	0.09598
1/2	0.40244	0.09175	0.50994	0.08774	0.72365	0.06007	0.80772	0.04542
1	0.25491	0.04244	0.36920	0.05275	0.63080	0.05276	0.74508	0.04246
2	0.17125	0.01431	0.27631	0.02552	0.55261	0.04349	0.68500	0.04125
4	0.13511	0.00621	0.22969	0.01456	0.50400	0.03699	0.64510	0.03957

Lemma 2.1

(a) $B_x(2,\,-1) = \dfrac{x^2}{1-x} + x + \ln(1-x), \quad 0 < x < 1.$

(b) $B_x(3,\,-1) = 2x\{1 + x^{-1}\ln(1-x)\} - \dfrac{x^2}{1-x}, \quad 0 < x < 1.$

(c) $B_{1-x}(\alpha,\,1-\alpha) = \alpha^{-1}(1-x)^{\alpha}F(\alpha,\alpha;\,\alpha+1;\,1-x)$, and
$B_{1-x}(\alpha,\,2-\alpha) = \alpha^{-1}(1-x)^{\alpha}F(\alpha-1,\,\alpha;\,\alpha+1;\,1-x), \quad 0 < x < 1.$

where $F(a,b;c;x)$ is the hypergeometric function.

By Lemma 2.1 and the recursion formulas of the hypergeometric function in Abramowitz and Stegun (1970, p.558), Table 2.2 shows mean and variance of ratio R for $\rho=1/4$, 1/2, 2, 4 and $\alpha=$ 1/4, 1/2, 1, 2, 4.

From Table 2.2 we observe the followings:

Fact 3.

(a) Mean gets larger as the value of ρ gets larger, but variance gets smaller.

(b) Mean and variance get smaller as the value of α gets larger.

3. Uniform and Half-Normal

In this section, we consider the case for the estimation of $P(Y < X)$ when $(X,\,Y)$ is a pair of uniform and half-normal random variables, respectively. As an application of this case X, representing time to sustain temperature enduring a state of equilibrium, is a uniform random variable and Y, representing a velocity of regular gas in a state of equilibrium, is a half-normal random variable.

3.1. Reliability

Let X and Y be two independent uniform random variable and half-normal random variable (see, Johnson *et al.*, 1994; Woo, 2007a) with the following densities:

$$f_X(x) = \frac{1}{\theta}, \qquad 0 < x < \theta,\;\; \theta > 0,$$

$$f_Y(y) = \sqrt{\frac{2}{\pi\beta}}e^{-\frac{y^2}{2}\beta}, \quad 0 < y,\;\; \beta > 0. \tag{3.1}$$

From the densities (3.1), and formula 8.250(1) in Gradshteyn and Ryzhik (1965, p.930), we obtain the reliability $P(X < Y)$:

Fact 4. When X and Y are independent uniform random variable and a half-normal random variable having the densities (2.1), then, for $\eta \equiv \theta^2/\beta$,

$$R \equiv P(X < Y) = \sqrt{\frac{2}{\pi}}\eta^{-\frac{1}{2}}\left(1 - e^{-\frac{\eta}{2}}\right) + 1 - \Phi\left(\sqrt{\frac{\eta}{2}}\right),$$

Where $\Phi(x) = 2/\sqrt{\pi}\int_0^x e^-t^2 dt$.

Since $\eta > 0$, we get $dR/d\eta < 0$ and hence we obtain Fact 5:

Fact 5. $R = P(X < Y)$ is a monotone decreasing function of η.

3.2. Estimating reliability $P(X < Y)$

We now consider estimation on the reliability $P(X < Y)$ when X and Y are independent uniform random variable and half-normal random variable, respectively having the densities (3.1). Because $R = P(X < Y)$ is a monotone function of η in Fact 5 and Remark, inference on the reliability is equivalent to inference on η (see, McCool, 1991). Hence we only consider estimation on $\eta \equiv \theta^2/\beta$ instead of estimating $R = P(X < Y)$.

Assume $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be two independent samples from X and Y with the densities in (3.1), respectively, then we have the followings:

Let $\hat{\theta}^2$ and $\hat{\beta}$ be the MLE of θ^2 and β, respectively. Then the MLE are given by:

$$\hat{\theta}^2 = X_{(m)}^2 = \{\max(X_1, X_2, \ldots, X_m)\}^2 \quad \text{and} \quad \hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2.$$

We need the following well-known results to evaluate expectation and variance of the MLE:

Fact 6.

(a) $\sum_{i=1}^{n} Y_i^2/2$ has a gamma distribution with the shape parameter $n/2$ and the scale parameter β.

(b) If Y has a gamma distribution with a shape α and a scale β, then $E(1/Y^k) = \Gamma(\alpha - k)/(\Gamma(\alpha)\beta^k)$, if $\alpha > k$.

From the result (2.3) and Fact 6 (a) & (b), we obtain the 1^{st} and 2^{nd} moments of the MLE $\hat{\eta} = \hat{\theta}^2/\hat{\beta} = nX_{(m)}^2/\sum_{i=1}^{n} Y_i^2$ of $\eta = \theta^2/\beta$.

$$E(\hat{\eta}) = \frac{mn}{(m+2)(n-2)}\eta, \qquad \text{if } \eta > 2, \tag{3.2}$$

$$E(\hat{\eta}^2) = \frac{mn^2}{(m+4)(n-2)(n-4)}\eta^2, \quad \text{if } n > 4. \tag{3.3}$$

From expectation (3.2), an unbiased estimator $\tilde{\eta}$ of $\eta = \theta^2/\beta$ is defined as:

$$\tilde{\eta} = \frac{(m+2)(n-2)}{m} \cdot \frac{X_(m)^2}{\sum_{i=1}^{n} Y_i^2}. \tag{3.4}$$

Table 3.1: Table 3. MSE of the MLE $\hat{\eta}$ and an unbiased estimator $\tilde{\eta}$ (unit: η^2)

m	10				20			
n	10	20	30	40	10	20	30	40
$\hat{\eta}$	0.40476	0.14021	0.09933	0.08104	0.46338	0.13721	0.08217	0.06078
$\tilde{\eta}$	0.37143	0.15714	0.10769	0.08571	0.34444	0.13438	0.08589	0.06435

m	30				40			
n	10	20	30	40	10	20	30	40
$\hat{\eta}$	0.49449	0.14216	0.08189	0.05831	0.51299	0.14622	0.08306	0.05825
$\tilde{\eta}$	0.33856	0.12941	0.08115	0.05969	0.33636	0.12756	0.07937	0.05796

By the same method, we obtain the 2^{nd} moment of $\tilde{\eta}$ by:

$$E(\hat{\eta}^2) = \frac{(m+2)^2(n-2)}{m(m+4)(n-4)}\eta^2, \quad \text{if } n > 4. \tag{3.5}$$

From the results (3.2), (3.3) and (3.4), we obtain the following Table 3.1.
From Table 3.1, we observe the following:

Fact 7.

(a) The MLE and an unbiased estimator can't dominate each other.

(b) The unbiased estimator performs better than the MLE in a sense of MSE, especially if $m = 40$ and $n = 10, 20, 30, 40$.

From Fact 7(a), since the MLE $\hat{\eta}$ and an unbiased estimator $\tilde{\eta}$ can't dominate each other, we can recommend the following estimator of η which is minimizing its mean squared error:

At first, as we find a constant "d" such that $E[(\hat{\hat{\eta}}-\eta)^2] \equiv E[(d\cdot X_{(m)}^2/\sum_{i=1}^{n} Y_i^2-\eta)^2]$ is minimized, and hence, the estimator $\hat{\hat{\eta}}$ which has the minimum MSE is recommended by:

$$\hat{\hat{\eta}} = \frac{(m+4)(n-4)}{(m+2)} \cdot \frac{X_{(m)}^2}{\sum_{i=1}^{n} Y_i^2}, \quad \text{if } n > 4, \tag{3.6}$$

which the proposed estimator $\hat{\hat{\eta}}$ is represented by the MLE $\hat{\eta}$:

$$\hat{\hat{\eta}} = \frac{(m+4)(n-4)}{(m+2)n} \cdot \hat{\eta}, \quad \text{if } n > 4. \tag{3.7}$$

By applying the MLE $\hat{\eta}$, an asymptotic $(1-\gamma)100\%$ confidence interval of is given by:

$$\hat{\eta} \pm z_{\frac{\gamma}{2}} \cdot \hat{\eta}\sqrt{\frac{mn^2}{(n-2)(n-4)(m+4)} - \frac{m^2n^2}{(m+2)^2(n-2)^2}}, \quad \text{if } n > 4, \tag{3.8}$$

where $\int_{z_{\gamma/2}}^{\infty} \phi(t)dt = \gamma/2$, $\phi(t)$ is the standard normal density.

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