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Cover Inequalities for the Robust Knapsack Problem

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ABSTRACT

Robust knapsack problem appears when dealing with data uncertainty on the knapsack constraint. This note presents a generalization of the cover inequality for the problem with its lifting procedure. Specifically, we show that the lifting can be done in a polynomial time as in the usual knapsack problem. The results can serve as a building block in devising an efficient branch-and-cut algorithm for the general robust (0, 1) IP problem.

Keywords: Knapsack problem, Cover inequality, Lifting

1. Introduction

The robust knapsack problem (RKP) was introduced in [2], of which the feasible set can be defined as follows:

$$\begin{aligned} S &= \{x \in B^n \mid \sum_{j \in N} a_j x_j + \max_{T \subseteq N, |T| = \Gamma} \sum_{j \in T} d_j x_j \le b\} \\ &= \{x \in B^n \mid \sum_{j \in N} a_j x_j + \sum_{j \in U} d_j x_j \le b, \text{ for all } U \subseteq N \text{ with } |U| = \Gamma\}, \end{aligned} \tag{1}$$

where $N = \{1, 2, \cdots, n\}$. For $j \in N$, a_j and d_j are nonnegative integers. Γ is a positive number used to control the conservatism of the solution, see Bertimas and Sim [3], where $1 \le \Gamma \le n$. Note that the number of constraints in (1) is ${}_nC_\Gamma$. Without loss

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of generality, we can assume $a_j + d_j \le b$, for all $j \in N$, and so, *conv*(*S*) is full-dimensional.

The (RKP) appears when there is uncertainty on the knapsack coefficients. Specifically, for each item $j \in N$, we are given a nominal weight a_j with its maximum possible deviation d_j from the nominal value. Γ is used to control the number of items of which weights simultaneously can take upper bounds. In [3], theoretical results are presented which can be used to determine the value of Γ to meet specific robustness requirement (the probability of infeasibility) of the optimal solution.

In this note, we present a generalization of the cover inequality to the case of (RKP). Also by giving a disjunctive representation of the set *S*, we show that the lifting of the cover inequality can be done in a polynomial time as in the case of ordinary knapsack problem. Since cover inequalities have been successfully applied to solve large-scale (0, 1) IP problems, the results can also be used to devise an efficient branch-and-cut procedure for solving robust (0, 1) IP problems.

2. Cover Inequalities and Sequential Lifting Procedure

As noted in the previous section, the number of constraints in (1) is very large in general. However, we can find an alternative representation of *S* which is more compact and can be used for deriving a sequential lifting procedure for the cover inequality for (RKP). Before presenting the result, we need some definitions. First, we assume that the indices are ordered such that $d_1 \ge d_2 \ge \cdots \ge d_n$ and also define $d_{n+1} = 0$. For a subset $K \subseteq N$, let us define $\Gamma(K)$ as follows:

$$\Gamma(K) = K \text{ if } |K| \le \Gamma$$

$$\Gamma(K) \subset K \text{ if } |K| \ge \Gamma + 1, \text{ where } |\Gamma(K)| = \Gamma \text{ and } \max_{j \in \Gamma(K)} j < \min_{j \in K \setminus \Gamma(K)} j \tag{2}$$

Note that when $|K| \ge \Gamma + 1$, $\Gamma(K)$ is the first Γ elements of *K* where *K* is ordered by the indices. Finally, for $l \in N \cup \{n+1\}$ and $K \subseteq N$, let $N_l = \{j \in N \mid j \le l\}$ and $K_l = K \cap N_l$.

The following proposition gives a disjunctive representation of *S*.

Proposition 1: $S = \bigcup_{l \in \{\Gamma, \Gamma+1, \ldots, n-1, n+1\}} S_l$, *where* $S_l = \{x \in B^n \mid \sum_{j \in N} a_j x_j + \sum_{j \in N_l} (d_j - d_l) x_j \le b - \Gamma d_l\}$.

Proof. First, we show that $S \subseteq \bigcup_{l \in \{\Gamma, \Gamma+1, \ldots, n-1, n+1\}} S_l$.

Let $\chi(K)$, the characteristic vector of a subset $K \subseteq N$, be feasible, that is,

$$\sum_{j \in K} a_j + \sum_{j \in \Gamma(K)} d_j \le b. \tag{3}$$

Suppose $|K| \le \Gamma$. Then $\Gamma(K) = K$, which implies $\chi(K) \in S_{n+1}$.

So let us assume $|K| \ge \Gamma + 1$. Let $l^* = \max\{j \mid j \in \Gamma(K)\}$. Then we have $K_{l^*} = \Gamma(K)$ and $|\Gamma(K)| = \Gamma$. Hence

$$\sum_{j \in K} a_j + \sum_{j \in K_{l^*}} (d_j - d_{l^*}) + \Gamma d_{l^*} = \sum_{j \in K} a_j + \sum_{j \in \Gamma(K)} d_j \le b, \tag{4}$$

Which implies $\chi(K) \in S_{l^*}$. Note that since $|K| \ge \Gamma + 1$, $l^* \ge \Gamma$. Also note that if $l^* = n$, then $|K| = \Gamma$ and so $\chi(K) \in S_{n+1}$ by the above result. Thus we have $S \subseteq \bigcup_{l \in \{\Gamma, \Gamma+1, \ldots, n-1, n+1\}} S_l$ and this completes the first part of the proof.

Now we show that $\bigcup_{l \in \{1, 2, \ldots, n+1\}} S_l \subseteq S$.

For $l^* \in N \cup \{n+1\}$, choose $\chi(K) \in S_{l^*}$. Suppose $l^* = n+1$. Then since $\Gamma(K) \subseteq K$,

$$\sum_{j \in K} a_j + \sum_{j \in \Gamma(K)} d_j \le \sum_{j \in K} (a_j + d_j) \le b, \tag{5}$$

which implies $\chi(K) \in S$. So let us assume $l^* \le n$. Then

$$\begin{aligned} \sum_{j \in K} a_j + \sum_{j \in \Gamma(K)} d_j &= \sum_{j \in K} a_j + \sum_{j \in \Gamma(K) \cap N_{l^*}} d_j + \sum_{j \in \Gamma(K) \setminus N_{l^*}} d_j \\ &\le \sum_{j \in K} a_j + \sum_{j \in K \cap N_{l^*}} (d_j - d_{l^*}) + \Gamma d_{l^*} \le b, \end{aligned} \tag{6}$$

since $|\Gamma(K)| \le \Gamma$ and $d_j \le d_{l^*}$ for all $j \in \Gamma(K) \setminus N_{l^*}$, which implies $\chi(K) \in S$. Hence we have $\bigcup_{l \in \{1, 2, \ldots, n, n+1\}} S_l \subseteq S \subseteq \bigcup_{l \in \{\Gamma, \Gamma+1, \ldots, n-1, n+1\}} S_l$. Thus we can conclude $S = \bigcup_{l \in \{\Gamma, \Gamma+1, \ldots, n-1, n+1\}} S_l$. ■

Now we can present the lifting procedure for the cover inequality for (RKP). First, let us define a cover for (RKP) as follows.

Definition 1: *$C \subseteq N$ is a cover if $\sum_{j\in C} a_j + \sum_{j\in\Gamma(C)} d_j > b$. A cover is minimal if none of its proper subset is a cover. For a minimal cover C, the cover inequality is*

$$\sum_{j\in C} x_j \leq |C|-1. \tag{7}$$

For a subset $K \subseteq N$, let us define $P(K) = conv\{S \mid x_j = 0, \text{ for all } j \in N \setminus K\}$. Then it can be easily shown that the inequality (7) defines a facet of $P(C)$ when C is a minimal cover. Hence by applying the sequential lifting procedure to it, we can obtain a facet-defining inequality for $P(N) = conv(S)$ [4]. As defined in (1), the number of constraints in S is very large in general. Hence at first glance, the lifting seems to be very difficult since we should deal with multiple knapsack problems simultaneously. However, by using the disjunctive representation of S presented in proposition 1, we can show that the lifting can be done in a polynomial time as is the case with an ordinary knapsack problem. To show the result, we need the following lemma [4].

Lemma 1: *If $\sum_{j\in N} \pi_j^1 x_j \leq \pi_0^1$ is valid for $S_1 \subseteq R_+^n$ and $\sum_{j\in N} \pi_j^2 x_j \leq \pi_0^2$ is valid for $S_2 \subseteq R_+^n$. then $\sum_{j\in N} \min(\pi_j^1, \pi_j^2) x_j \leq \max(\pi_0^1, \pi_0^2)$ is valid for $S_1 \cup S_2$.*

Let C be a minimal cover with cover inequality defined as (7). The following is a sequential lifting procedure for the inequality.

Sequential Procedure for Cover Inequality of (RKP)

Let $N \setminus C = \{j_1, j_2, \ldots, j_t\}$ and assume that the lifting is applied in that order. For $k \in \{1, \cdots, t\}$, let $\sum_{j\in C} x_j + \sum_{i=1}^{k-1} \alpha_i x_{j_i} \leq |C|-1$ be a current lifted cover inequality. Then α_k is determined as follows:

$$\alpha_k = \min_{l\in\{\Gamma,\Gamma+1,\cdots n-1,n+1\}}(|C|-1-Z_l\}) \text{ and} \tag{8}$$

$$Z_l = \max\{\sum_{j\in C} x_j + \sum_{i=1}^{k-1} \alpha_i x_{j_i} \mid x \in S_l, x_{j_k} = 1, x_j = 0, j \in N \setminus (C \cup \{j_1, \dots, j_k\})\} \quad (9)$$

Note that the maximization problem in (9) is an ordinary knapsack problem and it can be solved in a polynomial time [4].

Proposition 2: *Let $C \subseteq N$ be a minimal cover. Then the lifted cover inequality obtained by applying the above procedure is facet-defining for conv(S).*

Proof: The validity of the inequality can be proved by sequentially applying lemma 1 in the order of lifting. The facet-defining property can be proved by noting that at each step $k \in \{1, \cdots, t\}$, we can get a feasible solution with $x_{i_k} = 1$ that satisfies the lifted cover inequality at equality (by choosing the solution where the minimum in (8) is attained). ■

3. Concluding Remarks

In this note, we proved that the lifting for the cover inequality for (RKP) can be done in a polynomial time. However, the procedure would be computationally demanding since in general, we should solve $n - \Gamma + 1$ knapsack problems to determine each lifting coefficient. An enhancement of the procedure can be made when $|C| \leq \Gamma$. In this case, we can easily show that the minimum in (8) is attained when $l = n+1$. Hence in this case, we only need to solve single knapsack problem in each step of the sequential lifting procedure. Another interesting research topic is to further analyze the disjunctive representation of S within the framework of disjunctive programming given in [1]. The approach can lead to other strong valid inequalities and strong reformulations for (RKP).

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