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Multicriteria Quadratic Plant Location Problem*

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ABSTRACT

In this paper, we have considered the multicriteria quadratic plant location problem. In addition to the allocation costs, the maintenance costs of the plants are also considered. The objective functions considered in this paper are quadratic in nature. The given problem is reduced to the problem with linear objective function. The method of Fernandez and Puerto (2003) is applied to solve the reduced problem. It is illustrated with help of examples. The effect of the change in the allocation and maintenance costs on allocation of plants to the clients has also been discussed.

Keyword: Uncapacitated Plant Location Problem, Maintenance Costs

1. Introduction

Facility location decisions are often strategic in nature as they frequently involve large capital outlays and long term planning horizons. Factories, distribution centers, libraries and sewage treatment plants have expected life of 20 to 50 years. The impact of the facility is felt by providers of the facility, its users and neighbours. Since the location decisions often involve many stake holders, often multiple objectives have to be considered [Current *et al.* (2004)].

Many of the practical location problems faced by executives and public sector

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decisions makers are strategic problems having multiple objectives and structures. There is a growing recognition that many facility location problems are multiobjective in nature. For example in location of schools, cost, racial balance, number of students within walking distance etc. are important criteria in making locational decisions [Revelle and Eiselt (2005)].

Zero-One multiple criteria decision making problem has been dealt by Rasmussen (1986). Several 0-1 problems would be better models of the real world if they include more than one objective function and this is especially true for location decision problem [Rasmussen (1986)]. Ross and Soland (1980) have used a multicriteria approach to locate public facilities. Ireneasz and Zolkiewski (1985) have formulated fractional programming problem and propose a compromise procedure. Quotients like profit/cost could be used to solve location problems and in such cases it would be interesting to see how this procedure works. Pelegrin and Fernandez (1988) have given a method for determining the efficient points in multiple objective location problems.

The uncapacitated plant location problem consists of opening a set of plants among a potential set of locations to allocate a given set of customers in order to minimize the set up cost of opening the plants plus the cost of allocating the clients. According to Fernandez and Puerto (2003) scenario analysis for uncapacitated plant location problem is very useful in real applications to describe seasonal behaviour, to gather different managerial strategies, to handle uncertainty in parameter estimation. When decision makers interact and have to evaluate different scenarios Fernandez and Puerto (2003) have proposed Pareto optimal solutions with regards to criteria's controlled by decision makers. Scenario analysis can be performed by considering the problem from a multiobjective point of view.

In this paper a multicriteria quadratic plant location problem is considered where the maintenance cost of the services provided by the plants are also considered. Since the variables are binary (assuming values 0 or 1) it shall be seen that the quadratic function reduces to a linear function and solution provided by Fernandez and Puerto (2003) can be used to solve the problem.

A fair number of functional relationships occurring in the world are truly quadratic e.g. Kinetic energy carried by a rocket or an atomic particle is proportional to the square of its velocity. In statistics the variance of a given sample of observations is a quadratic function of the values that constitute the sample.

In the thesis by Archana Khurana (2004) 'Algorithms for non-convex optimization in Transportation and quadratic programs' product of costs of transporting goods from the i^{th} origin to j^{th} destination and per unit depreciation cost in transporting these goods has been considered as it is desired to minimize the two costs simultaneously.

In our paper, we have considered the product of allocation costs and maintenance costs as we wish to minimize them simultaneously.

We may have factories making various parts of a machine/electrical gadgets. We can think of shopping complexes with different kinds of shops/eateries etc. or hospitals providing different kinds of treatment or different nursing homes where only some special services are available. Also when these services are provided to customers some cost is also required to maintain these services and it is desirable to make these costs minimum.

There are two ways for a client to fulfill his requirements; (i) A client may want to avoid travel from one place to another and wishes to get all his requirements from one source. (ii) The client doesn't mind getting his various requirements fulfilled from different places provided his criteria are satisfied. For both these options we are proposing two algorithms.

In first kind of option a client wishes to get all his requirements from a single place. He may want to get all parts of his machine from a single factory or get all his shopping done from a single shopping complex or get his tests and treatment from a single hospital.

In the second kind of option a client is open to getting his various requirements fulfilled from different places. For his machine he doesn't mind getting parts from different factories or the shopping is done from different places to get the best product at optimum rate or get his test/treatment done from different nursing homes/ hospital for different ailments in his family.

2. Formulation of the Problem

Let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ respectively denote the set of indices for plants and for clients. Let $K = \{1, 2, \cdots, p\}$ denote the various kinds of commodities/services provided by the plants. Let $Q = \{1, 2, \cdots, q\}$ denote the indices for the considered criteria. Let $I \subseteq M$ denote the set of open plants.

Also for the r[th] criteria, $i \in M$, $j \in N$ and $k \in K$, let

m^r_{ijk} maintenance cost of the i^{th} plant providing the k^{th} commodity/service which is allocated to client j.

F^r_{ik} Denote the setup costs of the i^{th} plant providing the k^{th} commodity/service.

c^r_{ijk} Allocation cost of the i^{th} plant providing the k^{th} commodity/service which is allocated to client j.

y_{ik} = 1 if i^{th} plant providing k^{th} commodity/service is open
= 0 otherwise

x_{ijk} = 1 if j^{th} client is provided k^{th} commodity/service by the i^{th} plant
= 0 otherwise

The Multicriteria Quadratic Plant Location Problem is formulated as follows:

Minimize

$$Z = \left\{ \left(\sum_{i \in M} \sum_{j \in N} \sum_{k \in K} c^1_{ijk} x_{ijk} \right) \left(\sum_{i \in M} \sum_{j \in N} \sum_{k \in K} m^1_{ijk} x_{ijk} \right) + \sum_{i \in M} \sum_{k \in K} F^1_{ik} y_{ik}, \cdots, \right.$$

$$\left. \cdots \left(\sum_{i \in M} \sum_{j \in N} \sum_{k \in K} c^q_{ijk} x_{ijk} \right) \left(\sum_{i \in M} \sum_{j \in N} \sum_{k \in K} m^q_{ijk} x_{ijk} \right) + \sum_{i \in M} \sum_{k \in K} F^q_{ik} y_{ik} \right\} \quad (1)$$

such that

for a fixed $j^* \exists$ a unique $i(j^*)$ such that

$$x_{i(j^*)j^*k} = 1 \quad \forall k \in K \quad (2)$$

$$x_{ijk} \le y_{ik} \forall \quad i \in M, j \in N, k \in K \quad (3)$$

$$x_{ijk}, y_{ik} \in \{0,1\} \quad (4)$$

Consider the vector minimum of the objective function given by Eqn. (1). For the first possibility that we have considered, constraints (2) and (4) together ensure that each client is assigned one plant, while constraints (3) guarantee that no client is assigned to a non-open plant.

For the second possibility, the constraint (2) is as follows

For a fixed $k \in K$ and $j^* \in J \exists$ a unique $i(j^*)$ such that

$$x_{i(j^*)j^*k} = 1 \tag{2A}$$

The constraint (2A) and (4) together ensure that each client is assigned one plant for each type of facility and constraints (3) guarantee that no client is assigned a non-open plant.

In this problem, as done by Fernandez and Puerto (2003), two nested decisions are to be addressed. First the set of plants to be opened is to be selected. Then allocation of clients within this open set of plants has to be done. Along with the allocation cost we wish to minimize the maintenance cost simultaneously. This allows us to use the notation in multicriteria dynamic programming.

Our problem is PSI ⊕ PSII

where $A \oplus C = \{a + c : a \in A, c \in C\}$

In this case the structure of the problem has been exploited by using the dynamic programming techniques

PS I:

Minimize

$$Z = \left\{ \left(\sum_{i \in I} \sum_{j \in N} \sum_{k \in K} c^1_{ijk} x_{ijk} \right) \left(\sum_{i \in I} \sum_{j \in N} \sum_{k \in K} m^1_{ijk} x_{ijk} \right), \cdots \right.$$

$$\left. \cdots, \left(\sum_{i \in I} \sum_{j \in N} \sum_{k \in K} c^q_{ijk} x_{ijk} \right) \left(\sum_{i \in I} \sum_{j \in N} \sum_{k \in K} m^q_{ijk} x_{ijk} \right) \right\}$$

such that

For a fixed $j^* \in J \exists$ a unique $i(j^*)$ such that

$$x_{i(j^*)j^*k} = 1 \ \forall k \in K$$

$$x_{ijk} \in \{0,1\} \ \forall i \in I, j \in N, k \in K$$

where I denote the set of open plants.

PS II

Minimize

$$\left\{\sum_{i\in M}\sum_{k\in K}F_{ik}^{1}y_{ik},\cdots,\sum_{i\in M}\sum_{k\in K}F_{ik}^{q}y_{ik}\right\}$$

$y_{ik}=1\ \forall\ i\in I$ and $k\in K$

$y_{ik}\in\{0,1\}\ \forall\ i\in M\setminus I$ and $\forall\ k\in K$

where I denote the set of open plants.

We make an assumption that all plants are providing the same number of commodities/services.

PS I is allocation subproblem which minimizes the maintenance costs simultaneously. **PS II** is plant selection subproblem associated with the state I. To begin with the set of open plants is selected and then the allocation subproblem is dealt with. The x's are the decision variables for **PS I** and y's are decision variables for PS II. The objective function for the subproblem **PS II** is multiobjective linear while the objective function of **PS I** is multiobjective quadratic. We now prove the following result which is useful in reducing the multiobjective quadratic problem to a multiobjective linear problem.

Result 1: Suppose that the objective function is of the type $(ax_1+bx_2)(cx_1+dx_2)$. If $x_1,x_2\in\{0,1\}$ and x_1,x_2 assume the same value (either both 0 or both 1) then the resultant quadratic function will bear the same result as when the function is taken to be linear i.e. if $x_1^2=x_1x_2=x_2^2=x_1$ or $x_1^2=x_1x_2=x_2^2=x_2$. Since if $x_1=x_2=0$ then $x_1^2=x_1x_2=x_2^2=0$ and if $x_1=x_2=1$ then $x_1^2=x_1x_2=x_2^2=1$.

We shall consider two types of problems. In the first type we make an assumption that a plant will be open and allocated to client only if it provides all types of commodities/services that a client requires and the client fulfills all his requirements from one plant. In this type of problem the conditions discussed above will be used.

We have a second condition namely when the variables x_1 and x_2 take different values i.e. one of them say x_1 assume value 1 and x_2 assumes value 0. In that case $x_1^2=1, x_2^2=0$ and $x_1x_2=0$. The resultant quadratic objective function reduces to $acx_1^2=acx_1$. This will be the case when a plant needn't provide all facilities to a client. This type of condition will find its use in the second type of problem where client is allowed to choose different commodities from different plants.

In the following result, we see how the problem formulated by us reduces to the problem as solved by Fernandez and Puerto (2003) on application of algorithm 1. We consider the bicriteria case.

Result 2: The objective function

$$Z=\left\{\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}c_{ijk}^{1}x_{ijk}\right)\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}m_{ijk}^{1}x_{ijk}\right),\cdots\right.$$
$$\left.\cdots,\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}c_{ijk}^{2}x_{ijk}\right)\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}m_{ijk}^{2}x_{ijk}\right)\right\} \text{ for } l\geq 2$$

it can be reduced to the form:

$$\left(Z_{ij}^{1},Z_{ij}^{2}\right)=\left\{\left(c_{ij1}^{1}+\cdots+c_{ijl}^{1}\right)\left(m_{ij1}^{1}+\cdots+m_{ijl}^{1}\right),\left(c_{ij1}^{2}+\cdots+c_{ijl}^{2}\right)\left(m_{ij1}^{2}+\cdots+m_{ijl}^{2}\right)\right\}$$
$$\forall i\in M, j\in N, l\geq 2$$

when a customer patronizes the same plant to receive all the commodities.

Proof: Consider the objective function

$$Z=\left\{\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}c_{ijk}^{1}x_{ijk}\right)\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}m_{ijk}^{1}x_{ijk}\right),\right.$$
$$\left.\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}c_{ijk}^{2}x_{ijk}\right)\left(\sum_{i\in I}\sum_{j\in N}\sum_{k=1}^{l}m_{ijk}^{2}x_{ijk}\right)\right\} \quad (1(a))$$

If we fix the values of i and j, i.e. fix the client 1 and suppose that plant 1 is assigned to it. We now have i = j =1, then $x_{11k}=1\ \forall k$ and $x_{ijk}=0\ \forall\ i\neq 1, j\neq 1$.

Since $x_{111}=x_{112}=\cdots=x_{11l}=1\ =x_{11}$ (say), then the expression

$$\left[\left(c_{111}^{1}x_{111}+\cdots+c_{11l}^{1}x_{11l}\right)\left(m_{111}^{1}x_{111}+\cdots+m_{11l}^{1}x_{11l}\right),\left(c_{111}^{2}x_{111}+\cdots+c_{11l}^{2}x_{11l}\right)\left(m_{111}^{2}x_{111}+\cdots+m_{11l}^{2}x_{11l}\right)\right] \quad (2(a))$$

Becomes

$$=\left[\left(c_{111}^{1}+\cdots+c_{11l}^{1}\right)\left(m_{111}^{1}+\cdots+m_{11l}^{1}\right)x_{11},\left(c_{111}^{2}+\cdots+c_{11l}^{2}\right)\left(m_{111}^{2}+\cdots+m_{11l}^{2}\right)x_{11}\right] \quad (3(a))$$

Similarly for a fixed $i=i^{*}$ and a fixed $j=j^{*}$ we have

$$x_{i^{*}j^{*}k}=1 \quad \forall\, k \quad \text{and} \quad x_{ijk}=0 \quad \forall\, i\neq i^{*} \text{ and } j\neq j^{*} \quad (4(a))$$

Then we take $x_{i^{*}j^{*}k}=1=x_{i^{*}j^{*}}$ $\forall\, k$. We retain the subscript i^{*} and j^{*} to identity the plant and the client that is being referred to. Every time j varies and a plant is assigned to it, eqn. (4(a)) hold good. Eqn. (3(a)) becomes

$$\left[\left(c_{111}^{1}+\cdots+c_{11l}^{1}\right)\left(m_{111}^{1}+\cdots+m_{11l}^{1}\right),\left(c_{111}^{2}+\cdots+c_{11l}^{2}\right)\left(m_{111}^{2}+\cdots+m_{11l}^{2}\right)\right] \quad (5(a))$$

$$=\left[Z_{11}^{1},Z_{11}^{2}\right] \quad (6(a)]$$

Similarly the other values are calculated and we get a new set of values

$$\left[Z_{ij}^{1},Z_{ij}^{2}\right]=\left\{\left(c_{ij1}^{1}+\cdots+c_{ijl}^{1}\right)\left(m_{ij1}^{1}+\cdots+m_{ijl}^{1}\right),\left(c_{ij1}^{2}+\cdots+c_{ijl}^{2}\right)\left(m_{ij1}^{2}+\cdots+m_{ijl}^{2}\right)\right\} \quad \forall i,j \quad (7(a))$$

Now we have two new cost matrices

$$Z^{1}=\begin{bmatrix} Z_{11}^{1} & Z_{21}^{1} & - & - \\ Z_{12}^{1} & Z_{22}^{1} & - & - \\ Z_{13}^{1} & Z_{23}^{1} & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \quad Z^{2}=\begin{bmatrix} Z_{11}^{2} & Z_{21}^{2} & - & - \\ Z_{12}^{2} & Z_{22}^{2} & - & - \\ Z_{13}^{2} & Z_{23}^{2} & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \quad (8(a))$$

The problem is now reduced to the problem of Fernandez and Puerto(2003)

Corollary: When a customer patronizes different plants for different commodities the objective function

$$Z = \left\{ \left(\sum_{i \in I} \sum_{j \in N} \sum_{k=1}^{l} c_{ijk}^{1} x_{ijk} \right) \left(\sum_{i \in I} \sum_{j \in N} \sum_{k=1}^{l} m_{ijk}^{1} x_{ijk} \right), \left(\sum_{i \in I} \sum_{j \in N} \sum_{k=1}^{l} c_{ijk}^{2} x_{ijk} \right) \left(\sum_{i \in I} \sum_{j \in N} \sum_{k=1}^{l} m_{ijk}^{2} x_{ijk} \right) \right\} \quad l \geq 2$$

becomes $= \left[Z_{ij}^{1}, Z_{ij}^{2} \right]$

$$\left[\left(c_{ijk}^{1} x_{ijk} \right) \left(m_{ijk}^{1} x_{ijk} \right), \left(c_{ijk}^{2} x_{ijk} \right) \left(m_{ijk}^{2} x_{ijk} \right) \right] \text{ for a fixed } k$$

Proof: If we fix $k = 1$, then for $i = j = 1$

$$\left[Z_{11}^{1}, Z_{12}^{2} \right] = \left[\left(c_{111}^{1} x_{111} \right) \left(m_{111}^{1} x_{111} \right), \left(c_{111}^{2} x_{111} \right) \left(m_{111}^{2} x_{111} \right) \right]$$

But $x_{111}^{2} = 1 = x_{11}$ (say)

$$\left[Z_{11}^{1}, Z_{12}^{2} \right] = \left[\left(c_{111}^{1} \right) \left(m_{111}^{1} \right), \left(c_{111}^{2} \right) \left(m_{111}^{2} \right) \right]$$

when a customer patronizes different plants for different commodities, calculations are done for each k.

3. Algorithm to Solve the Facility Location Problems Involving Maintenance Costs

Type 1: At first we present an algorithm for the type of problems where a plant is allocated to a client only if it is able to provide all types of commodities/services. Here the client does not avail different commodities/services from different plants. Hence allocation is done after costs of all types of commodities/services have been considered.

Algorithm 1

Step 1: The fixed open costs, the allocations costs and the maintenance costs of each type of commodity/service are listed.

Step 2: Since the problem can be divided into two subproblems and the fixed open problem is straight forward once the set of open plants has been selected, we deal with the allocation subproblem. We select the set of open warehouses.

Step 3: Find the value of the objective function as of subproblem I (PS-I).

Step 4: By taking into account the conditions given in the above result, we notice that the objective function which is quadratic reduces to a linear objective function.

Step 5: Obtain a new set of values for the selected set of open warehouses for the number of criteria that we had started with.

Step 6: Find scalarized version of the allocation subproblem and proceed in a similar manner as we do for the linear case given by Fernandez and Puerto (2003).

Algorithm 2

Type 2: This algorithm is for the type of problems where the plants are allocated to the clients commodity/service-wise *i.e.* the client can avail different commodities/ services from different plants. In such cases it is not necessary that the plants must be able to provide all kinds of commodities/services. The problem will have to be solved commodity/service wise.

Step 1: The fixed open costs, the allocation costs and the maintenance costs of each type of commodities/services are listed.

Step 2: The structure of the problem allows us to break the problem into two subproblems-the fixed open cost problem and the allocation subproblem.

Step 3: The problem involving fixed open costs is straightforward. The allocation subproblem is dealt with in detail. Here each type of facility that the clients wishe to make use of are dealt with separately taking into consideration only one commodity/service at a time. We begin with the first commodity/service.

We fix $k = 1$ so that $x_{ijk} = 0$ for remaining k's

Step 4: The selection of set of open warehouses is done. We name this set I.

Step 5: Find the values of the objective function in PS-I for $k = 1$ *i.e.* for the first commodity/service.

Step 6: Repeat step 5 for the remaining commodities/services.

Step 7: The problem reduces to a linear one and we get a new set of values for the different objective functions and for the set of open warehouses.

Step 8: Using the scalarized version of the allocation subproblem, we can proceed by the algorithm given by Fernandez and Puerto (2003).

4.

In this section, we illustrate of the two types of algorithms given above with the help of a numerical example. Consider an example of 5 plants, 3 clients, 2 types of commodities and 2 objectives. We first discuss problem based on algorithm 1

Step 1:

$C^1=$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$j=1$	20 / 15	30 / 10	10 / 5	20 / 12	40 / 10
$j=2$	50 / 40	10 / 20	60 / 30	10 / 5	80 / 20
$j=3$	40 / 20	40 / 15	30 / 10	20 / 8	30 / 12

We list the allocation cost and the maintenance cost of each plant *w.r.t.* each facility.

$C^2=$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$j=1$	50 / 10	30 / 12	30 / 15	20 / 12	50 / 30
$j=2$	20 / 10	60 / 20	30 / 20	30 / 10	40 / 30
$j=3$	40 / 10	20 / 15	20 / 15	40 / 10	30 / 20

$F^1=$

7 / 8	3 / 2	5 / 6	8 / 6	2 / 3

$F^2=$

12 / 10	6 / 8	13 / 10	14 / 12	15 / 10

$M^1=$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$j=1$	15 / 10	20 / 5	8 / 4	10 / 10	30 / 5
$j=2$	40 / 30	5 / 10	50 / 20	5 / 4	60 / 12
$j=3$	30 / 10	30 / 10	20 / 8	10 / 6	20 / 10

$M^2 =$

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$j=1$	40 / 5	20 / 10	20 / 10	10 / 10	40 / 20
$j=2$	10 / 5	40 / 10	20 / 10	20 / 5	20 / 10
$j=3$	20 / 8	10 / 10	10 / 10	30 / 6	10 / 10

The entries on the top left corner of each cell of C^1 and C^2 represent costs of first commodity and entries on the bottom right corner represent the costs of the second commodity. Similarly the entries on the top left corner of each cell of M^1 and M^2 represent the maintenance costs of first facility and entries on the bottom right corner represent the maintenance costs of second facility.

F^1 and F^2 gives the set up costs for the two objectives. The top left entries corresponds to the set up costs of first facility while the entries at the bottom correspond to those of second facility.

Step 2: Suppose that warehouse 1 and 2 are open i.e. $I = \{1,2\}$

Step 3 and **Step 4:**

Let $i=1,\ j=1$

Here $\left\{\left(\sum_{k=1}^{2} c_{ijk}^{1} x_{ijk}\right)\left(\sum_{k=1}^{2} m_{ijk}^{1} x_{ijk}\right), \left(\sum_{k=1}^{2} c_{ijk}^{2} x_{ijk}\right)\left(\sum_{k=1}^{2} m_{ijk}^{2} x_{ijk}\right)\right\}$

$= \left(\left(c_{111}^{1} x_{111} + c_{112}^{1} x_{112}\right)\left(m_{111}^{1} x_{111} + m_{112}^{1} x_{112}\right)\left(c_{111}^{2} x_{111} + c_{112}^{2} x_{112}\right)\left(m_{111}^{2} x_{111} + m_{112}^{2} x_{112}\right)\right)$

$= \{(35 \times 25\, x_{11}, 60 \times 45\, x_{11})\} = \{(875\, x_{11}, 2700\, x_{11})\}$

Similarly the remaining results are as follows

For

$i=2,\ j=1$ $(1000\, x_{21}, 1260\, x_{21})$

$i=1,\ j=2$ $(6300\, x_{12}, 450\, x_{12})$

$i=2,\ j=2$ $(450\, x_{22}, 4000\, x_{22})$

$i=1,\ j=3$ $(2400\, x_{13}, 2200\, x_{13})$

$i=2,\ j=3$ $(2200\, x_{23}, 700\, x_{23})$

Step 5:

We now have the following set of new values

$$Z^1 = \begin{pmatrix} 875 & 1000 \\ 6300 & 450 \\ 2400 & 2200 \end{pmatrix} \quad Z^2 = \begin{pmatrix} 2700 & 1260 \\ 450 & 4000 \\ 2200 & 700 \end{pmatrix}$$

Step 6:

Find the non-dominated solutions by

Minimizing $\sum_{i \in I} \left\{ \left(Z_{ij}^2 + \lambda \left(Z_{ij}^1 - Z_{ij}^2 \right) \right) x_{ij} \right\}$

where $x_{ij} = 1$ or 0 depending on whether plant i is allocated to customer j or not.

The various intervals of λ in which the allocations of plants are made to the customers are obtained using the following proposition (see Fernandez and Puerto (2003)).

Proposition For a fixed value of λ $i^* \in I$ is the optimal allocation for client $j \in N \Leftrightarrow$

$$\underset{c_{ij}^1 - c_{i^*j}^1 < c_{ij}^2 - c_{i^*j}^2}{\text{Max}} \frac{c_{ij}^2 - c_{i^*j}^2}{\left(c_{ij}^2 - c_{i^*j}^2\right) - \left(c_{ij}^1 - c_{i^*j}^1\right)} \le \lambda \le \underset{c_{ij}^2 - c_{i^*j}^2 < c_{ij}^1 - c_{i^*j}^1}{\text{Min}} \frac{c_{ij}^2 - c_{i^*j}^2}{\left(c_{ij}^2 - c_{i^*j}^2\right) - \left(c_{ij}^1 - c_{i^*j}^1\right)}$$

For $i = 1, \quad j = 1, \quad \lambda = \frac{1}{2} \quad 2700 + \frac{1}{2}(875 - 2700) = 1787.50$

$Z_{11}^2 + \frac{1}{2}\left(Z_{11}^1 - Z_{11}^2\right) = 1787.50$

For $i = 2, \quad j = 1, \quad \lambda = \frac{1}{2} \quad 1260 + \frac{1}{2}(1000 - 1260) = 1180.00$

$Z_{21}^2 + \frac{1}{2}\left(Z_{21}^1 - Z_{21}^2\right) = 1180.00$

Hence for $\lambda = \frac{1}{2}$ warehouse 2 is allocated

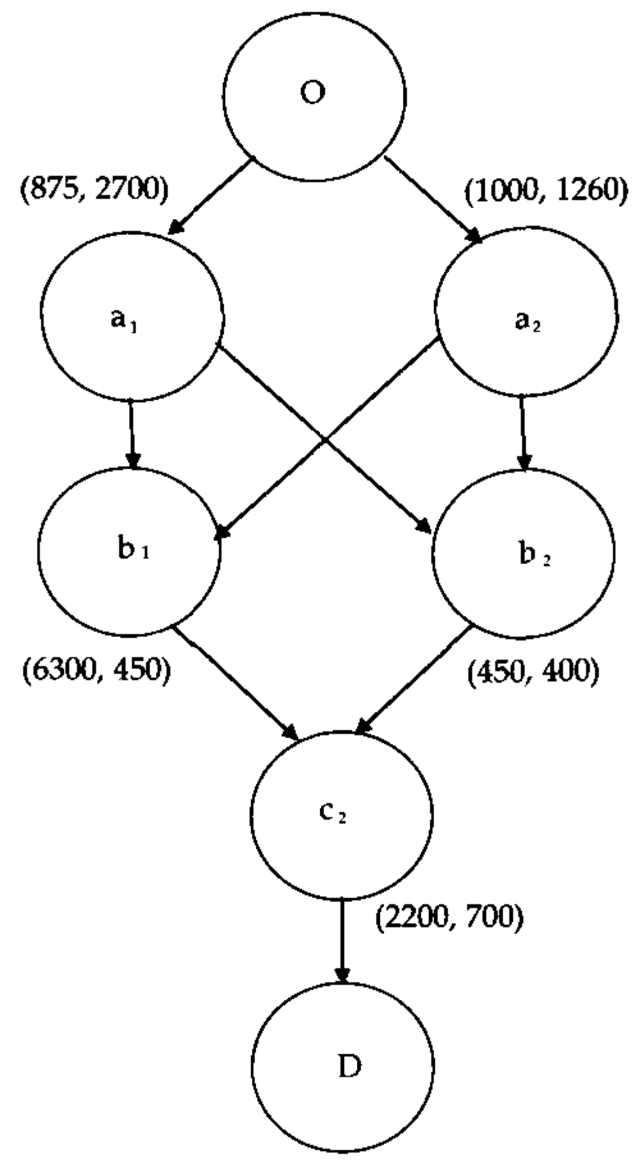
Using the proposition stated above we get the following results

$$j = 1 \quad a(j) = \begin{cases} 2 & 0 \le \lambda \le 0.93 \\ 1 & 0.93 \le \lambda \le 1 \end{cases}$$

$$j = 2 \quad a(j) = \begin{cases} 2 & 0 \le \lambda \le 0.17 \\ 1 & 0.17 \le \lambda \le 1 \end{cases}$$

$$j = 3 \quad a(j) = \; 2 \quad 0 \le \lambda \le 1$$

where $a(j)$ is the allocation of the j^{th} client. The supported non-dominated solutions, to allocation problem, are obtained using the following network



The solutions are

(1) $0 \to a_1 \to b_1 \to c_2 \to D$ with value $(9375, 3850)$

(2) $0 \to a_2 \to b_1 \to c_2 \to D$ with value $(9500, 2410)$

(3) $0 \to a_2 \to b_2 \to c_2 \to D$ with value $(3650, 5960)$

(4) $0 \to a_1 \to b_2 \to c_2 \to D$ with value $(3525, 7450)$

We shall now discuss problem of type 2

Step 1: Consider the same allocation costs, the fixed open costs and maintenance cost as for type 1 problem.

Step 2: Consider the allocation subproblem

Minimize

$$Z = \left\{ \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk}^{1} x_{ijk} \right) \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} m_{ijk}^{1} x_{ijk} \right), \right.$$

$$\left. \cdots, \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} c_{ijk}^{2} x_{ijk} \right) \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} m_{ijk}^{2} x_{ijk} \right) \right\}$$

such that

For a fixed $k \in K$ and $j^* \in J \ni$ a unique $i(j^*)$ such that $x_{i(j^*)j^*k} = 1$

$x_{ijk} \in \{0,1\} \ \forall i \in I, j \in N, k \in K$

Step 3: Let k =1 i.e. we begin with the first facility.

Step 4: Let $I = \{1,2\}$

Step 5 and Step 6: For $i = 1, \quad j = 1$ we find the value of the objective function

$$\left\{\left(c^1_{111}\ x_{111}\right)\left(m^1_{111}\ x_{111}\right), \left(c^2_{111}\ x_{111}\right)\left(m^2_{111}\ x_{111}\right)\right\}$$

Since $x^2_{111} = x_{111} = x_{11}$, we have the objective function taking the following value.

(1) $\left(300x_{11},\ 2000x_{11}\right)$ for $i = 1, j = 1, k = 1$

(2) $\left(150x_{11},\ 50x_{11}\right)$ for $i = 1, j = 1, k = 2$

(3) $\left(2000x_{12},\ 200x_{12}\right)$ for $i = 1, j = 2, k = 1$

(4) $\left(1200x_{12},\ 50x_{12}\right)$ for $i = 1, j = 2, k = 2$

(5) $\left(600x_{21},\ 600x_{21}\right)$ for $i = 2, j = 1, k = 1$

(6) $\left(50x_{21},\ 120x_{21}\right)$ for $i = 2, j = 1, k = 2$

(7) $\left(50x_{22},\ 2400x_{22}\right)$ for $i = 2, j = 2, k = 1$

(8) $\left(200x_{22},\ 200x_{22}\right)$ for $i = 2, j = 2, k = 2$

(9) $\left(1200x_{13},\ 800x_{13}\right)$ for $i = 1, j = 3, k = 1$

(10) $\left(250x_{13},\ 80x_{13}\right)$ for $i = 1, j = 3, k = 2$

(11) $\left(1200x_{23},\ 200x_{23}\right)$ for $i = 2, j = 3, k = 1$

(12) $\left(150x_{23},\ 150x_{23}\right)$ for $i = 2, j = 3, k = 2$

Step 7: For k=1 we have the following set of values

$$A^1 = \begin{pmatrix} 300 & 600 \\ 2000 & 50 \\ 1200 & 1200 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 2000 & 600 \\ 200 & 2400 \\ 800 & 200 \end{pmatrix}$$

Step 8: To find the supported non-dominated solutions we consider the scalarized version of the subproblem

Minimize:

$$\sum_{i\in I}\left[a_{ij}^{2}+\lambda\left(a_{ij}^{1}-a_{ij}^{2}\right)x_{ij}\right]$$

For $j=1$, $\lambda=\frac{1}{2}$ and $i=1$, the objective function takes the value

$$2000+\frac{1}{2}(300-2000)=1150$$

For $j=1$, $\lambda=\frac{1}{2}$ and $i=2$, the value is

$$600+\frac{1}{2}(600-600)=600$$

Warehouse 2 is allocated to customer 1 for facility 1.

According to the proposition by Fernandez and Puerto (2003) allocations are made as follows for k = 1 (i.e. for the first facility)

$$j=1 \quad a(j)=\begin{cases}2 & 0\le\lambda\le 0.82\\ 1 & 0.82\le\lambda\le 1\end{cases}$$

$$j=2 \quad a(j)=\begin{cases}2 & 0\le\lambda\le 0.52\\ 1 & 0.52\le\lambda\le 1\end{cases}$$

$$j=3 \quad a(j)=\ 2 \quad 0\le\lambda\le 1$$

We now allocate the plants to the clients considering $k=2$ (i.e. for second facility)

For $k=2$ we have the following set of values

$$B^1=\begin{pmatrix}150 & 50\\ 1200 & 200\\ 200 & 150\end{pmatrix} \qquad B^2=\begin{pmatrix}50 & 120\\ 50 & 200\\ 80 & 150\end{pmatrix}$$

For $j=1$, $\lambda=\frac{1}{2}$ and $i=1$

The objective function takes the value $50+\frac{1}{2}(150-50)=100$

For $j=1$, $\lambda=\frac{1}{2}$ and $i=2$, the value is $120+\frac{1}{2}(50-120)=85$

Warehouse 2 is allocated to customer 1 for facility 2.

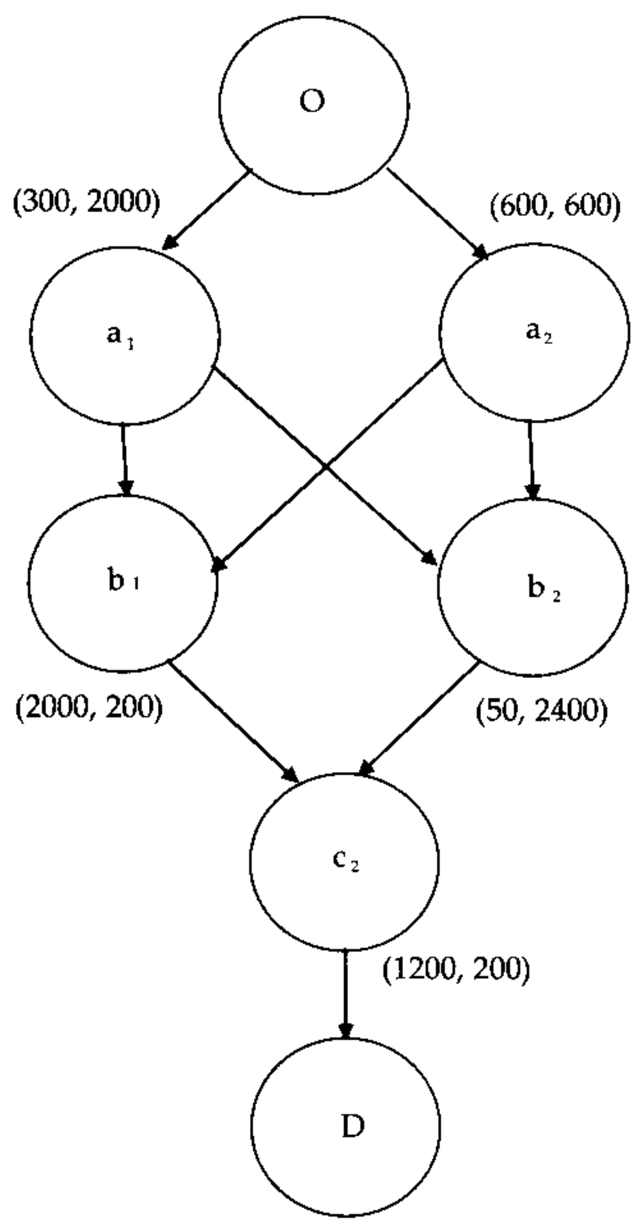
Hence, as for facility 1 we have the following results

$$j=1 \quad a(j)=\begin{cases}2 & 0\le\lambda\le 0.41\\ 1 & 0.41\le\lambda\le 1\end{cases}$$

$$j=2 \quad a(j)=\begin{cases}2 & 0\le\lambda\le 0.13\\ 1 & 0.13\le\lambda\le 1\end{cases}$$

$$j=3 \quad a(j)=\begin{cases}1 & 0\le\lambda\le 0.583\\ 2 & 0.583\le\lambda\le 1\end{cases}$$

In this case we get two sets of supported non-dominated solutions
For k = 1



The solutions are:

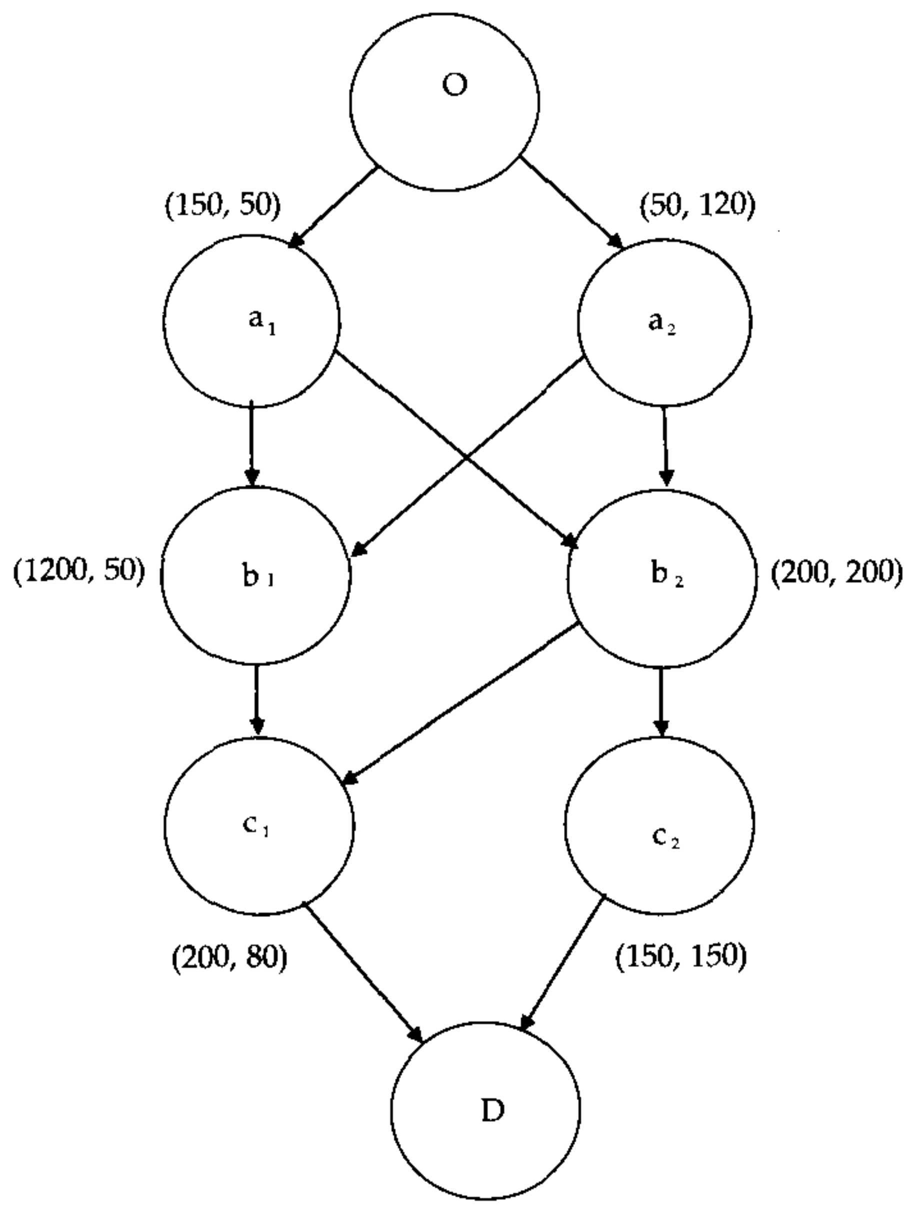
$0\to a_1\to b_1\to c_2\to D$ value $(3500, 600)$

$0\to a_2\to b_2\to c_2\to D$ value $(1850, 3200)$

$0\to a_1\to b_2\to c_2\to D$ value $(1550, 4600)$

$0\to a_2\to b_1\to c_2\to D$ value $(3800, 1000)$

Similarly for k = 2



The solutions are:

1. $0 \to a_1 \to b_1 \to c_1 \to D$ value $(1550, 180)$
2. $0 \to a_2 \to b_2 \to c_2 \to D$ value $(400, 470)$
3. $0 \to a_1 \to b_2 \to c_2 \to D$ value $(500, 400)$
4. $0 \to a_1 \to b_2 \to c_1 \to D$ value $(550, 330)$
5. $0 \to a_1 \to b_1 \to c_2 \to D$ value $(1400, 250)$
6. $0 \to a_2 \to b_1 \to c_1 \to D$ value $(1450, 250)$
7. $0 \to a_2 \to b_2 \to c_1 \to D$ value $(450, 400)$
8. $0 \to a_2 \to b_1 \to c_2 \to D$ value $(1400, 320)$

Once the allocation problem is solved for the selected set of open warehouses, another set of open warehouses is selected and the problem is solved. The search is performed until we get an optimum solution. The criterion for selection of new plant and ways to minimize this search is in accordance with that given by Fernandez and Puerto (2003). Once the optimum set of open warehouses is found, the fixed open cost problem can be solved and the value added to the value of allocation problem.

Seven problems including the above example were considered. The allocation costs and the maintenance costs were taken in different proportions to the values as given in the above example. The effect of increase or decrease in the allocation costs/maintenance costs on the allocation of plants to the clients was studied. The values of allocation costs and maintenance costs were taken to be twice thrice and four times the values in the illustrated example. They were also taken to be half, one fourth and one fifth of the values in the illustrated example. Both the algorithms were applied and allocations of the clients were observed when the first two plants were fixed open. The observations are given in Table 1 below.

Table 1. In this table, the allocations that were made to each client for maximum number of times are mentioned. This procedure has been followed for both the algorithms

	Client	Allocation	No. of times the allocation was made (out of Seven)
Algorithm 1	j = 1	$a(j)=\begin{cases}2 & 0\le\lambda\le 0.9\\ 1 & 0.9\le\lambda\le 1\end{cases}$	3
	j = 2	$a(j)=\begin{cases}1 & 0\le\lambda\le 0.3\\ 2 & 0.3\le\lambda\le 1\end{cases}$	3
	j = 3	$a(j)=2 \quad \lambda\in[0,1]$	6
Algorithm 2 k = 1	j = 1	$a(j)=\begin{cases}2 & 0\le\lambda\le 0.8\\ 1 & 0.8\le\lambda\le 1\end{cases}$	3
	j = 2	$a(j)=\begin{cases}1 & 0\le\lambda\le 0.5\\ 2 & 0.5\le\lambda\le 1\end{cases}$	3
	j = 3	$a(j)=2$	7
Algorithm 2 k = 2	j = 1	$a(j)=\begin{cases}1 & 0\le\lambda\le 0.4\\ 2 & 0.4\le\lambda\le 1\end{cases}$	3
	j = 2	$a(j)=\begin{cases}1 & 0\le\lambda\le 0.1\\ 2 & 0.1\le\lambda\le 1\end{cases}$	3
	j = 3	$a(j)=\begin{cases}1 & 0\le\lambda\le \eta\\ 2 & \eta\le\lambda\le 1\end{cases}$ $\{0.2\le\eta\le 0.6\}$	3

We shall now analyze the allocations for each client and for each algorithm.

Algorithm 1

Client 1: In the illustrative example the allocation to the first client is

$$a(j)=\begin{cases}2 & 0\le\lambda\le 0.8\\ 1 & 0.8\le\lambda\le 1\end{cases}$$

It is observed that the deviations in allocations are maximum when values of allocation/maintenance costs are taken to be four times, one fourth and twice the values in the illustrative example.

Client 2: For the second client, the differences in allocations are observed when cost values are taken twice, four times and one fourth the values in the above example. The allocation in the above example is

$$a(j) = \begin{cases} 2 & 0 \le \lambda \le 0.5 \\ 1 & 0.5 \le \lambda \le 1 \end{cases}$$

which is different from the allocations in Table 1.

Client 3: In this case hardly any deviations were observed. It was only when values of allocation/maintenance costs are taken thrice the values in the above example, the allocation is different. Rests of the allocations are the same.

Algorithm 2

The following observations were made when algorithm 2 was applied to k = 1.

Client 1: Allocation of plants to clients for the cases when values of allocation and maintenance costs were taken to be one fourth and twice of those in the illustrative example were

$$a(j) = \begin{cases} 2 & 0 \le \lambda \le 0.5 \\ 1 & 0.5 \le \lambda \le 1 \end{cases}$$

The upper limit of λ when the client is allocated plant 2 is 0.5 which is 0.2 more than the upper limit of in table 1 for the same case. Deviations were seen when cost values were taken thrice of those in the example.

Client 2: When values of allocation and maintenance costs were taken to be one fourth, half and thrice the values in the illustrative example the upper limit is varied from 0.3 to 0.8 for the case when plant 1 was allocated to client 2.

Client 3: No deviations were observed in this case.

Algorithm 2 ($k = 2$)

Client 1: When allocation and maintenance costs were taken to be one fifth, twice and four times those in the illustrative example, deviations were observed.

Client 2: The allocations in the illustrative example are different from those in the table. Deviations were also observed when cost values were taken to be twice, thrice and one fourth of the values in the example.

Cleent 3: Here the upper limit of λ when the client was allocated plant 1varied from 0.2-0.6. Maximum deviation was observed when cost values were one fourth and thrice the cost values in the above example.

On the whole, maximum deviations in the allocation of plants to clients occurred; when cost values were taken to be one fourth, thrice or four times the original cost values. When a single plant was allocated to a client, there was hardly any deviation when cost values were changed. However out of the seven problems that were tested the allocations were almost the same at least three times in all the cases.

5. Conclusions

In this paper we have discussed two types of problem. In the first type of problem a client fulfills all his requirements from a single place while in the second kind of problem a client is open to getting his requirement fulfilled from different places.

The main advantage of the approach used by us is that we have reduced a multiobjective quadratic plant location problem to a problem with linear objective function. This simplifies the problem and it can be solved using methods already known.

We have provided two algorithms where the multicriteria quadratic uncapacitated plant location problem reduces to the multicriteria linear uncapacitated plant location problem. Instead of directly solving the quadratic multiobjective plant location problem it is always easier to solve the multiobjective linear plant location problem.

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