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Optimal Admission and Pricing Control Problem with Sideline Profit, Customer Order Cancellation, and No Waiting Room*

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ABSTRACT

We discuss the problem of selecting profitable customer orders arriving at a company providing two classes of services. For an arriving customer order for the first class, the company 1) makes a decision whether to accept or reject it (admission control), or 2) decides a price of the order to offer to an arriving customer (pricing control). The customer order undergoing processing in the system may be cancelled owing to unavoidable circumstances with the customer. The second class of service is provided as a sideline, which prevents the server from being idle when all the customer orders for the first class are completed and delivered. This yields the sideline profit. We discuss both admission control and pricing control problems in an identical framework as well as examine the structure of the optimal policies maximizing the total expected present discounted net profit gained over an infinite planning horizon.

Keywords: Admission Control, Pricing Control, Sideline, Order Cancellation

1. Introduction

We consider a stochastic decision problem of selecting profitable customer orders sequentially arriving at a company operating in service industries providing two classes of services. The second class is provided as a sideline, which prevents servers from being idle when the customer orders for the first class are all served and the sys-

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tem is empty, while the first class is designed to meet the particular needs of customers; and the company makes a decision whether to accept or reject the customer orders for this service (admission control), or decides a proper price to offer to the customer requesting this service (pricing control). In admission control [3, 8, 9, 11, 14, 15, 21], a customer offers a price for his order, and judging from this, the company decides whether to accept it. In pricing control [1, 5, 10, 12, 13, 16, 22, 25], an arriving customer is assumed to have a maximum permissible ordering price, called the *reservation price*. To an arriving customer, the company proposes a price for the order, and if and only if this price is lower than or equal to his reservation price, the customer then is willing to place his order with the company; hence the company decides the offering price to maximize its expected profit.

Yoon and Lewis [23] formulated both admission control and pricing control problems, separately. They also separately investigated the properties of optimal policies for both problems. Gans and Savin [2] considered a rental firm with two types of customers; one is controlled by the admission policy and the other is by the pricing policy. Contrary to this, in our problem there exists only one type of customer to be controlled, and although the two problems with, respectively, the admission policy and pricing policy are separately defined, we show that both problems can be analyzed within an identical framework. Further, we introduce two basic assumptions that have not been taken into consideration in the conventional models.

First, it should be noted that, in the conventional model, when the system gets empty, the system is assumed to become idle and yields no profit during this idle period. Contrary to this, we assume that when all the accepted orders are completed, and the system becomes idle, it can generate an expected profit by offering potentially existing services as a sideline during the idle period; let us call this the *sideline profit*. For example, let us consider a design office that places more emphasis on providing strategic design services (first class of service; logo designs, advertising designs, brand identities, and etc.) than providing subsidiary services (second class of service; making leaflets and handbills) as a sideline. The second class of service as a sideline can be completed in just a little while, say one or two hours, compared to long period of the first class, say several weeks or months. Suppose that the expected sideline profit per period is relatively small compared with that from the first class. When all the accepted orders for the first class were completed and delivered, the design office will be willing to engage in making leaflets and handbills as a sideline.

When an order for the first class arrives and is accepted, the design office however can quickly complete the sideline and get back to normal business.

Next, from the practical point of view, the company must conduct a variety of business activities to find customer orders by paying some cost, called the *search cost*. Business activities include, for example, advertisements, preparing of materials for presentations, the demonstration of sample products, and so on. The search cost has been introduced in the conventional optimal stopping problems [4, 6, 18, 24] but not in both optimal admission and pricing control problems. Generally, if the search cost is larger than the profit from a customer order, the company incurs a loss. Therefore, the company should make a prudent decision on whether to conduct the search for customer orders at any given time. In other words, the introduction of the search cost eventually creates the option whether to conduct the search or not. To the best of our knowledge, so far the search cost and this new option have not been taken into consideration in the conventional models of the literature.

The basic model with the aforementioned assumptions was introduced by Son [19] and Son and Ikuta [20]. In the former where multiple customer orders are allowed to be held in the system, the properties of the optimal policies in the number of customer orders were clarified without considering the due date and the penalty. In the latter, they considered the due date as well as the penalty where only one customer order is allowed to be held in the system. Our model is an extension of [20] in the following sense. In addition to the assumptions of [20], the model of this paper considers the uncertainty that a customer order undergoing processing in the system may be cancelled due to unavoidable circumstances with the customer. For example, let us consider a company developing a new product and requesting a brand identity to a design office. The company often encounters its competitor's faster branching out. In this case, if business prospects of the processing project are unprofitable, the top manager would decide to quit the processing project and cancel the placed order with a design office. Of course, the penalty is paid to the design office. This customer order cancellation is not considered in [20].

The objective of our work is to examine and clarify the structure of the optimal search policies as well as the optimal admission/pricing policies to maximize the total expected present discounted net profit gained over an infinite planning horizon.

The rest of this paper is organized as follows. Section 2 defines some notations of the model as well as some functions that will be used in analyzing the properties of the optimal policies. In Section 3, we provide a definition of the model and describe the optimality equations for both admission control and pricing control problems. Using some of functions defined in Section 2, in Section 4 we transform the optimality equations of both problems in an identical form. In Section 5, we clarify the properties of the optimal policies and present the implications of results obtained. Finally, Section 6 summarizes the conclusions and suggests some subjects of future study.

2. Preliminary

In this section we define some notations and functions. On the basis of the notations, our model will be formulated in Section 3. The functions are used in transforming the optimality equations in Section 4 as well as examining the properties of the optimal policies of the model in Section 5. First, let us define the following notations.

λ : customer arrival probability
q : service completion probability
s : sideline profit
c : search cost
θ: delay cost
τ : due date
$\bar{\nu}$: order cancellation probability after τ
ν : order cancellation probability before τ
$\bar{\vartheta}$: order cancellation penalty before τ
ϑ : order cancellation penalty before τ
β : discount factor
$p(z)$: probability of customer's placing order for a proposed price z
C, K : decision of, respectively, continuing the search and skipping the search
$<\mathrm{C}>_i, <\mathrm{K}>_i$: corresponding decision is *optimal* for state i
h : optimal admission criterion
$z(h)$: optimal offering price

For convenience, let us define

$$\eta = (1-q)(1-\nu)\beta, \quad (0<\eta<1), \quad \bar{\eta} = (1-q)(1-\bar{\nu})\beta, \quad (0<\bar{\eta}<1),$$

$$\rho = (1-q)\beta\left(\frac{\eta^{\tau-1}(\theta-\bar{v}\bar{\vartheta})}{1-\bar{\eta}} - \frac{(1-\eta^{\tau-1})v\vartheta}{1-\eta}\right), \tag{2.1}$$

$$\varphi = \frac{1-\eta^{\tau-1}}{1-\eta}(1+(1-q)v) + \frac{1-\eta^{\tau-1}}{1-\bar{\eta}}(1+(1-q)\bar{v}), \quad \kappa = \frac{1-\eta^{\tau-1}}{1-\eta} + \frac{\eta^{\tau-1}}{1-\bar{\eta}}. \tag{2.2}$$

Between κ and φ the following relations hold.

$$\beta\varphi = 1-(1-\beta)\kappa, \quad \kappa > \varphi, \quad 1 > \varphi, \tag{2.3}$$

the proofs of which will be given in Appendix. For any real number x let us define following function.

$$T(x) = \begin{cases} \mathrm{E}[\max\{w-x, 0\}] & \text{for the admission control problem} \\ \max_z p(z)(z-x) & \text{for the pricing control problem} \end{cases} \tag{2.4}$$

where E[·] represents the expectation of a given function as to w. In the pricing control problem, by $z(x)$ let us designate z attaining the maximum of $\max_z p(z)(z-x)$ for a given x. Further, let us define

$$L(x) = \lambda\beta T(x) - c, \quad J(x) = \lambda\beta(q+(1-q)\nu)T(x) - c,$$
$$\overline{J}(x) = \lambda\beta(q+(1-q)\overline{\nu})T(x) - c. \tag{2.5}$$

Here, regarding x as a function of s, i.e., $x = x(s)$, by s_L^*, s_J^*, and $\overline{s}_J^*$ let us denote the solutions of, respectively, $L(x(s)) = 0$, $J(x(s)) = 0$, and $\overline{J}(x(s)) = 0$ if they exist, i.e.,

$$L(s_L^*) = 0, \quad J(s_J^*) = 0, \quad \overline{J}(\overline{s}_J^*) = 0. \tag{2.6}$$

If multiple solutions are exist in the above equations, let us define the smallest of them as s_L^*, s_J^*, and $\overline{s}_J^*$. For convenience in later discussions, we define

$$G(x) = \kappa\max\{L(x), 0\} - (1-\eta^{\tau-1})\max\{J(x), 0\}/(1-\eta)$$
$$-\eta^{\tau-1}\max\{\overline{J}(x), 0\}/(1-\eta) - x + \kappa s + \rho, \tag{2.7}$$
$$B_1(x) = \lambda\beta\kappa(1-q)(1-\nu)T(x) - x + \kappa s + \rho,$$
$$\overline{B}_1(x) = \lambda\beta\kappa(1-q)(1-\overline{\nu})T(x) - x + \kappa s + \rho, \tag{2.8}$$
$$B_2(x) = \lambda\beta(\kappa-\varphi)T(x) - x + \kappa s + \rho, \tag{2.9}$$

By x_G, x_{B_1}, $\overline{x}_{B_1}$, and x_{B_2} let us designate solutions of the equation $G(x) = 0$, $B_1(x) = 0$, $\overline{B}_1(x) = 0$, and $B_2(x) = 0$, respectively, if they exist, i.e., $G(x_G) = 0$, $B_1(x_{B_1}) = 0$, $\overline{B}_1(\overline{x}_{B_1}) = 0$, and $B_2(x_{B_2}) = 0$.

Lemma 2.1: *For both admission control and pricing control problems we have:*

(a) $T(x)$ *is nondecreasing in* x *on* $(-\infty, \infty)$ *and strictly decreasing in* x *on* $(-\infty, b]$.

(b) $T(x) = 0$ *for* $x \geq b$ *and* $T(x) > 0$ *for* $x < b$.

(c) *The equation* $T(x) = v$ *has a unique solution less than* b *for any constant* $v > 0$.

(d) $vT(x) + x$ *is strictly increasing in* x if $0 \leq v < 1$.

(e) $\lim_{x\to\infty} T(x) = 0$ *and* $\lim_{x\to-\infty} vT(x) + x = -\infty$.

(f) $\lim_{x\to\infty} T(x) + x = \infty$, *and if* $v < 1$, *then* $\lim_{x\to-\infty} vT(x) + x = -\infty$.

Proof: The proof is given in [20]. ∎

Lemma 2.2: *For the pricing control problem we have:*

(a) $x < z(x) < b$ *for* $x < b$, $x \geq z(x) = b$ *for* $x \geq b$, *and* $a \leq z(x) \leq b$ *for any* x.

(b) $z(x)$ *is nondecreasing in* x *on* $(-\infty, \infty)$.

Proof: The proof is given in [20][1]. ∎

Lemma 2.3: *For the pricing control problem we have:*

(a) $G(x)$ *is strictly decreasing in* x, and $G(x) < (>) 0$ *for any sufficiently large (small)* x, *hence* x_G *uniquely exists.*

(b) $G(x)$ *is nondecreasing in* c, ϑ, $\overline{\vartheta}$ *and strictly increasing in* s *and* ϑ *and nondecreasing in* λ *for all* x.

Proof: The proof is seen in Appendix. ∎

Lemma 2.4: *For the pricing control problem we have:*

(a) $B_1(x)$, $\overline{B}_1(x)$, *and* $B_2(x)$ *are all strictly decreasing in* x, *and* $B_1(x) > (<) 0$, $\overline{B}_1(x) > (<) 0$, $B_2(x) > (<) 0$ *for any sufficiently small (large)* x, *hence* x_{B_1}, $\overline{x}_{B_1}$, *and* x_{B_2} *are all uniquely exist.*

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[1] The lemma was originally proposed and proven by You [24]. However, the definition of the distribution function $F(w)$ in You's work is subtly different from that in our paper, and this creates some differences in the process of the proof. From this reason, readers should refer to [20].

(b) *Let* $v > (<) \overline{v}$. *If* $s < (b-\rho)/\kappa$, *then* $x_{B_1}(\overline{x}_{B_1}) < b$, *or else* $x_{B_1}(\overline{x}_{B_1}) = x_{B_2} \geq b$.

(c) *Let* $v = \overline{v}$. *Then* $x_{B_1} = \overline{x}_{B_1} = x_{B_2}$.

Proof: The proof is seen in Appendix. ▮

3. Problem Formulation

We consider the following discrete-time sequential stochastic decision process with an infinite planning horizon where points in time are equally spaced on the axis of the planning horizon. Let the time interval between two successive points in time be called the period.

Note that the term "order" or "customer order" throughout the paper means the order for the first class of service. At every point in time, the company makes a decision whether to conduct or skip the search for customers; in order to conduct the search a fixed cost $c \geq 0$ must be paid, called the search cost. Only when the search is conducted at a point in time, a customer arrives at the next point in time with a probability λ ($0 < \lambda \leq 1$). The prices offered by subsequently arriving customers, $w, w', \cdots$, in the admission control problem and the maximum permissible ordering prices (reservation price) of subsequently appearing customers, $w, w', \cdots$, in the pricing control problem are both independent identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ. In both problems, for certain given numbers a and b ($0 < a < b < \infty$) let us define the probability density function as follows:

$$f(w) = 0, \quad w < a, \qquad f(w > 0, \quad a \leq w \leq b, \qquad f(w) = 0, \quad b < w.$$

In the admission control problem the company makes a decision on whether to accept or reject an arriving customer order, judging from the price offered by the customer. In the pricing control problem an appearing customer is willing to place his order with the company if and only if the customer's reservation price is greater than or equal to the price proposed by the company; accordingly, in this case, when the company proposes a price z to an appearing customer, the probability of the customer's placing the order with the company is given by $p(z) = \Pr\{z \le x\}$.

A customer order in the system at a certain point in time is completed and goes out of the system at the next point in time with a probability q ($0 < q < 1$). A contract is assumed to be signed with all customers that any accepted order is delivered within $\tau \ge 1$ periods, and then a clause is added to the contract that if it can not be honored, a delay cost $\theta \ge 0$ is paid for the period delayed. This due date τ is assumed to be constant and an equivalent value among all customers. Accordingly, for an order accepted $l \ge 0$ periods ago, when $l \ge \tau - 1$, a delay cost θ must be paid if it is not completed in the next point in time with a probability $1-q$, otherwise not. As all arriving orders are assumed to be homogenous in this paper, it is natural that τ could be the same for all orders. However, if there exist multiple classes of orders, then τ may depend on the class. Further, if τ is determined through the negotiation with each arriving customer, it should be assumed as a random variable. This extension is a subject of future study, in which the structure of state space becomes more complicated, and hence its analysis is expected to be more complex.

A customer order undergoing processing in the system may be cancelled owing to unavoidable circumstances with the customer with known probabilities ν and $\bar{\nu}$ when, respectively, $l < \tau - 1$ and $l \ge \tau - 1$. If a customer cancels the order in the next point in time when $l < \tau - 1$ or when $l \ge \tau - 1$, he must pay a penalty, respectively, $\vartheta > 0$ or $\bar{\vartheta} > 0$ to the company.

When there exists no customer order accepted in the system for a period, an expected profit $s \ge 0$ is yielded by engaging in a sideline for the period, called the *sideline profit*. It is assumed that any sideline work can be completed within one period. Moreover, let the discount factor be denoted by β, assumed to be $0 < \beta < 1$.

In this paper, we propose a model with single server and no waiting room as a basic model with aforementioned assumptions. A model with multiple servers and waiting room remains as a future research subject.

In both admission control and pricing control problems, by ϕ let us denote the state in which there is no order in the system, and by l the state in which there is an order accepted l periods ago. The state $l = 0$ implies that an order has just been accepted in the system. Let $V(\phi)$ and $V(l)$ be the maximum total expected present discounted net profits in states ϕ and l, respectively. Then we can write the optimality equations for both problems as follows.

3.1 Admission control problem

$$V(\phi)=\max\begin{Bmatrix}\mathrm{C}:\beta\big(\lambda \mathrm{E}[\{w+V(0),V(\phi)\}]+(1-\lambda)V(\phi)\big)-c+s,\\ \mathrm{K}:\beta V(\phi)+s\end{Bmatrix}, \tag{3.1}$$

$$V(l)=\max\begin{Bmatrix}\mathrm{C}:(1\text{-}q)\beta\big(v(\vartheta+\lambda \mathrm{E}[\{w+V(0),V(\phi)\}]+(1-\lambda)V(\phi))+(1-v)V(l+1)\big)\\ \qquad +q\beta\big(\lambda \mathrm{E}[\{w+V(0),V(\phi)\}]+(1-\lambda)V(\phi)\big)-c,\\ \mathrm{K}:(1\text{-}q)\beta\big((v(\vartheta+V(\phi))+(1-v)V(l+1)\big)+q\beta V(\phi)\end{Bmatrix},\ 0\le l<\tau-1, \tag{3.2}$$

$$V(l)=\max\begin{Bmatrix}\mathrm{C}:(1\text{-}q)\beta\Big(-\theta+\overline{v}\big(\overline{\vartheta}+\lambda \mathrm{E}[\{w+V(0),V(\phi)\}]+(1-\lambda)V(\phi)\big)+(1-\overline{v})V(l+1)\Big)\\ \qquad +q\beta\big(\lambda \mathrm{E}[\{w+V(0),V(\phi)\}]+(1-\lambda)V(\phi)\big)-c,\\ \mathrm{K}:(1\text{-}q)\beta\Big(-\theta\big(\overline{v}(\overline{\vartheta}+V(\phi)\big)+(1-\overline{v})V(l+1)\Big)+q\beta V(\phi)\end{Bmatrix},\ \tau-1\le l. \tag{3.3}$$

In Eq. (3.1), whether the search is conducted or skipped, the sideline profit s is yielded for a period up to the next point in time. Assume that the search is conducted, i.e., C, by paying the search cost c. When a customer arrives at the next point in time with probability λ and proposes a price w, if it is accepted, the profit w is obtained and the system is in state 0, otherwise the system remains in state ϕ. Contrary to this, when no customer arrives in the next point in time with probability $1-\lambda$, the system remains in state ϕ. Assume that the search is skipped, i.e., K. Then since no customer arrives at the next point in time by the assumption, the system still remains in state ϕ. In Eq. (3.2), no sideline profit s is yielded even though any of C and K is taken. Assume that the search is conducted, i.e., C, by paying the search cost c. If an order in the system is completed and leaves the system with probability q at the next point in time, the system is in state ϕ, following the same status as in Eq. (3.1) with the search being conducted. However, when the order is not completed up to the next point in time with probability $1-q$, if it is cancelled by the customer with probability v, the penalty ϑ is paid by the customer and the system is in state ϕ, following the same status as in Eq. (3.1) with the search, otherwise the system is in state $l+1$. Assume that the search is skipped, i.e., K. Then no search cost is incurred and no customer arrives at the next point in time, which can be considered to be the case with $c=0$ and $\lambda=0$. As a consequence, this can be obtained by setting $c=0$ and $\lambda=0$ in the above case of the search being conducted.

The description of Eq. (3.3) is quite the same as that of Eq. (3.2) except that the delay cost θ is paid when the order is not completed in the next point in time with probability $1-q$ and that the cancellation probability and its penalty are changed from v and ϑ to $\bar{v}$ and $\bar{\vartheta}$, respectively.

3.2 Pricing control problem

$$V(\phi)=\max\begin{Bmatrix}\mathrm{C}:\beta\left(\lambda\max_z\{p(z)(z+V(0))+(1-p(z))V(\phi)\}+(1-\lambda)V(\phi)\right)-c+s,\\ \mathrm{K}:\beta V(\phi)+s\end{Bmatrix}, \tag{3.4}$$

$$V(l)=\max\begin{Bmatrix}\mathrm{C}:(1\text{-}q)\beta\left(v(\vartheta+\lambda\max_z\{p(z)(z+V(0))+(1-p(z))V(\phi)\}+(1-\lambda)V(\phi))+(1-v)V(l+1)\right)\\ +q\beta\left(\lambda\max_z\{p(z)(z+V(0))+(1-p(z))V(\phi)\}+(1-\lambda)V(\phi)\right)-c,\\ \mathrm{K}:(1\text{-}q)\beta\left((v(\vartheta+V(\phi))+(1-v)V(l+1)\right)+q\beta V(\phi)\end{Bmatrix},\ 0\le l<\tau-1, \tag{3.5}$$

$$V(l)=\max\begin{Bmatrix}\mathrm{C}:(1\text{-}q)\beta\left(-\theta+\bar{v}\left(\bar{\vartheta}+\lambda\max_z\{p(z)(z+V(0))+(1-p(z))V(\phi)\}+(1-\lambda)V(\phi)\right)+(1-\bar{v})V(l+1)\right)\\ +q\beta\left(\lambda\max_z\{p(z)(z+V(0))+(1-p(z))V(\phi)\}+(1-\lambda)V(\phi)\right)-c,\\ \mathrm{K}:(1\text{-}q)\beta\left(-\theta+\left(\bar{v}(\bar{\vartheta}+V(\phi)\right)+(1-\bar{v})V(l+1)\right)+q\beta V(\phi)\end{Bmatrix},\ \tau-1\le l. \tag{3.6}$$

The explanation of the above equations can be given in the same way as those of Eqs. (3.1) to (3.3) except the term multiplied by λ. The term implies the following. Let a price z be offered to an arriving customer. Then if the customer accepts the price and places his order with the company with probability $p(z)$, the company obtains the profit z and the maximum expected present discounted profit after that is given by $V(0)$. If the customer rejects the price and does not place his order with the company with probability $1-p(z)$, the company can obtain no profit from the customer and the maximum expected present discounted profit after that is given by $V(\phi)$. Accordingly, the company plans to offer a price z attaining the maximum of the expected present discounted profit, i.e., $\max_z\{p(z)(z+V(0))+(1-p(z))V(\phi)\}$.

4. Transformation of Optimality Equations

Note that our model can be considered as Markov decision process with infinite planning horizon and discounting. Since the expectation of immediate reward at any point in time is finite, by using the conventional way outlined in Markov decision process (see [17], pp.23-30), one can easily verify that $|V(\phi)| \le M/(1-\beta)$ and $|V(l)| \le M/(1-\beta)$, for a sufficiently large $M>0$, i.e., $V(\phi)$ is finite and $V(l)$ is bounded in $l \ge 0$. Therefore, we see that each of systems of Eqs. (3.1) to (3.3) as well as Eqs. (3.4) to (3.6) has a unique solution whatever each system's optimal decisions are (refer to [17], Proposition 2.3). Now, noting that the two terms $\max\{w+V(0), V(\phi)\}$ and $\max_z\{p(z)(z+V(0))+(1-p(z))V(\phi)\}$ can be represented as $\max\{w+V(0)-V(\phi), 0\}+V(\phi)$ and $\max_z p(z)(z+V(0)-V(\phi))+V(\phi)$, respectively, we can transform the system of optimality equations of each problem as follows.

$$V(\phi) = \beta V(\phi) + \max\{\lambda\beta \mathrm{E}[\max\{w+V(0)-V(\phi), 0\}] - c, 0\} + s, \tag{4.1}$$

$$\begin{aligned} V(l) &= (1\text{-}q)\beta\big(v(\vartheta+V(\phi))+(1-v)V(l+1)\big)+q\beta V(\phi) \\ &\quad + \max\{\lambda\beta(q+(1-q)v)\mathrm{E}[\max\{w+V(0)-V(\phi), 0\}] - c, 0\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} V(l) &= (1\text{-}q)\beta\big(-\theta+\overline{v}(\overline{\vartheta}+V(\phi))+(1-\overline{v})V(l+1)\big)+q\beta V(\phi) \\ &\quad + \max\{\lambda\beta(q+(1-q)\overline{v})\mathrm{E}[\max\{w+V(0)-V(\phi), 0\}] - c, 0\}, \end{aligned} \tag{4.3}$$

and

$$V(\phi) = \beta V(\phi) + \max\{\lambda\beta \max_z p(z)(z+V(0)-V(\phi)) - c, 0\} + s, \tag{4.4}$$

$$\begin{aligned} V(l) &= (1\text{-}q)\beta\big(v(\vartheta+V(\phi))+(1-v)V(l+1)\big)+q\beta V(\phi) \\ &\quad + \max\{\lambda\beta(q+(1-q)v)\max_z p(z)(z+V(0)-V(\phi)) - c, 0\}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} V(l) &= (1\text{-}q)\beta\big(-\theta+\overline{v}(\overline{\vartheta}+V(\phi))+(1-\overline{v})V(l+1)\big)+q\beta V(\phi) \\ &\quad + \max\{\lambda\beta(q+(1-q)\overline{v})\max_z p(z)(z+V(0)-V(\phi)) - c, 0\}. \end{aligned} \tag{4.6}$$

In the admission control problem, the marginal profit from accepting an order w compared with rejecting it is given by the difference of $w+V(0)$ and $V(\phi)$, i.e., $w+V(0)-V(\phi) = w-h$ where

$$h = V(\phi) - V(0). \tag{4.7}$$

This implies that if the marginal profit is greater than or equal to 0 or equivalently $w \ge h$, accept the order, otherwise reject it. For this season let us refer to h as the *optimal admission criterion*. In the pricing control problem, provided that a price z is proposed by the company, the marginal profit becomes $z+V(0)-V(\phi) = z-h$. In this case, the marginal profit is obtained with probability $p(z)$ and not obtained with probability $1-p(z)$. Accordingly, the expected marginal profit gained is given by $p(z)(z-h)+(1-p(z))\times 0=p(z)(z-h)$. Then the company will determine the optimal price as the price z maximizing the expected marginal profit $p(z)(z-h)$, i.e., $\max_z p(z)(z-h)$.

Now, using functions $L(x)$, $J(x)$, and $\overline{J}(x)$, we can easily see that the system of Eqs. (4.1) to (4.3) and the system of Eqs. (4.4) to (4.6) are rewritten in an identical form as follows.

$$V(\phi)=(\max\{L(h),0\}+s)/(1-\beta)\geq 0, \tag{4.8}$$

$$V(l)=\eta V(l+1)+(q+(1-q)v)\beta V(\phi)+\max\{J(h),0\}+(1-q)\beta v\vartheta,\quad 0\leq l<\tau-1, \tag{4.9}$$

$$V(l)=\overline{\eta}V(l+1)+(q+(1-q)\overline{v})\beta V(\phi)+\max\{\overline{J}(h),0\}+(1-q)\beta(\overline{v}\overline{\vartheta}-\theta),\quad \tau-1\leq l. \tag{4.10}$$

From the above discussion we see that the optimal policies for any given state can be prescribed as follows.

<Optimal Policies>

(a) *Optimal Admission/Pricing Policies*: In the admission control problem, when a customer appears and proposes a price w for his order, if $w\geq h$, then admission is optimal (see Eq. (4.1)), while in the pricing control problem, the optimal offering price is given by z maximizing $p(z)(z-h)$, denote by $z(h)$ (see Eq. (4.4))

(b) *Optimal Search Rules*: For both problems it is immediate from Eq. (4.8) to (4.10) that

$$L(h)\geq 0\Rightarrow\langle \mathrm{C}\rangle_{\phi},\qquad J(h)\geq 0\Rightarrow\langle \mathrm{C}\rangle_{l<\tau-1},\qquad \overline{J}(h)\geq 0\Rightarrow\langle \mathrm{C}\rangle_{l\geq\tau-1},$$
$$L(h)\leq 0\Rightarrow\langle \mathrm{K}\rangle_{\phi},\qquad J(h)\leq 0\Rightarrow\langle \mathrm{K}\rangle_{l<\tau-1},\qquad \overline{J}(h)\leq 0\Rightarrow\langle \mathrm{K}\rangle_{l\geq\tau-1}.$$

When decisions are indifferent, to avoid confusion in the subsequent discussions we define the following.

1. In case of $c=0$, if each of $L(h)$, $J(h)$, and $\overline{J}(h)$ is *greater than or equal to* zero, to continue the search in each corresponding state, i.e., $\langle \mathrm{C}\rangle_{\phi}$, $\langle \mathrm{C}\rangle_{l<\tau-1}$ and $\langle \mathrm{C}\rangle_{l\geq\tau-1}$, otherwise not.
2. In case of $c>0$, if each of $L(h)$, $J(h)$, and $\overline{J}(h)$ is *greater than* zero, it is optimal to continue the search in each corresponding state, i.e. $\langle \mathrm{C}\rangle_{\phi}$, $\langle \mathrm{C}\rangle_{l<\tau-1}$,

and $< C >_{l \geq \tau-1}$, otherwise not.

5. Results

In this section we examine the properties of Optimal Policies described in the previous section and also provide their implications. We shall prove many assertions related to the sideline profit s. Then it is often that h and $G(x)$ are regarded as a function of s, i.e., $h = h(s)$ and $G(x,s)$. However, for explanatory simplicity let us employ the notations " h and $G(x)$ " in the theorem or in the lemma. Further, for convenience let us denote x_{B_1}, $\overline{x}_{B_1}$, and x_{B_2} for $s = 0$ by $x_{B_1}(0)$, $\overline{x}_{B_1}(0)$, and $x_{B_2}(0)$, respectively.

5.1 Optimal Admission/Pricing Policies

The admission/pricing policies operate only when a customer appears by conducting the search in state ϕ. The result below states the properties of the optimal admission criterion h and the optimal offering price $z(h)$.

Theorem 5.1

(a) *h is given by the $G(x) = 0$, x_G, i.e., $h = x_G$ and $h \geq \rho$.*

(b) *h is nondecreasing in c, ϑ, and $\overline{\vartheta}$ and nondecreasing in λ. In addition, h is strictly increasing in θ and s with $\lim_{s\to\infty} h(s) = \infty$ and $\lim_{s\to-\infty} h(s) = -\infty$.*

(c) *$z(h)$ is nondecreasing in c, ϑ, and $\overline{\vartheta}$ and nondecreasing in λ, θ and s.*

(d) *If $s < (b-\rho)/\kappa$ (hence $b > \rho$ due to $s \geq 0$), then $h < z(h) < b$, otherwise $h \geq z(h) = b$.*

Proof:

(a) For simplicity, let $R = (q+(1-q)v)\beta V(\phi) + \max\{J(h),0\} + (1-q)\beta v\vartheta$ and $\overline{R} = (q+(1-q)\overline{v})\beta V(\phi) + \max\{\overline{J}(h),0\} + (1-q)\beta(\overline{v}\overline{\vartheta} - \theta)$. Then from Eq. (4.9) with $l = 0$ we have

$$V(0) = \eta V(1) + R = \eta(\eta V(2) + R) + R = \cdots = \eta^{\tau-1} V(\tau-1) + (1-\eta^{\tau-1})R/(1-\eta). \qquad (5.1)$$

Further, from Eq. (4.10) with $l \geq \tau - 1$ we obtain $V(\tau-1) = \bar{\eta}V(\tau) + \bar{R} = \bar{\eta}^2 V(\tau+1) + (1+\bar{\eta})\bar{R}$, which becomes $V(\tau-1) = \bar{\eta}^j V(\tau+j-1) + (1-\bar{\eta}^j)\bar{R}/(1-\bar{\eta})$ after j^{th} iteration. Accor-dingly, since $\bar{\eta}^j \to 0$ as $j \to \infty$ due to $\bar{\eta} < 1$ and $V(l)$ is bounded in $l \geq 0$, the above equation becomes $V(\tau-1) = \bar{R}/(1-\bar{\eta})$. Substituting this equation to Eq. (5.1) produces

$$V(0) = \beta\varphi V(\phi) + (1-\eta^{\tau-1})\max\{J(h),0\}/(1-\eta) + \eta^{\tau-1}\max\{\bar{J}(h),0\}/(1-\bar{\eta}) - \rho,$$

which can be expressed as

$$V(0) = (1-(1-\beta)\kappa)V(\phi) + (1-\eta^{\tau-1})\max\{J(h),0\}/(1-\eta) + \eta^{\tau-1}\max\{\bar{J}(h),0\}/(1-\bar{\eta}) - \rho$$

due to Eq. (2.3). Therefore, Eq. (4.7) can be rewritten as

$$h = (1-\beta)\kappa V(\phi) - (1-\eta^{\tau-1})\max\{J(h),0\}/(1-\eta) - \eta^{\tau-1}\max\{\bar{J}(h),0\}/(1-\bar{\eta}) + \rho.$$

Rearranging this equation by substituting Eq. (4.8) yields

$$\begin{aligned} h = {} & \kappa\max\{L(h),0\} - (1-\eta^{\tau-1})\max\{J(h),0\}/(1-\eta) \\ & -\eta^{\tau-1}\max\{\bar{J}(h),0\}/(1-\bar{\eta}) + \kappa s + \rho. \end{aligned} \tag{5.2}$$

Noting Eq. (2.7), we can eventually rewrite Eq. (5.2) as $G(h) = 0$; in other words, h is given by the unique solution of $G(x) = 0$, i.e., $h = x_G$ due to the Lemma 2.3(a). Next, form Eq. (2.7) we have

$$\begin{aligned} G(\rho) = {} & (1-\eta^{\tau-1})\big(\max\{L(h),0\} - \max\{J(h),0\}\big)/(1-\eta) \\ & +\eta^{\tau-1}\big(\max\{L(h),0\} - \max\{\bar{J}(h),0\}\big)/(1-\bar{\eta}) + \kappa s \geq 0, \end{aligned}$$

implying $h \geq \rho$ due to $L(\rho) \geq J(\rho)$ and $L(\rho) \geq \bar{J}(\rho)$ from Eq. (2.5).

(b) The former half is immediate from Lemma 2.3(b). Suppose $h(s)$ converges to a finite $\bar{h}$ as $s \to \infty$. Then since $h(s) < \bar{h}$ for any s, we have $G(\bar{h},s) < G(h(s),s) = 0$ from the fact that $G(x,s)$ is strictly decreasing in x due to Lemma 2.3(a). Thus, $\lim_{s\to\infty} G(\bar{h},s) \leq 0$ due to Lemma 2.3(b). However, $\lim_{s\to\infty} G(\bar{h},s) = \infty$ from Lemma 2.3(b), which is a contradiction. Thus $h(s)$ must diverge as $s \to \infty$. Similarly also proven $\lim_{s\to-\infty} h(s) = -\infty$.

(c) Immediate from Lemma 2.2(b).

(d) Since $L(b) = J(b) = \overline{J}(b) = -c$ due to Lemma 2.1(b) and Eq. (2.5), we obtain $G(b) = -b + \kappa s + \rho$ from Eq. (2.7). If $s < (b-\rho)/\kappa$, then $G(b) < 0$, implying $h < b$, or else $G(b) \geq 0$, implying $h \geq b$. Therefore, from Lemma 2.2(a) the assertion holds. ▮

We can see that as the customer arrival probability λ, the sideline profit s, or the delay cost θ increases, the optimal admission criterion h and the optimal offering price $z(h)$ increase. In this case, it is reasonable to accept orders with higher value in the admission control problem as well as to offer higher prices to arriving customers in the pricing control problem; and vice versa. In addition, as the search cost c, or the penalty ϑ and $\overline{\vartheta}$ for canceling the order increases, the optimal admission criterion h and the optimal offering price $z(h)$ decrease. In this case, it is reasonable to accept orders even with low price in the admission control problem as well as to propose lower prices in the pricing control problem; and vice versa.

5.2 Optimal Search Policies

Below, by s_b let us denote the solution of $h(s) = b$ if it exists. In fact, s_b uniquely exists from the fact that $h(s)$ is strictly increasing in s due to Theorem 5.1(b).

The result below is very useful in examining the properties of optimal search policies.

Lemma 5.1

(a) *$L(h)$, $J(h)$, and $\overline{J}(h)$ are strictly decreasing in $s < s_b$ and nonincreasing in s.*

(b) *$L(h)$, $J(h)$, and $\overline{J}(h)$ are greater than zero for any sufficiently small s and less than zero for any sufficiently large s if $c > 0$.*

(c) If $v > \overline{v}$, then $L(h) > J(h) > \overline{J}(h)$ for $s < s_b$ and $L(h) = J(h) = \overline{J}(h) = -c$ for $s \geq s_b$.

(d) If $v < (=)\overline{v}$, then $L(h) > J(h) > \overline{J}(h)$ for $s < s_b$ and $L(h) = J(h) = \overline{J}(h) = -c$ for $s \geq s_b$.

Proof: The proof is seen in Appendix. ▮

From Lemma 5.1 we can depict Figure 5.1. The following result provides the optimal search policies.

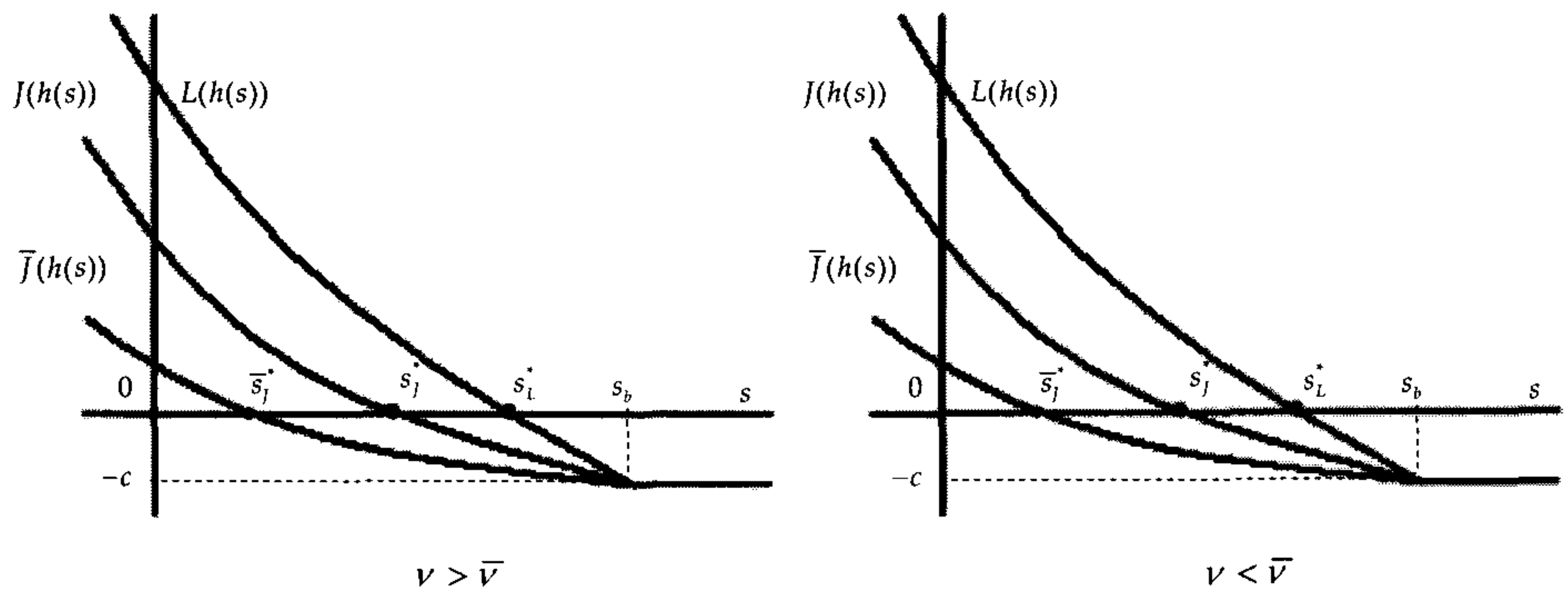

Figure 5.1. Graphs of $L(h(s))$, $J(h(s))$, and $\bar{J}(h(s))$ where $c>0$ and $J(h(s))=\bar{J}(h(s))$ if $v=\bar{v}$

Theorem 5.2

(a) *Let* $c=0$. *Then* $<\mathrm{C}>_{\phi}, <\mathrm{C}>_{l<\tau-1}$ *and* $<\mathrm{C}>_{l\geq\tau-1}$.

(b) *Let* $c>0$. *Then* s_L^*, s_J^*, *and* $\bar{s}_L^*$ *uniquely exist with* $s_J^*<s_L^*$ *and* $\bar{s}_J^*<s_L^*$. *If* $s_L^*\leq s$, *then* $<\mathrm{K}>_{\phi}$, $<\mathrm{K}>_{l<\tau-1}$, *and* $<\mathrm{K}>_{l\geq\tau-1}$, *otherwise we have:*

1. *Let* $v>\bar{v}$. *Then* $\bar{s}_J^*<s_J^*<s_L^*$.
   - i. *If* $s<\bar{s}_J^*$, *then* $<\mathrm{C}>_{\phi}$, $<\mathrm{C}>_{l<\tau-1}$, *and* $<\mathrm{C}>_{l\geq\tau-1}$.
   - ii. *If* $\bar{s}_J^*\leq s<s_J^*$, *then* $<\mathrm{C}>_{\phi}$, $<\mathrm{C}>_{l\geq\tau-1}$, *and* $<\mathrm{K}>_{l\geq\tau-1}$.
   - iii. *If* $s_J^*\leq s<s_L^*$, *then* $<\mathrm{C}>_{\phi}$, $<\mathrm{K}>_{l<\tau-1}$, *and* $<\mathrm{K}>_{l\geq\tau-1}$.
2. *Let* $v<(=)\bar{v}$. *Then* $s_J^*<(=)\bar{s}_J^*<s_L^*$.
   - i. *If* $s<s_J^*$, *then* $<\mathrm{C}>_{\phi}$, $<\mathrm{C}>_{l<\tau-1}$, *and* $<\mathrm{C}>_{l\geq\tau-1}$.
   - ii. *If* $s_J^*\leq s<\bar{s}_J^*$, *then* $<\mathrm{C}>_{\phi}$, $<\mathrm{K}>_{l<\tau-1}$, *and* $<\mathrm{C}>_{l\geq\tau-1}$.
   - iii. *If* $\bar{s}_J^*\leq s<s_L^*$, *then* $<\mathrm{C}>_{\phi}$, $<\mathrm{K}>_{l<\tau-1}$, *and* $<\mathrm{K}>_{l\geq\tau-1}$.

Proof:

(a) If $c=0$, then $L(h)=\lambda\beta T(h)\geq 0$, $J(h)=\lambda\beta(q+(1-q)v)T(h)\geq 0$, and $\bar{J}(h)=\lambda\beta(q+(1-q)\bar{v})T(h)\geq 0$ due to Lemma 2.1 (b), hence from Optimal Policies (b1) defined in Section 4 we have $<\mathrm{C}>_{\phi}$, $<\mathrm{C}>_{l<\tau-1}$, and $<\mathrm{C}>_{l\geq\tau-1}$.

(b) Let $c\geq 0$. The unique existence of s_L^*, s_J^*, and $\bar{s}_J^*$ are immediate from Lemma 5.1 (a, b). Therefore, the proof of the assertion is evident from Lemma 5.1 and Optimal Policies (b2). ∎

The assertion (a) in the above lemma implies that when no cost incurs in the search for customer orders, i.e., $c = 0$, conducting the search is always optimal. This can be said to be a reasonable conclusion.

The assertion (b) implies that if the sideline profit s is sufficiently large with $s_L^* \le s$, it would be better to engage in the sideline by making the process empty through skipping the search in all states ϕ and $l \ge 0$, i.e., $<\mathrm{K}>_\phi$ and $<\mathrm{K}>_{l\ge 0}$. As a result, the system is always empty and the company is always engaged in the sideline. In other words, the sideline becomes the main work for the company. On the other hand, the assertion (b1i, b2i) implies that if the sideline profit s is sufficiently small, it is optimal to conduct the search in all states ϕ and $l \ge 0$, i.e., $<\mathrm{C}>_\phi$ and $<\mathrm{C}>_{l\ge 0}$. The result represents that it would be better to obtain profits from arriving orders by conducting the search than to obtain the sideline profit by making the process empty through skipping the search.

Although the result above provides the perfect optimal search policies, we shall further investigate the detailed optimal search policies focusing on the relationship of the search cost c with the sideline profit s. To do so, we regard s_L^*, s_J^*, and $\overline{s}_J^*$ as functions of c, i.e., $s_L^*(c)$, $s_J^*(c)$, and $\overline{s}_J^*(c)$. Further, by c_L^*, c_J^*, and $\overline{c}_J^*$ let us denote the solution of $s_L^*(c) = 0$, $s_J^*(c) = 0$, and $\overline{s}_J^*(c) = 0$, respectively, if they exist, i.e.,

$$s_L^*(c_L^*) = 0, \qquad s_J^*(c_J^*) = 0, \qquad \overline{s}_J^*(\overline{c}_J^*) = 0. \tag{5.3}$$

Lemma 5.2

(a) $s_L^*(c)$, $s_J^*(c)$, *and* $\overline{s}_J^*(c)$ *are strictly decreasing in* c *with* $s_L^*(0) = s_J^*(0) = \overline{s}_J^*(0) = (b - \rho)/\kappa$.

(b) $c_L^* = \lambda\beta T(\rho)$.

(c) *Let* $s < (b-\rho)/\kappa$. *Then we have*:

1. *If* $v > \overline{v}$, *then* $\overline{c}_J^* < c_J^* < c_L^*$ *where* $\overline{c}_J^* = (q+(1-q)\overline{v})(x_{B_2}(0)-\rho)/(\kappa-\varphi)$ *and* $c_J^* = (q+(1-q)v)(x_{B_1}(0)-\rho)/\kappa(1-q)(1-v)$.
2. *If* $v < \overline{v}$, *then* $c_J^* < \overline{c}_J^* < c_L^*$ *where* $c_J^* = (q+(1-q)v)(x_{B_2}(0)-\rho)/(\kappa-\varphi)$ *and* $\overline{c}_J^* = (q+(1-q)\overline{v})(\overline{x}_{B_1}(0)-\rho)/\kappa(1-q)(1-v)$.

3. *If* $v=\bar{v}$, *then* $\bar{c}_J^* = c_J^* < c_L^*$ *where* $c_J^* = \bar{c}_J^* = (q+(1-q)v)(x_{B_1}(0)-\rho)/(\kappa-\varphi)$.

Proof: The proof is seen in Appendix. ▮

When $\rho < b$, depicting the optimal search policies obtained in Lemma 5.2 in the relationship with the search cost c and the sideline profit s, we have Figure 5.2. The regions $\Omega(K,K,K)$, $\Omega(C,K,K)$, $\Omega(C,C,K)$, $\Omega(C,C,C)$, and $\Omega(C,K,C)$ in Figure 5.2 correspond to the optimal decisions as follows. $\Omega(K,K,K)$ → skipping in all states ϕ and $l \geq 0$, $\Omega(C,K,K)$ → continuing in state ϕ and skipping in state $l \geq 0$, $\Omega(C,C,K)$ → continuing in states ϕ and $l < \tau-1$ and skipping in state $l \geq \tau-1$, $\Omega(C,C,C)$ → continuing in all states ϕ and $l \geq 0$, $\Omega(C,K,C)$ → continuing in states ϕ and $l \geq \tau-1$, and skipping in state $l < \tau-1$.

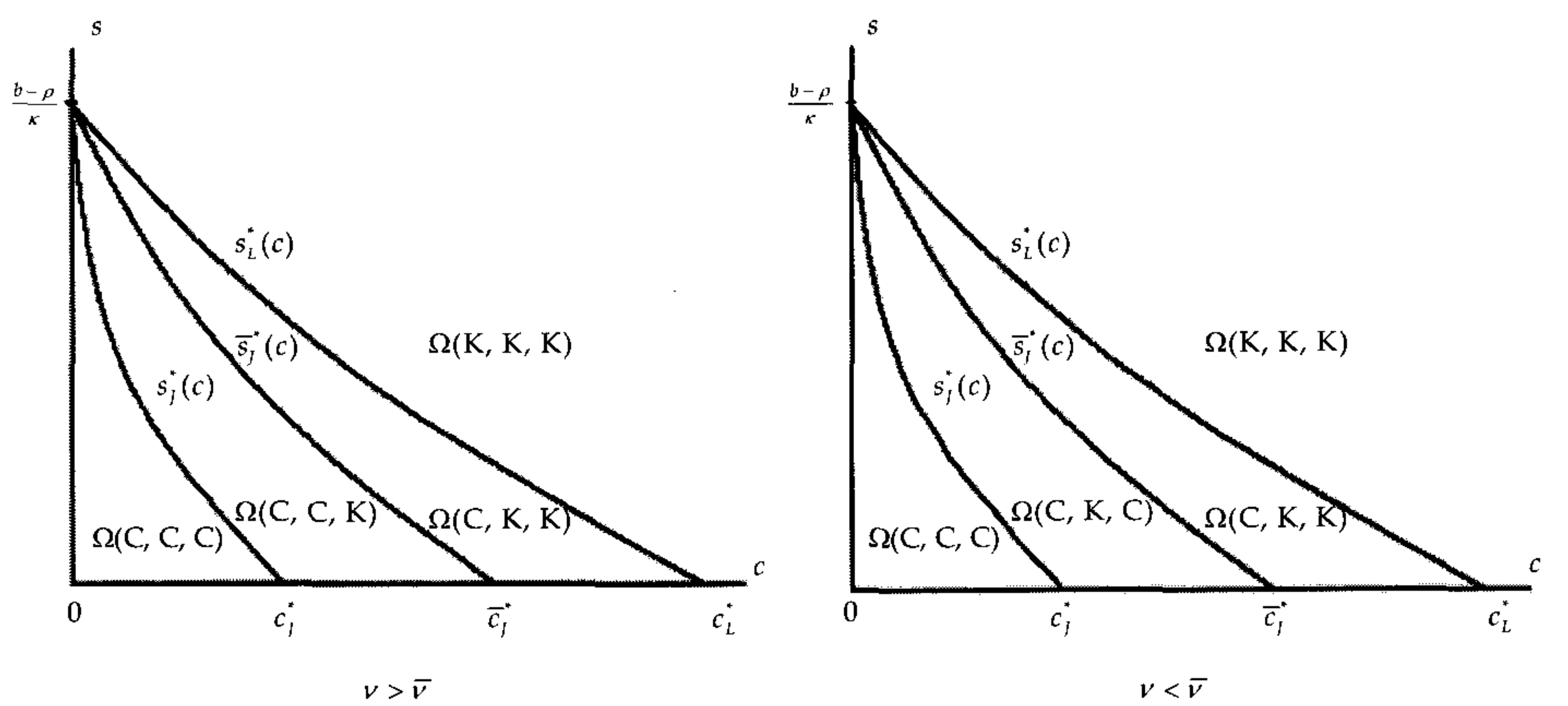

Figure 5.2. Four regions encircled by the functions $s_L^*(c)$, $s_J^*(c)$, and $\bar{s}_J^*(c)$ and the axes c, s when $b > \rho$. Note $s_J^*(c) = \bar{s}_J^*(c)$ *if* $v = \bar{v}$

From Figure 5.2 we see that the optimal search policies are given according to which region the set of search cost and sideline profit (c,s) is involved in. Moreover, if the search cost is sufficiently large with $c_L^* \leq c$ or the sideline profit is sufficiently large with $(b-\rho)/\kappa \leq s$, it is always optimal to skip the search for customer orders in all states, as a result the company enjoys the sideline profit. In this case, the sideline becomes the main business for the company.

6. Conclusions and Considerations

In this paper, discussing both admission control and pricing control problems in an identical framework, we clarified the optimal admission/pricing policies as well as the optimal search policies specially focusing on the relationships of the search cost and the sideline profit.

The model in the paper can be extended to a model that more than one customer can be held in the system. In this case, since the optimal admission criterion and the optimal offering price may depend on the number of customer orders in the system, an analysis of the monotonicity will be needed. As a general framework for the derivation of monotonicity properties, Koole [7] proposes a unified treatment of the various queueing models by concentrating on system events and the form of the value function instead of on the value function itself. It would be an interesting topic for further research to investigate if and to what extent the extended model above where the notions of search skipping and customer order cancellation are introduced and both admission control and pricing control are defined fits within Koole's framework.

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Appendix – Proofs

Proof of Eq. (2.3) To begin with, note that $v\bar{\eta}-\bar{v}\eta=(1-q)\beta(v-\bar{v})=\bar{\eta}-\eta \cdots (*)$. From Eq. (2.2) we have

$$\begin{aligned}\beta\varphi &= \beta\left(\frac{(1-\eta^{\tau-1})(q+(1-q)v)}{1-\eta}+\frac{\eta^{\tau-1}(q+(1-q)\bar{v})}{1-\bar{\eta}}\right)\\ &=\frac{q\beta+(1-q)\beta v}{1-\eta}+\frac{q\beta(\eta-\bar{\eta})-(1-q)\beta(v\bar{\eta}-\bar{v}\eta)}{(1-\eta)(1-\bar{\eta})}\eta^{\tau-1}\\ &=1+\frac{-1+\eta+q\beta+(1-q)\beta v}{1-\eta}+\frac{q\beta(\bar{\eta}-\eta)-(\bar{\eta}-\eta)+(1-q)\beta(\bar{\eta}-\eta)}{(1-\eta)(1-\bar{\eta})}\eta^{\tau-1}\quad \text{(from (*))}\\ &=1+\frac{-1+(1-q)\beta(1-v)+q\beta+(1-q)\beta v}{1-\eta}-\frac{(\bar{\eta}-\eta)(1-q\beta-(1-q)\beta)}{(1-\eta)(1-\bar{\eta})}\eta^{\tau-1}\\ &=1-\frac{1-\beta}{1-\eta}-\frac{(1-\beta)(\bar{\eta}-\eta)}{(1-\eta)(1-\bar{\eta})}\eta^{\tau-1}=1-(1-\beta)\kappa\,.\end{aligned}$$

Furthermore $\kappa-\varphi=\dfrac{(1-\eta^{\tau-1})(1-q)(1-v)}{1-\eta}+\dfrac{\eta^{\tau-1}(1-q)(1-\bar{v})}{(1-\bar{\eta})}>0$. Regarding φ as a function of $\eta^{\tau-1}$, i.e., $\varphi(\eta^{\tau-1})$, with $0<\eta^{\tau-1}<1$, we obtain $\varphi(0)=(q+(1-q)v)/(1-\eta)$. Hence $\varphi(0)<1$ because $1-\eta-(q+(1-q)v)=(1-q)(1-\beta)(1-v)>0$. And we also have $\varphi(1)=(q+(1-q)\bar{v})/(1-\bar{\eta})<1$, which can be proven in quite the same way as the above. Since $\varphi(\eta^{\tau-1})$ is a linear function of $\eta^{\tau-1}$, it follows that $\varphi(\eta^{\tau-1})<1$ for $0\le\eta^{\tau-1}\le 1$ or equivalently $\varphi<1$. ∎

Proof of Lemma 2.3

(a) We first prove the monotonicity of $G(x)$ in x. For any $x'<x$, from Eq. (2.7) we have

$$\begin{aligned}G(x)-G(x')\le\kappa\left(\max\{L(x)-L(x'),0\}\right)+(1-\eta^{\tau-1})\left(\max\{J(x')-J(x),\ 0\}\right)/(1-\eta)\\ +\eta^{\tau-1}\left(\max\{\bar{J}(x')-\bar{J}(x),0\}\right)/(1-\bar{\eta})+(x'-x)\,.\end{aligned}$$

Noting that $L(x)$, $J(x)$, and $\bar{J}(x)$ are all nonincreasing in x due to Lemma 2.1(a) and Eq. (2.5), we obtain $G(x)-G(x')\le\lambda\beta\varphi T(x')+x'-(\lambda\beta\varphi T(x)+x)$ (due to Eq. (2.2)), which is less than zero due to Eq. (2.3) with $\varphi < 1$ and Lemma 2.1(d). Next, applying Lemma 2.1(e) to Eq. (2.7) leads to

$$\lim_{x\to\infty} G(x) = \kappa(\max\{-c, 0\}) + (1-\eta^{\tau-1})(\max\{-c, 0\})/(1-\eta)$$
$$+\eta^{\tau-1}(\max\{-c, 0\})/(1-\overline{\eta}) + \lim_{x\to\infty} x + \kappa s + \rho = \infty.$$

Now, from Eq. (2.7) we have

$$G(x) = -(1-\eta^{\tau-1})(\max\{\lambda\beta(1+(1-q)v)T(x) - c, 0\})/(1-\eta)$$
$$-\eta^{\tau-1}(\max\{\lambda\beta(1+(1-q)\overline{v})T(x) - c, 0\})/(1-\overline{\eta}) - x + \kappa s + \rho. \qquad \text{(A.1)}$$

Let $c = 0$. Then since $T(x) \ge 0$ for all x due to Lemma 2.1(b), the inequality above becomes $G(x) = -(\lambda\beta\varphi T(x) + x) + \kappa s + \rho$. Hence since $\varphi < 1$ from Eq. (2.3), applying Lemma 2.1(f) into the above equation produces $\lim_{x\to-\infty} G(x) \ge -\lim_{x\to-\infty}(\lambda\beta\varphi T(x) + x) + \kappa s + \rho = \infty$. Let $c > 0$. Then there exists an $\overline{x} < b$ such that if $\overline{v} > v$, then $T(\overline{x}) = c/\lambda\beta(q+(1-q)v) > c/\lambda\beta(q+(1-q)\overline{v})$, otherwise $T(\overline{x}) = c/\lambda\beta(q+(1-q)\overline{v}) > c/\lambda\beta(q+(1-q)v)$ due to Lemma 2.1(c). For $x < \overline{x}$ we have $T(x) > T(\overline{x}) = c/\lambda\beta(q+(1-q)v) > c/\lambda\beta(q+(1-q)\overline{v}) > 0$ or $T(x) > T(\overline{x}) = c/\lambda\beta(q+(1-q)\overline{v}) > c/\lambda\beta(q+(1-q)v) > 0$, im-plying $\lambda\beta(q+(1-q)v)T(x) - c > 0$ and $\lambda\beta(q+(1-q)\overline{v})T(x) - c > 0$, hence Eq. (A.1) be-comes $G(x) \ge -(\lambda\beta\varphi T(x) + x) + \kappa c + \kappa s + \rho$. Accordingly, we can also have $\lim_{x\to-\infty} G(x) = \infty$ in quite the same way as when $s = 0$. Therefore, from the two facts above we can see that x_G uniquely exists.

(b) We first prove the monotonicity of $G(x)$ in the search cost c. For convenience, let us rewrite Eq. (2.7) as follows.

$$G(x) = \kappa\lambda\beta(\max\{T(x) - c/\lambda\beta, 0\}) - \frac{\lambda\beta(1-\eta^{\tau-1})(q+(1-q)v)}{1-\eta}(\max\{T(x) - c/\lambda\beta(q+(1-q)v), 0\})$$
$$-\frac{\lambda\beta(1-\eta^{\tau-1})(q+(1-q)\overline{v})}{1-\eta}(\max\{T(x) - c/\lambda\beta(q+(1-q)\overline{v}), 0\}) - x + \kappa s + \rho.$$

Let $v \ge \overline{v}$. For any given $c > 0$ let $x_1(c)$, $x_2(c)$, and $x_3(c)$ be the solution of, respectively, $T(x) = c/\lambda\beta(q+(1-q)\overline{v})$, $T(x) = c/\lambda\beta(q+(1-q)v)$, and $T(x) = c/\lambda\beta$. The tively, $T(x)=c/\lambda\beta(q+(1-q)\overline{v})$, $T(x)=c/\lambda\beta(q+(1-q)v)$, and $T(x)=c/\lambda\beta$. Then since $c/\lambda\beta(q+(1-q)\overline{v})\geq c/\lambda\beta(q+(1-q)v)>c/\lambda\beta>0$ due to $(q+(1-q)\overline{v})\leq(q+(1-q)\overline{v})<1$, clearly $x_1(c)$, $x_2(c)$, and $x_3(c)$ uniquely exist from Lemma 2.1(c) where $x_1(c)<b$, $x_2(c)<b$, and $x_3(c)<b$. In addition, we have $x_1(c)<x_2(c)<x_3(c)$ due to Lemma 2.1(a). It is evident that $x_1(c)$, $x_2(c)$, and $x_3(c)$ are strictly decreasing in c. Now, let $c'=c+\varepsilon$ for any infinitesimal $\varepsilon>0$, hence $c'>c$. Then $x_1(c')<x_1(c)<x_2(c')<x_2(c)<x_3(c')<x_3(c)$ (see Figure 6.3). Below, describing $G(x)$ and $G(x,c)$, let us examine the relationship of $G(x,c)$ and c. First, Eq. (2.7) for each c and c' can be rewritten as follows, respectively.

$$G(x,c)=\begin{cases}\lambda\beta(\kappa-\varphi)T(x)-x+\kappa s+\rho, & \text{on } \mathrm{I}\cup\mathrm{II} \quad \cdots (1)\\ \lambda\beta\left(\kappa-(1-\eta^{\tau-1})(q+(1-q)v)/(1-\eta)\right)T(x) & \\ \qquad -\eta^{\tau-1}c/(1-\overline{\eta})-x+\kappa s+\rho, & \text{on } \mathrm{III}\cup\mathrm{IV} \cdots (2)\\ \lambda\beta\kappa T(x)-x-\kappa c+\kappa s+\rho, & \text{on } \mathrm{V}\cup\mathrm{VI} \cdots (3)\\ -x+\kappa s+\rho, & \text{on } \mathrm{VII} \quad \cdots (4)\end{cases} \tag{A.2}$$

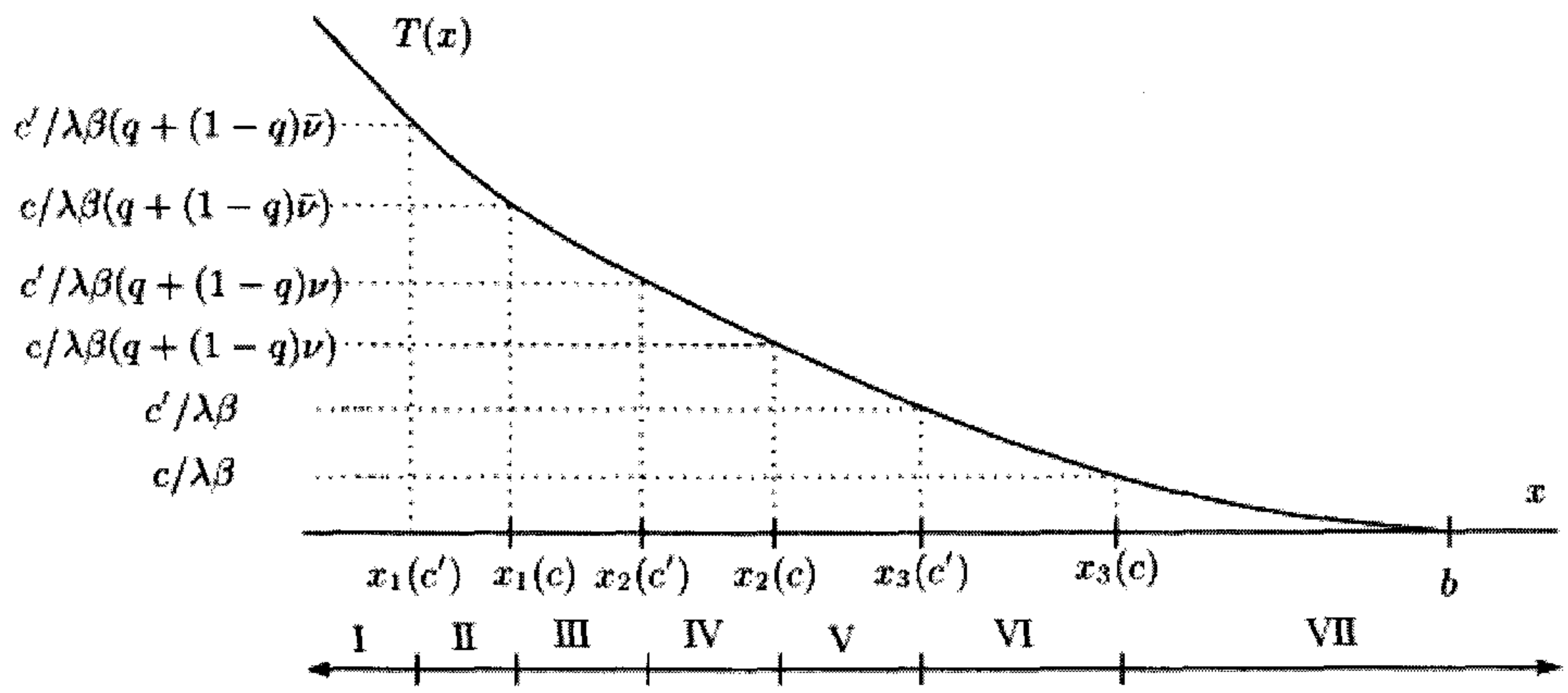

Figure 6.3. The relationship between $x_1(c')$, $x_1(c)$, $x_2(c')$, $x_2(c)$, $x_3(c)$, and $x_3(c')$

$$G(x,c')=\begin{cases}\lambda\beta(\kappa-\varphi)T(x)-x+\kappa s+\rho, & \text{on } \mathrm{I} \quad \cdots (1')\\ \lambda\beta\left(\kappa-(1-\eta^{\tau-1})(q+(1-q)v)/(1-\eta)\right)T(x) & \\ \qquad -\eta^{\tau-1}c'/(1-\overline{\eta})-x+\kappa s+\rho, & \text{on } \mathrm{II}\cup\mathrm{III} \quad \cdots (2')\\ \lambda\beta\kappa T(x)-x-\kappa c'+\kappa s+\rho, & \text{on } \mathrm{IV}\cup\mathrm{V} \quad \cdots (3')\\ -x+\kappa s+\rho, & \text{on } \mathrm{VI}\cup\mathrm{VII} \cdots (4')\end{cases} \tag{A.3}$$

On the intervals I and VII, we have $G(x,c)=G(x,c')$ from Eqs. (A.2(1)) and (A.3(1')) and from Eqs. (A.2(4)) and (A.3(4')), respectively. On the interval II, noting Eq. (2.2), from Eqs. (A.2(1)) and (A.3(2')) we get $G(x,c)-G(x,c')\geq -\eta^{\tau-1}(\lambda\beta(q+(1-q)\overline{v})T(x)-c')/(1-\overline{\eta})$, which is greater than zero due to $\lambda\beta(q+(1-q)\overline{v})T(x)-c'<0$ on $x_1(c')<x$, hence $G(x,c)>G(x,c')$. On the interval III, from Eqs. (A.2(2)) and (A.3(2')) we have

$$G(x,c)=\lambda\beta\big(\kappa-(1-\eta^{\tau-1})(q+(1-q)v)/(1-\eta)\big)T(x)-\eta^{\tau-1}c/(1-\overline{\eta})-x+\kappa s+\rho$$
$$>\lambda\beta\big(\kappa-(1-\eta^{\tau-1})(q+(1-q)v)/(1-\eta)\big)T(x)-\eta^{\tau-1}c/(1-\overline{\eta})-x+\kappa s+\rho=G(x,c').$$

On the interval IV, from Eqs. (A.2(2)) and (A.3(3')) we obtain

$$G(x,c)-G(x,c')=-\lambda\beta(1-\eta^{\tau-1})(q+(1-q)v)T(x)/(1-\eta)-\eta^{\tau-1}c/(1-\overline{\eta})+\kappa c'$$
$$>-(1-\eta^{\tau-1})(\lambda\beta(q+(1-q)v)T(x)-c')/(1-\eta)>0$$

due to $\lambda\beta(q+(1-q)v)T(x)-c'<0$ on $x_2(c')<x$. On the interval V, from Eqs. (A.2(3)) and (A.3(3')) we have $G(x,c)=\lambda\beta\kappa T(x)-x-\kappa c+\kappa s+\rho>\lambda\beta\kappa T(x)-x-\kappa c'+\kappa s+\rho=G(x,c')$. On the interval VI, from (A.2(3)) and (A.3(4')) we have $G(x,c)-G(x,c')=\lambda\beta\kappa T(x)-\kappa c=\kappa(\lambda\beta T(x)-c)>0$.

From all the above, it eventually follows that $G(x,c)\geq G(x,c')$ for all x, that is $G(x,c)$ is nonincreasing in c for all x. When $v<\overline{v}$, in the almost same way as the above we can prove that $G(x)$ is nonincreasing in c. The proof of the monotonicity of $G(x)$ in λ can also be proven in quite the same way as the above, hence we abbreviate it. Finally, since $G(x)$ is strictly increasing in ρ and that the ρ is strictly decreasing in ϑ and $\overline{\vartheta}$ and strictly increasing in θ from Eq. (2.1). In addition, it is obvious from Eq. (2.7) that $G(x)$ is strictly increasing in s. ∎

Proof of Lemma 2.4

(a) First, $B_1(x), \overline{B}_1(x),$ *and* $B_2(x)$ are all strictly decreasing in due to the fact that $T(x)$ is nonincreasing in x due to Lemma 2.1(a) and that $-x$ is strictly decreasing in x. Next, $B_1(x)>(<)0$, $\overline{B}_1(x)>(<)0$, $B_2(x)>(<)0$ for any sufficiently small (large)

x due to Lemma 2.1(e). Therefore, from the two facts above x_{B_1}, $\overline{x}_{B_1}$, and x_{B_2} are all uniquely exist.

(b) First, let $v > \overline{v}$. From Eqs. (2.8) and (2.9) we have $B_1(x) - B_2(x) = -\lambda\beta\eta^{\tau-1}(1-q)(v-\overline{v})T(x)/(1-\overline{\eta})$. If $x < b$, then $T(x) > 0$ due to Lemma 2.1(b), hence $B_1(x) < B_2(x)$, otherwise $T(x) = 0$, hence $B_1(x) = B_2(x)$. Now, if $s < (b-\rho)/\kappa$, then $B_1(b) = B_2(b) = -b + \kappa s + \rho < 0$. Accordingly, x_{B_1} and x_{B_2} are less than b, implying $x_{B_1} < x_{B_2} < b$ due to $B_1(x) < B_2(x)$ for $x < b$. If $s \geq (b-\rho)/\kappa$, then $B_1(b) = B_2(b) = -b + \kappa s + \rho \geq 0$. Therefore, x_{B_1} and x_{B_2} are greater than or equal to b, implying $x_{B_1} = x_{B_2} \geq b$ due to $B_1(x) = B_2(x)$ for $x \geq b$. Next, let $v < \overline{v}$. From Eqs. (2.8) and (2.9) we have $\overline{B}_1(x) - B_2(x) = \lambda\beta(1-\eta^{\tau-1})(1-q)(v-\overline{v})T(x)/(1-\kappa)$. Therefore, we can prove the assertion in the same procedure as the above.

(c) Let $v = \overline{v}$. Then since $\eta = \overline{\eta}$, we immediately have $\kappa = (1-\eta)$ and $\varphi = (q+(1-q)v)/(1-\eta)$ from Eq. (2.2), hence $\kappa - \varphi = \kappa(1-q)(1-v)$. Accordingly, $B_1(x) = \overline{B}_1(x) = B_2(x)$ due to Eqs. (2.8) and (2.9). ▮

Proof of Lemma 5.1

(a) For $s < s' < s_b$ since $h(s) < h(s') < h(s_b) = b$ due to Theorem 5.1(b), we get $L(h(s)) > L(h(s'))$, $J(h(s)) > J(h(s'))$, and $\overline{J}(h(s)) > \overline{J}(h(s'))$ from Lemma 2.1(a) and Eq. (2.5), hence the former half of the assertion holds. From Theorem 5.1(b) and Lemma 2.1(a) it can be immediately seen that $L(h(s))$, $J(h(s))$, and $\overline{J}(h(s))$ are nonincreasing in s.

(b) Let $c > 0$. Then from Theorem 5.1(b) and Lemma 2.1(c) we easily see that there exists an $\overline{s}$ such that $T(h(s)) = c/\lambda\beta > 0$ and $h(\overline{s}) < b$. Now, for $s < \overline{s}$ we have $h(s) < h(\overline{s}) < b$ due to Theorem 5.1(b). Accordingly, from Lemma 2.1(a) we get $T(h(s)) > T(h(\overline{s})) = c/\lambda\beta$, hence $0 < \lambda\beta T(h(s)) - c = L(h(s))$. Let $c = 0$. Then for $s < s_b$ we obtain $h(s) < h(s_b) = b$, hence $L(h(s)) = \lambda\beta T(h(s)) > \lambda\beta T(b) = 0$ due to Lemma 2.1(a, b). The proofs of $J(h(s)) > 0$ and $\overline{J}(h(s)) > 0$ are also shown in quite the same way as the above. Next, the latter half of the assertion can be proven as follows. Let $c > 0$. For $s_b \leq s$ we have $b = h(s_b) \leq h(s)$ from Theorem 5.1(b), hence $L(h(s)) = J(h(s)) = \overline{J}(h(s)) = -c < 0$ due to Lemma 2.1(b) and Eq. (2.5).

(c, d) Note that $1 > q + (1-q)v$ and $1 > q + (1-q)\overline{v}$. For $s < s_b$ we have $h(s) < h(s_b)$

$= b$ due to Theorem 5.1(b), hence $T(h(s)) > 0$ due to Lemma 2.1(b). Therefore, if $v > \overline{v}$, then $\lambda\beta T(h(s)) - c > \lambda\beta(q+(1-q)v)T(h(s)) - c > \lambda\beta(q+(1-q)\overline{v})T(h(s)) - c$, or equivalently $L(h(s)) > J(h(s)) > \overline{J}(h(s))$. Moreover, if $v < (=)\overline{v}$, we immediately have $L(h(s)) > \overline{J}(h(s)) > (=) J(h(s))$. For $s_b \le s$ we get $b = h(s_b) \le h(s)$, hence $L(h(s)) = J(h(s)) = \overline{J}(h(s)) = -c < 0$ due to Lemma 2.1(b). ∎

Proof of Lemma 5.2

(a) The former half of the assertion is proven as follows. Let $c < c'$. Then since $L(h(s_L^*(c))) = 0$ and $L(h(s_L^*(c'))) = 0$ due to Eq. (2.6), we have $c = \lambda\beta T(h(s_L^*(c)))$ and $c' = \lambda\beta T(h(s_L^*(c')))$. Accordingly, $\lambda\beta T(h(s_L^*(c))) = c < c' = \lambda\beta T(h(s_L^*(c')))$. Hence, $h(s_L^*(c)) > h(s_L^*(c'))$ because if not so, we have the contradiction of $T(h(s_L^*(c))) \ge T(h(s_L^*(c')))$ due to Lemma 2.1(a). Therefore, we obtain $s_L^*(c) > s_L^*(c')$ from Lemma 5.1(b). Similarly also proven for $s_J^*(c)$ and $\overline{s}_J^*(c)$. The latter half of the assertion is proven as follows. Let $c = 0$. Then $L(x) = \lambda\beta T(x) \ge 0$, $J(x) = \lambda\beta(q+(1-q)v)T(x) \ge 0$, and $\overline{J}(x) = \lambda\beta(q+(1-q)\overline{v})T(x) \ge 0$ for all x due to Lemma 2.1(b), from Eq. (2.7) we have $G(x,c) = \lambda\beta(\kappa-\varphi)T(x) - x + \kappa s + \dot{\rho}$ due to Eq. (2.2). Let $\overline{s} = (b-\rho)/\kappa$. Then from the equation above we have $0 = G(h(\overline{s}), c) = \lambda\beta(\kappa-\varphi)T(h(\overline{s})) - h(\overline{s}) + b$, hence $\lambda\beta(\kappa-\varphi)T(h(\overline{s})) = h(\overline{s}) - b \cdots$(**). Since $\kappa > \varphi$ from Eq. (2.3) and $T(h(\overline{s})) > 0$ from Lemma 2.1(b), we obtain $h(\overline{s}) \ge b$. Noting $h(s_b) = b$ from the definition of s_b, we have $s_b \le \overline{s}$ due to Theorem 5.1(b). Here, suppose $s_b < \overline{s}$. Then $b = h(s_b) < h(\overline{s})$, hence $T(h(\overline{s})) > 0$ from (**), so that $h(\overline{s}) < b$ due to the Lemma 2.1(b), which is a contradiction. Therefore it must be $s_b = \overline{s}$. Now, if $s < s_b$, then $h(s) < h(s_b)$ from Theorem 5.1(b), hence $L(h(s)) > L(h(s_b)) = 0$ due to Lemma 5.1 (a, c, d) with $c = 0$, and if $s \ge s_b$, then $L(h(s)) = 0$ due to Lemma 5.1(c, d) with $c = 0$. Thus we have $s_L^* = s_b$ due the definition of s_L^*. The proof of $s_J^* = \overline{s}_J^* = s_b$ is the same as the above.

(b) From Eq. (2.6) we have

$$c = \lambda\beta T(h(s_L^*)), \tag{A.4}$$

$$s = s_L^* \; c = \lambda\beta(q+(1-q)v)T(h(s_J^*)), \tag{A.5}$$

$$c = \lambda\beta(q+(1-q)\overline{v})T(h(\overline{s}_J^*)) . \tag{A.6}$$

Since $1>(q+(1-q)v)$ and $1>(q+(1-q)\overline{v})$ due to the assumption of $q<1$, $v<1$, and $\overline{v}<1$, we have $0=L(h(s_L^*))\geq \overline{J}(h(s_L^*))$. Now, noting $h(s_L^*)$ is the solution of $G(x)=0$ with $s=s_L^*$, we obtain $G(h(s_L^*),s_L^*)=h(s_L^*)+\kappa s_L^*+\rho$, hence $h(s_L^*)=\kappa s_L^*+\rho$ for any c. Accordingly, if $c=c_L^*$, then $s_L^*=s_L^*(c_L^*)=0$, thus since $h(s_L^*(c_L^*))=\rho$, from Eq. (A.4) we have $c_L^*=\lambda\beta T(h(s_L^*(c_L^*)))=\lambda\beta T(\rho)$.

(c) Let $s<(b-\rho)/\kappa$.

(c1) Let $v>\overline{v}$. We can find out c_J^* as follows. If $s=s_J^*$, then $L(h)>0$, $J(h)=0$, and $\overline{J}(h)\leq 0$ due to Theorem 5.2(b1iii). Hence from Eq. (2.7) we have, for $x=h(s_J^*)$, $0=G(h(s_J^*),s_J^*)=\kappa(\lambda\beta T(h(s_J^*))-c)-h(s_J^*)+ks_J^*+\rho$. Substituting Eq. (A.5) into this equation yields $0=\kappa\lambda\beta(1-q)(1-v)T(h(s_J^*))-h(s_J^*)+ks_J^*+\rho$, implying $B_1(h(s_J^*))=0$, hence $x_{B_1}(s_J^*)=h(s_J^*)$ from Lemma 2.4(a). If $c=c_J^*$, then $s_J^*=s_J^*(c_J^*)=0$ due to Eq. (5.3), thus $x_{B_1}(0)=h(0)$. Accordingly,

$$T(x_{B_1}(0))=(x_{B_1}(0)-\rho)/\kappa\lambda\beta(1-q)(1-v). \tag{A.7}$$

Now, from Eq. (A.5) we have

$$c_J^*=\lambda\beta(q+(1-q)v)T(h(s_J^*))=\lambda\beta(q+(1-q)v)T(h(0))=\lambda\beta(q+(1-q)v)T(x_{B_1}(0)),$$

leading to $T(x_{B_1}(0))=c_J^*/\lambda\beta(q+(1-q)v)$, hence from Eq. (A.7) we obtain $c_J^*/\lambda\beta(q+(1-q)v)=(x_{B_1}(0)-\rho)/\kappa\lambda\beta(1-q)(1-v)$, or equivalently $c_J^*=(q+(1-q)v)(x_{B_1}(0)-\rho)/\kappa(1-q)(1-v)$.

We can find out $\overline{c}_J^*$ as follows. If $s=\overline{s}_J^*$, then $L(h)>0$, $J(h)>0$, and $\overline{J}(h)=0$ due to Theorem 5.2(b1, b1ii). Hence from Eq. (2.7) we have, for $0=\lambda\beta(\kappa-(1-\eta^{\tau-1})(q+(1-q)v)/(1-\eta)T(h(\overline{s}_J^*))-\eta^{\tau-1}c/(1-\overline{\eta})-h(\overline{s}_J^*)+\kappa\overline{s}_J^*+\rho$ due to Eq. (2.2). Substituting Eq. (A.6) into the above equations produces $0=\lambda\beta(\kappa-\varphi)T(h(\overline{s}_J^*))-h(\overline{s}_J^*)+\kappa\overline{s}_J^*+\rho$ due to Eq. (2.2), implying $B_2(h(\overline{s}_J^*))=0$, hence $x_{B_2}(\overline{s}_J^*)=h(\overline{s}_J^*)$

from Lemma 2.4(a). If $c = \overline{c}_J^*$, then $\overline{s}_J^* = \overline{s}_J^*(\overline{c}_J^*) = 0$ due to Eq. (5.3), thus $x_{B_2}(0) = h(0)$. Accordingly,

$$T(x_{B2}(0)) = (x_{B_2}(0) - \rho) / \lambda\beta(\kappa - \varphi). \tag{A.8}$$

Now, from Eq. (A.6) we have

$$\overline{c}_J^* = \lambda\beta(q + (1-q)\overline{v})T(h(\overline{s}_J^*)) = \lambda\beta(q + (1-q)\overline{v})T(h(0)) = \lambda\beta(q + (1-q)\overline{v})T(x_{B_2}(0)),$$

from which $T(x_{B_2}(0)) = \overline{c}_J^* / \lambda\beta(q + (1-q)\overline{v})$, hence from Eq. (A.8) we get $\overline{c}_J^* / \lambda\beta(q + (1-q)\overline{v}) = (x_{B_2}(0) - \rho) / \lambda\beta(\kappa - \varphi)$, or equivalently $\overline{c}_J^* = (q + (1-q)\overline{v})(x_{B_2}(0) - \rho) / (\kappa - \varphi)$.

The relationship of $\overline{c}_J^*$, c_J^*, and c_L^* can be proven as follows. Noting $x_{B_1}(0)$ and $x_{B_2}(0)$ are solution of $B_1(x) = 0$ and $B_2(x) = 0$, from Eqs. (2.8) and (2.9) we immediately have the same equations as Eqs. (A.7) and (A.8), respectively. Since $T(x_{B1}(0)) \ge 0$ from the fact that $T(x) \ge 0$ for all x due to Lemma 2.1(b), we have $x_{B1}(0) \ge \rho$ from Eq. (A.7), thus $T(\rho) \ge T(x_{B1}(0))$ c due to Lemma 2.1(a). Accordingly, from Eqs. (A.7) and (c1) we have

$$T(\rho) \ge \frac{x_{B1}(0) - \rho}{\kappa\lambda\beta(1-q)(1-v)} = \frac{(q + (1-q)v)(x_{B1}(0) - \rho) / \kappa(1-q)(1-v)}{\lambda\beta(q + (1-q)v)} = \frac{c_J^*}{\lambda\beta(q + (1-q)v)}.$$

Let $s < (b - \rho)/\kappa$. Then $b < \rho$ due to the assumption of $s \ge 0$. Therefore, since $T(\rho) > 0$ from Lemma 2.1(b), we obtain $c_J^* \le \lambda\beta(q + (1-q)v)T(\rho) < \lambda\beta T(\rho) = c_L^*$ due to (b) and the assumption of $q < 1$ and $v < 1$. Further, $x_{B_1} < x_{B_2} < b$ from Lemma 2.4(b), hence $T(x_{B_1}) < T(x_{B_2})$ due to Lemma 2.1(a). Therefore, from Eqs. (A.7) and (A.8) we can immediately show that $\overline{c}_J^* < c_J^*$. Thus we eventually obtain $\overline{c}_J^* < c_J^* < c_L^*$.

(c2) Proven in the same way as in the proof of (c1).

(c3) Immediate from the fact that $x_{B_1} = \overline{x}_{B_1} = x_{B_2}$ due to Lemma 2.4(c) and $\kappa(1-q)(1-v) = \kappa(1-q)(1-\overline{v}) = \kappa - \varphi$, which was shown in the proof of Lemma 2.4(c). ∎