J. Appl. Math. & Informatics Vol. 26(2008), No. 3 - 4, pp. 721 - 732
Website: http://www.kcam.biz

A SINGLE SERVER RETRIAL QUEUE WITH VACATION

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ABSTRACT. A single server infinite capacity queueing system with Poisson arrival and a general service time distribution along with repeated attempt and server vacation is considered. We made a comprehensive analysis of the system including ergodicity and limiting behaviour. Some operating characteristics are derived and numerical results are presented to test the feasibility of the queueing model.

AMS Mathematics Subject Classification: 60K25 and 60K30
Key words and phrases: Bernoulli process, single vacation, retrial, stationary distribution, queue length, waiting time.

1. Introduction

The queueing literature exposes many useful results for both vacation and retrial queues. However, relatively few results are available for queues that allow both retrials and vacations. The single server queueing systems with vacation are analysed by many authors including Levy et al [7,8], Cooper [4], Doshi [5] and Teghem [9] have made a comprehensive survey of queueing system with vacation. Models with repeated attempts arise frequently in the analysis of telephone and other communication systems. A review of the main results can be found in Yang et al [10], Falin [6] and Choi et al [2]. Artalejo [1] have made a comprehensive analysis on $M/G/1$ retrial queue with vacation.

In this article we study an $M/G/1$ queue with repeated attempts and with Bernoulli vacation (model I), and with single vacation (model II). The rest of the article is organized as follows. The mathematical definition of model I is introduced in section 2 and also the steady state analysis is carried out. In section 3, we study a mathematical definition and the steady state analysis of model II. A numerical study is carried out for model I and II in section 4. In section 5, we are given a conclusion.

For both the models the arrival process is Poisson with parameter λ. The successive service time S are independent identically distributed random variables

Received September 28, 2006. *Corresponding author.

with distribution function $B(t)$, $(B(0) = 0)$. Conditional completion rate is $\eta(x)$, first and second moments are β_1 and β_2, and Laplace-Stieltjes transform of $B(t)$ is $\beta(\theta)$. Any arriving customer who finds the server busy upon arrival leaves the service area immediately and joins a group of blocked customers called orbit. The orbit customers try their luck in a random manner. The corresponding random variable follows negative exponential with mean μ. If the orbit customer finds the server free, he occupies the server and this policy does not depends on the number of customers in the orbit, therefore it is called constant retrial policy.

2. Model I

2.1. Bernoulli vacation

In addition to assumptions in section I, we also assume that the server takes a vacation when the system becomes empty after a service completion or a vacation completion. However after each service completion the server takes a vacation with probability p and starts a new service (if a customer is present) with probability q, $(p + q = 1)$. The decision about taking vacation after each service completion or vacation completion are independent. In addition, if the server finds a customer upon return from a vacation, he always starts the service otherwise he will again take a vacation.

2.2. Steady state analysis

Assume that the input flow of primary arrivals, intervals between repeated attempts and service times are mutually independent. The state of the system at time t can be described by the process

$$X(t) = (C(t), N(t), \xi(t))$$

where

$$C(t) = \begin{cases} 0 & \text{if the server is idle,} \\ 1 & \text{if the server is busy,} \\ 2 & \text{if the server is on vacation, and} \end{cases}$$

$N(t)$ is the number of customers in the orbit at time t. If $C(t) = 1$ or $C(t) = 2$, then $\xi(t)$ represents the elapsed time of the customer being served or the elapsed time of the vacation in progress, respectively.

Let B be a Bernoulli random variable with

$$B = \begin{cases} 1, & \text{the server take vacation after service with probability } p, \\ 0, & \text{the server continues service after service with probability } q. \end{cases}$$

The evolution of this queue can be described in terms of alternating sequence of idle and busy periods for the server. At any service completion epoch the server becomes free. Then the next idle period for the server is of two different types viz.,

(i) The orbit becomes empty so the server takes a proper vacation.

(ii) After the service completion epoch $n(\geq 1)$ customers were in orbit, then a competition between two exponential laws with rate λ and μ respectively determine the next customer and the server continue service with probability q here we have some idle period, and we take the idle period also a vacation period, and the server may take vacation with probability p.

Let $\{t_n; n \in N\}$ be the sequence of service completion or vacation completion epochs. The sequence of random variables,

$$Z_n = (C(t_{n^-}), N(t_{n^+})) = (i_n, j_n)$$

forms a Markov chain which is embedded Markov chain for this model. The state space is

$S = \{1,2\} \times N$, where $N = \{0,1,2,\ldots\}$. The law of transition is given by

$$(i_{n+1}, j_{n+1}) = \begin{cases} (2, X), & \text{if } j_n = 0, \\ (1, j_n + Y - B_n + B), & \text{if } j_n \geq 1. \end{cases} \tag{2.1}$$

Also $B_n = 0, 1 : B = 0, 1$ where X is the number of arrivals during a vacation: Y the number of arrivals during a service period and $B_n = 1$ if the customer who leaves the system at time t_{n+1}, proceeds from the orbit and $B_n = 0$ otherwise.

First we investigate the ergodicity of the chain.

Theorem 2.1. *The embedded Markov chain is ergodic*

$$\textit{iff } (pE(X) + q\beta_1\lambda)(\lambda + \mu)\mu^{-1} < 1. \tag{2.2}$$

Proof. It is easy to prove that the chain $\{Z_n : n \in N\}$ is irreducible and aperiodic. To prove (2.2) we use Foster's criterion which states that $\{Z_n; n \in N\}$ is ergodic if there exists a non-negative function $f(s)$, $s \in S$ and $\in > 0$, such that the mean drift $D_{ij} = E\Big[f(Z_{n+1}) - f(Z_n)/Z_n = (i,j)\Big]$ is finite for all states and $D_s \leq - \in$ for all s except a finite number.

Choose $f(i,j) = \dfrac{j}{1-\rho}$ for $(i,j) \in S$ then

$$D_{ij} = \begin{cases} E(X)p + \beta_1\lambda q - \dfrac{\mu}{\lambda + \mu}, & \text{if } j \geq 1, \\ \dfrac{E(X)}{1-\rho}, & \text{if } j = 0 \end{cases}$$

and condition (2.2) is satisfied then Foster's criterion guarantees that $\{Z_n : n \in N\}$ is ergodic.

In [3] it can be shown that the limiting probabilities of $Z(t) = (C(t), N(t))$ exist and are positive if the embedded Markov chain $\{Z_n : n \in N\}$ is ergodic. $\square$

Now we write

$$\rho = (pE(X) + q\beta_1\lambda)(\lambda + \mu)\mu^{-1} \tag{2.3}$$

and we assume that $\rho < 1$.

The stationary distribution of the embedded Markov chain is obtained in the following theorem.

Theorem 2.2. *The probability generating function of the stationary distribution is*

$$\pi_{ij} = \lim_{n\to\infty} Pr\{Z_n = (i,j)/(i,j) \in S\}. \tag{2.4}$$

$$\Pi_1(z) = \frac{\gamma(X(z)-1)(\mu+\lambda z)(qK(z)+pX(z))}{(\lambda+\mu)z-(\mu+\lambda z)(qK(z)+pX(z))} \tag{2.5}$$

$$\Pi_2(z) = \gamma X(z) \tag{2.6}$$

where

$$\left.\begin{array}{l} \Pi_i(z) = \sum\limits_{j=0}^{\infty} \pi_{ij} z^j \\ K(z) = \beta(\lambda - \lambda z) \\ X(z) = \sum\limits_{j=0}^{\infty} Pr\{X = j\} z^j \end{array}\right\} \tag{2.7}$$

and

$$\gamma = \pi_{10} + \pi_{20} = (1-\rho)[1-\rho+(1+\lambda\mu^{-1})E(X)]^{-1}. \tag{2.8}$$

Proof. The stationary distribution $\{\pi_{ij}\}$ satisfies the following stationary equations.

$$\begin{aligned}\pi_{1j} &= (\lambda+\mu)^{-1}\Bigg[\mu \sum_{k=1}^{j+1}(qk_{j+1-k} + pPr\{X = j+k-1\})(\pi_{1k}+\pi_{2k}) \\ &\quad + \lambda \sum_{k=1}^{j}(qk_{j-k} + pPr\{X = j-k\})(\pi_{1k}+\pi_{2k})(1-\delta_{jo})\Bigg],\ j \geq 0 \end{aligned}\tag{2.9}$$

$$\pi_{2j} = (\pi_{10}+\pi_{20})Pr\{X = j\},\ j \geq 0 \tag{2.10}$$

where $k_j = \int\limits_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} dB(t), j \geq 0$ and δ denotes the Kronecker's function, multiplying (2.9) and (2.10) by z^j and taking summation over $j = 0$ to ∞ and then applying (2.7). We find $\Pi_1(z)$ and $\Pi_2(z)$ are as in (2.5) and (2.6) respectively.
Now

$$\begin{aligned}\Pi(z) &= \Pi_1(z) + \Pi_2(z) \\ &= \frac{\gamma[\{(\lambda+\mu)z - p(\mu+\lambda z)\}X(z) - q(\mu+\lambda z)K(z)]}{(\lambda+\mu)z-(\mu+\lambda z)(qK(z)+pX(z))}.\end{aligned}\tag{2.11}$$

Apply L'Hospital rule on (2.11) and using $\Pi(1) = 1$, we get the form of γ as in (2.8). Finally we have $\Pi_1(1) = 1-\gamma$ and $\Pi_2(1) = \gamma$. □

Particular case:
For $p = 0$, then we get a result which coincides with the result of Artalejo [1], page 496.

2.3. Operating characteristics

By a straightforward calculation on (2.11), we obtain the following operating characteristics. Note that the first moment of N is

$$\frac{\gamma}{2(\mu-AC_1)^2}\Big[(\mu-AC_1)(2AX'(1)-2\lambda C_1+qAB_2)+(\mu+qAB_1)(AC_2+2\lambda C_1)\Big],$$

and the second moment of N is

$$\begin{aligned}&\frac{\gamma}{6(\mu-AC_1)^3}\Big[(\mu-AC_1)^2(6AX''(1)-6\lambda C_2+2qAB_3)\\&+(\mu-AC_1)(2AX'(1)-2\lambda C_1+qAB_2)(3AC_2+6\lambda C_1)\\&+(\mu-AC_1)(\mu+qAB_1)(2AC_3+6\lambda C_2)\\&+(\mu+qAB_1)(AC_2+2\lambda C_1)^2\Big].\end{aligned}$$

Using Little's formula $L=\lambda W$, we can find that the average waiting time in the system W is given by

$$\begin{aligned}E(W) &= \frac{\gamma}{2\lambda[\mu-AC_1]^2}\Big\{[\mu_1-AC_1][2AX'(1)-2\lambda C_1+qAB_2]\\&\quad+[\mu+qAB_1][AC_2+2\lambda C_1]\Big\}\end{aligned}$$

and the second moment of the waiting time in the system is given by

$$\begin{aligned}E[W^2] &= \frac{\gamma}{6\lambda^2[\mu-AC_1]^3}\Big\{(\mu_1-AC_1)^2(6AX''(1)-6\lambda C_2+2qAB_3)\\&\quad+(\mu-AC_1)(2AX'(1)-2\lambda C_1+qAB_2)(3AC_2+6\lambda C_1)\\&\quad+(\mu-AC_1)(\mu+qAB_1)(2AC_3+6\lambda C_2)\\&\quad+(\mu+qAB_1)(AC_2+2\lambda C_1)^2\Big\}\end{aligned}$$

where $A=\lambda+\mu$, $B_1=X'(1)-K'(1)$, $B_2=X''(1)-K''(1)$, $B_3=X'''(1)-K'''(1)$, $C_1=pX'(1)+qK'(1)$, $C_2=pX''(1)+qK''(1)$, $C_3=pX'''(1)+qK'''(1)$, $X'(1)=E(X), X''(1)=E(X^2)-E(X)$, $X'''(1)=E(X^3)-3E(X^2)+2E(X)$, $K'(1)=\lambda\beta_1$, $K''(1)=\lambda^2\beta_2$, and $K'''(1)=\lambda^3\beta_3$.

3. Model II

3.1. Single vacation

In addition to the assumptions in section I, the server takes a vacation when the system becomes empty, after finishing his vacation the server returns to the main system. If, on returning, the server finds customer waiting for him he starts service immediately and keeps busy until the system becomes idle and he leaves for another vacation. If no customer has arrived during the vacation period the server waits for the first customer to arrive when an ordinary $M/G/1$ busy period is initiated. At the termination of the busy period the server takes another vacation. This type of vacation policy is called single vacation policy. The vacation period is general with distribution function $V(t)$.

3.2. Steady state analysis

Assume that the input flow of primary arrivals, intervals between repeated attempts and service times are mutually independent. The state of the system at time t can be described by the process $X(t) = (C(t), N(t))$ where $C(t)$, the state of the server and takes the value $0, 1$, or 2 according as the server is idle, busy or on vacation. $N(t)$ denotes the number of customers in orbit at time t. Let $\{t_n, n \in N\}$ be a sequence of epochs at which a service completion occurs or a vacation period ends. The sequence of bivariate random variables $X_n = (C(t_n-), N(t_n+))$ forms a Markov chain, which is the embedded Markov chain of our queueing system. The state space of Markov chain is $S = \{1,2\} \times N$, where N= {1,2,3,...} and the transitions are given by

$$(i_{n+1}, j_{n+1}) = \begin{cases} (1, j_n + k - B_n), & j_n \geq 1,\ j_{n+1} \geq j_n - 1, \\ (1, \xi), & (i_n, j_n) = (0,0), \\ (2, \xi_v), & (i_n, j_n) = (1,0) \end{cases} \tag{3.1}$$

where ξ_v is the number of customers arriving during a vacation period, ξ is the number of arrivals that occurs during the service time which ends at time t_{n+1}, and $B_n = 1$ if the customer who leaves the system at time t_{n+1} is from orbit and $B_n = 0$ otherwise.

It can be shown from the results in Cinlar [3] that the limiting probabilities of $X(t) = (C(t), N(t))$ exist and are positive if the embedded Markov chain $\{X_n, n \in N\}$ is ergodic.

Theorem 3.1. *The Markov chain is ergodic if and only if*

$$\lambda\beta_1 - \mu(\lambda+\mu)^{-1} < 0 \tag{3.2}$$

where β_1 is the first moment of service time distribution.

Proof. It is easy to prove that the chain $\{X_n, n \in N\}$ is irreducible and aperiodic. To prove (3.2) we use Foster's criterion. Choose $f(i,j) = j$ for $(i,j) \in S$. Then

$$\begin{aligned} D_{ij} &= E\Big\{f(X_{n+1}) - f(X_n) | X_n = (i,j)\Big\} \\ &= \begin{cases} \lambda\beta_1 - \dfrac{\mu}{\lambda+\mu} & , \quad if\ j_n \geq 1 \\ \lambda\beta_1 & , \quad if\ (i_n, j_n) = (0,0) \\ E(\xi_v) & , \quad if\ j_n = 0. \end{cases} \end{aligned}$$

Thus, if condition (3.2) is fulfilled then Foster's criterion guarantees that $\{X_n, n \in N\}$ is ergodic. □

In the steady state region, define

$$\pi_{ij} = \lim_{n\to\infty} Pr\{X_n = (i,j) | (i,j) \in S\}, \quad i = 1,2 \text{ and } j \geq 0.$$

and,

$$\Pi_i(z) = \sum_{j=0}^{\infty} \pi_{ij} z^j, \qquad i = 1,2 \tag{3.3}$$

the generating function of $\{\pi_{ij} : i = 1,2 \textit{ and } j \geq 0\}$. The following theorem describes the stationary distribution of the embedded Markov chain.

Theorem 3.2. *The stationary distribution is obtained from (3.1)*

$$\Pi_2(z) = \pi_{10}X(z) and \quad (3.4)$$

$$\Pi_1(z) = \frac{\{\lambda(\lambda+\mu)z\pi_{00} + (\lambda z+\mu)[\pi_{10}(X(z)-1) - \pi_{20}]\}K(z)}{(\lambda+\mu)z - (\lambda z+\mu)K(z)} \quad (3.5)$$

where $c_0 = Pr\{\xi_v = 0\}, \pi_{00} = \frac{c_0}{\lambda}\pi_{10}$ *and* $\pi_{20} = \pi_{10}c_0$ *and*

$$\pi_{10} = \frac{1-\rho}{1-\rho+c_0+(\lambda+\mu)\mu^{-1}E(X)}$$

$$K(z) = \beta(\lambda-\lambda z), \quad X(z) = \sum_{j=0}^{\infty} P_r\{\xi_v = j\}z^j$$

Proof. From (3.1) the stationary probabilities satisfies the following difference equations.

$$\pi_{2j} = \pi_{10}Pr\{\xi_v = j\} \quad (3.6)$$

$$\pi_{1j} = \lambda\pi_{00}k_j + \frac{\lambda}{\lambda+\mu}\sum_{k=1}^{j}(\pi_{1k}+\pi_{2k})k_{j-k} + \frac{\mu}{\lambda+\mu}\sum_{k=1}^{j+1}(\pi_{1k}+\pi_{2k})k_{j-k+1} \quad (3.7)$$

Using (3.3), we can obtain (3.4) and (3.5). □

Define $\Pi(z) = \Pi_1(z) + \Pi_2(z)$

$$\Pi(z) = \frac{(\lambda+\mu)z[c_0\pi_{10}K(z) + \pi_{10}X(z)] - (\lambda z+\mu)(1+c_0)\pi_{10}K(z)}{(\lambda+\mu)z - (\lambda z+\mu)K(z)} \quad (3.8)$$

as $z \to 1$, $\Pi(z) \to 1$.

From (3.8) we get $\pi_{10} = \dfrac{1-\rho}{1-\rho+c_0+(\lambda+\mu)\mu^{-1}E(X)}$ where $c_0 = P_r\{\xi_v = 0\}$, $\pi_{00} = \frac{c_0}{\lambda}\pi_{10}$, and $\pi_{20} = \pi_{10}c_0$.

3.3. Operating characteristics

By a straight forward calculation on (3.8) we obtain the following operating characteristics. The first moment of N is

$$\frac{\pi_{10}}{2D^2}\{D[AB_2 + 2AB_1 - (1+c_0)D_2] + [A(1+c_0) + AB_1 - (1+c_0)D_1]D_2\}.$$

The second moment of N is

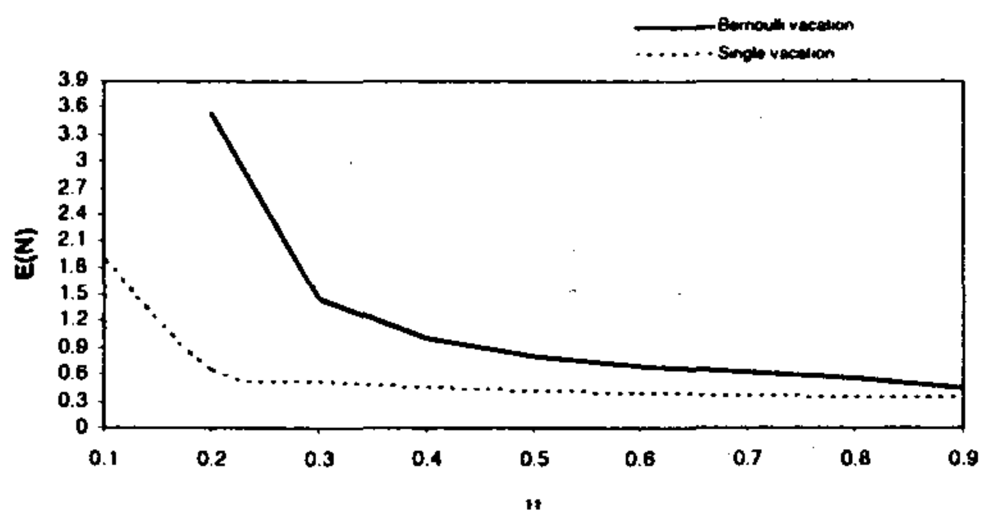
$$\frac{\pi_{10}}{6D^3}\{2D^2[AB_3 + 3AB_2 - (1+c_0)D_3] + 3D[AB_2 + 2AB_1 - (1+c_0)D_2]D_2 + 3[A(1+c_0) + AB_1 - (1+c_0)D_1][D_2]^2 + 2D[A(1+c_0) + AB_1 - (1+c_0)D_1]D_3\}.$$

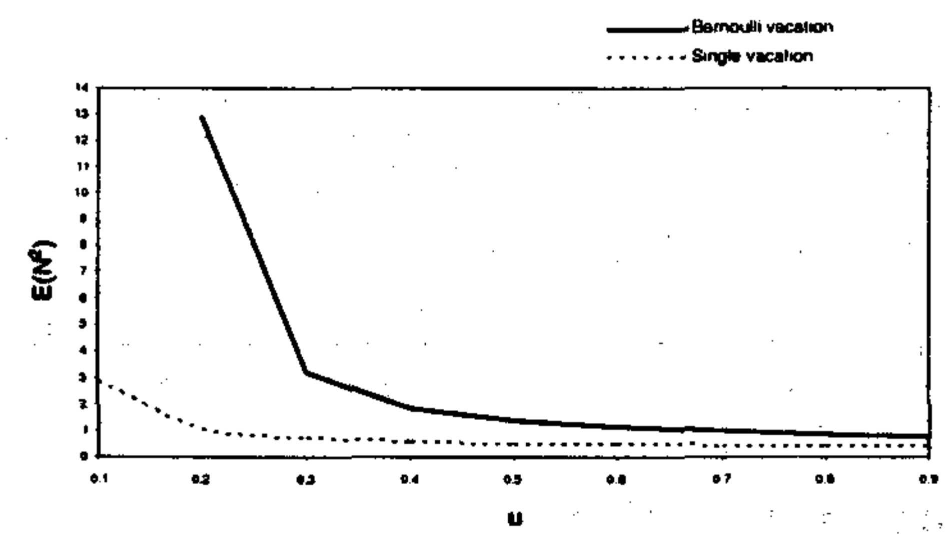
Using Little's formula $L = \lambda W$, we can find that W, the average waiting time in the system is given by $E[W] = \frac{\pi_{10}}{2\lambda D^2}\{D[AB_2 + 2AB_1 - (1+c_0)D_2] + [A(1+$

$c_0) + AB_1 - (1 + c_0)D_1]D_2\}$ and the second moment of the waiting time in the system is given by

$$\begin{aligned} E[W^2] &= \frac{\pi_{10}}{6\lambda^2 D^3}\Big\{2D^2[AB_3 + 3AB_2 - (1 + c_0)D_3] \\ &\quad +3D[AB_2 + 2AB_1 - (1 + c_0)D_2]D_2 \\ &\quad +3[A(1 + c_0) + AB_1 - (1 + c_0)D_1][D_2]^2 \\ &\quad +2D[A(1 + c_0) + AB_1 - (1 + c_0)D_1]D_3\Big\} \end{aligned}$$

where $A = \lambda + \mu$, $D = \mu(1 - \rho)$, $D_1 = AK^{'}(1) + \lambda$, $D_2 = AK^{''}(1) + 2\lambda K^{'}(1)$, $D_3 = AK^{'''}(1) + 3\lambda K^{''}(1)$, $B_1 = c_0 K^{'}(1) + X^{'}(1)$, $B_2 = c_0 K^{''}(1) + X^{''}(1)$, $B_3 = c_0 K^{'''}(1) + X^{'''}(1)$, $K^{'}(1) = \lambda\beta_1$, $K^{''}(1) = \lambda^2\beta_2$, $K^{'''}(1) = \lambda^3\beta_3$, $X^{'}(1) = E(X)$, $X^{''}(1) = E(X^2) - E(X)$, $X^{'''}(1) = E(X^3) - 3E(X^2) + 2E(X)$, $c_0 = P_r\{\xi_v = 0\}$, $\pi_{10} = \dfrac{1-\rho}{1 - \rho + c_0 + A\mu^{-1}E(X)}$, and $\rho = \mu^{-1}(\lambda + \mu)\lambda\beta_1$.





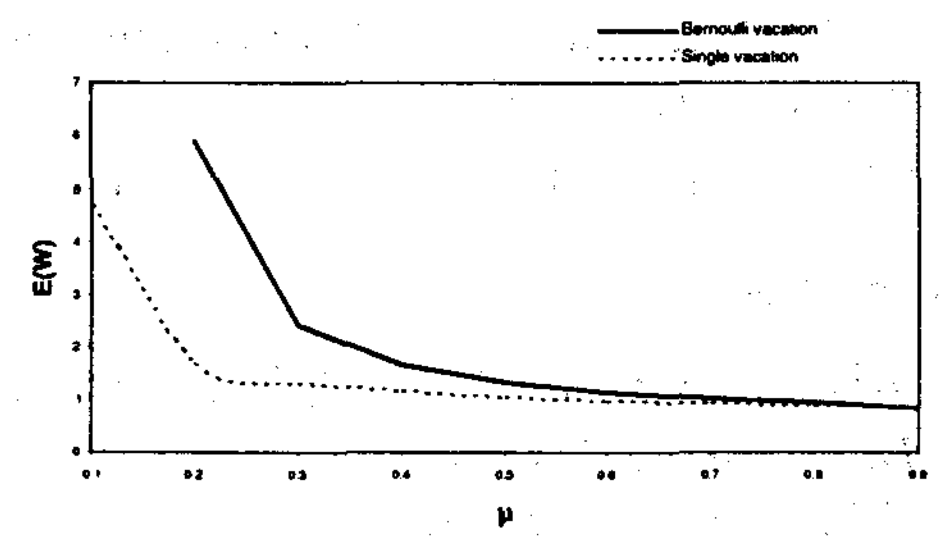


Figure 3. μ versus E(W)

4. Numerical study

In this section, we numerically analyze the first and second moments of orbit length and waiting time for the models studied in section 2 and 3. For particular values of the system parameters, the results are shown in Figure 1-12.

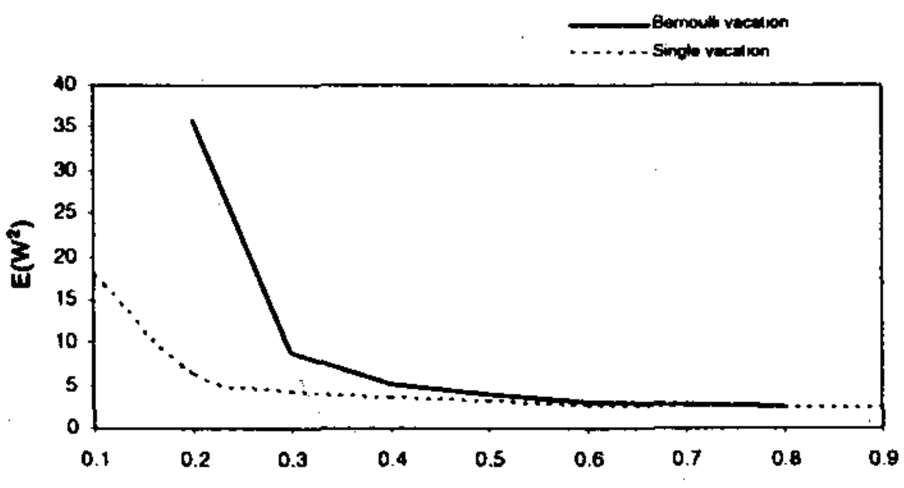


Figure 4. μ versus $E(W^2)$

Figure 1-4 shows μ versus the moments studied in section 2 and 3. Our computation experience shows that as μ increases values of the moments decreases uniformly.

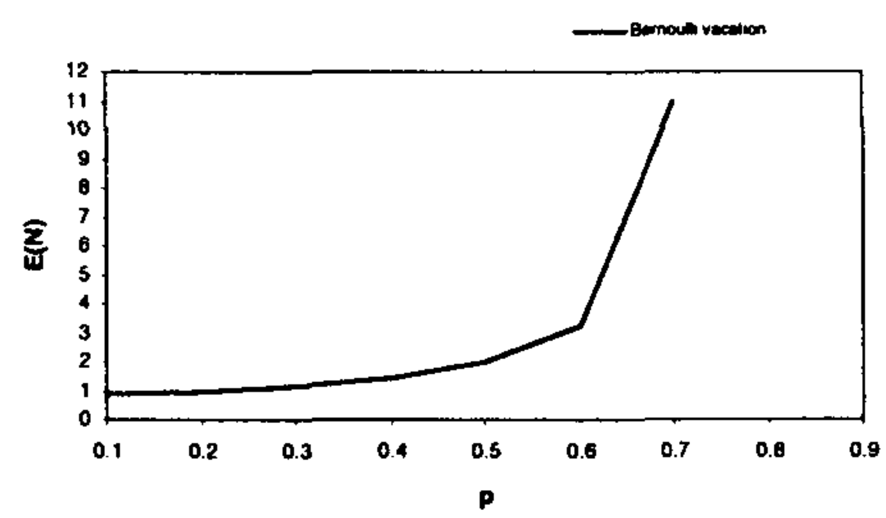


Figure 5. p versus E(N)

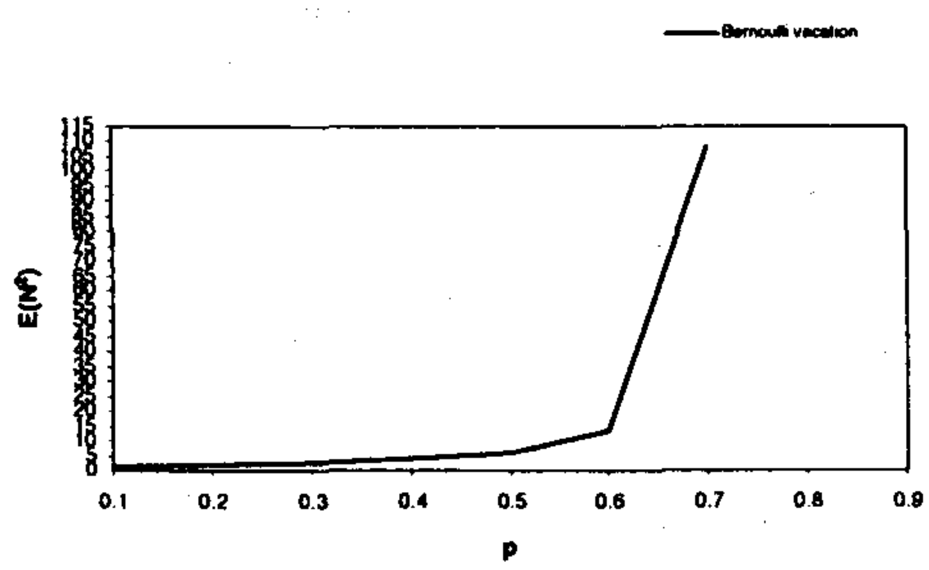


Figure 6. p versus $E(N^2)$

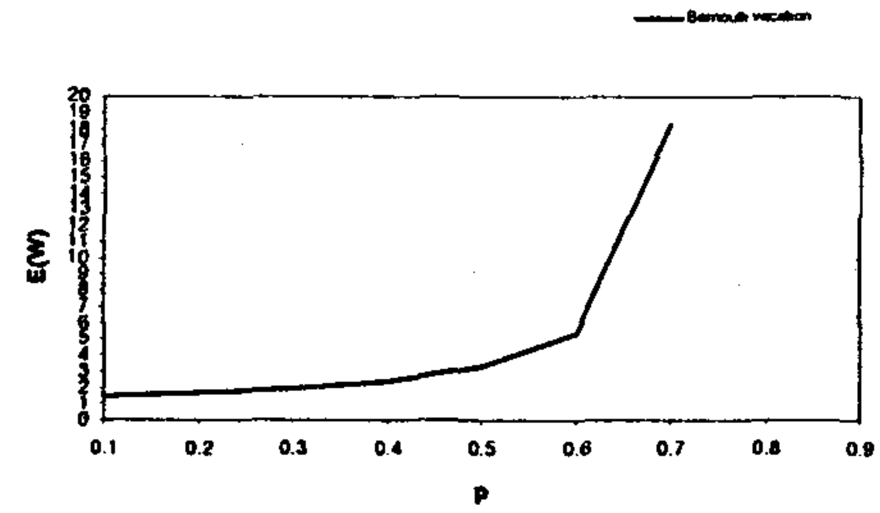


Figure 7. p versus E(W)

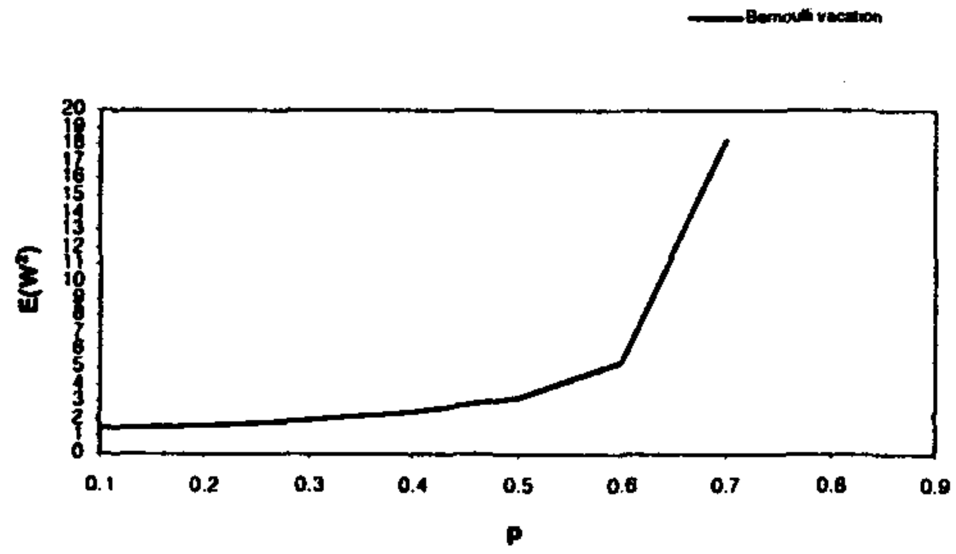


Figure 8. p versus E(W^2)

Figure 5-8 shows p versus moments studied in section 2. From the figures we conclude that as p increases the moments also increases.

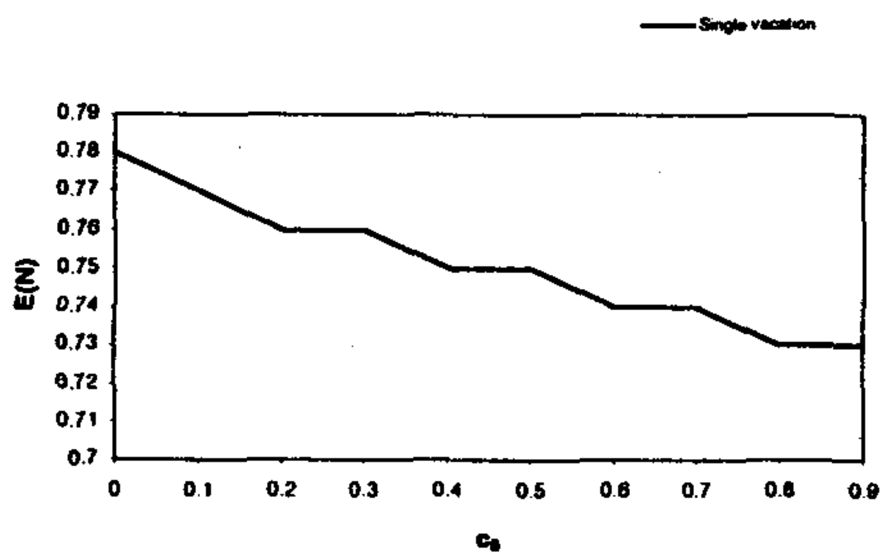


Figure 9. c_0 versus E(N)

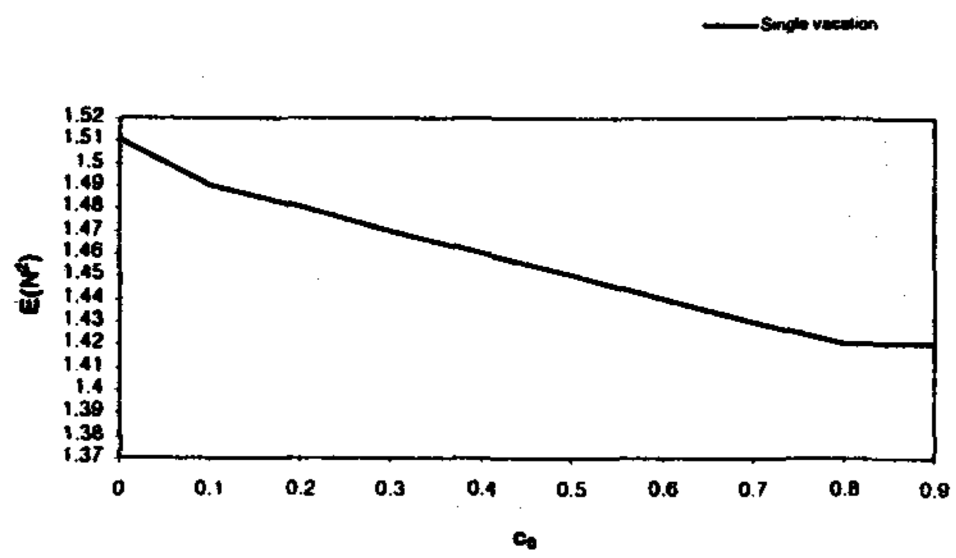
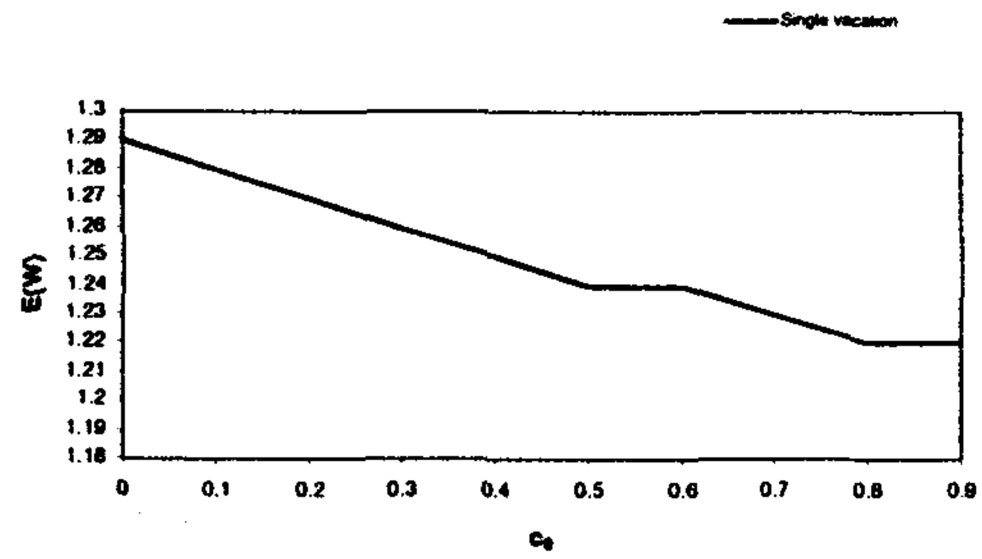


Figure 10. c_0 versus E(N^2)

Figure 9-12 shows c_0 versus moments studied in section 3. From the figures we conclude that as c_0 increases the moments decreases. The two models analyzed in section 2 and 3 differ from each other only by the vacation policy. From Figures 1-4 we conclude that the operating characteristics of model II is always smaller than the operating characteristics of model I.



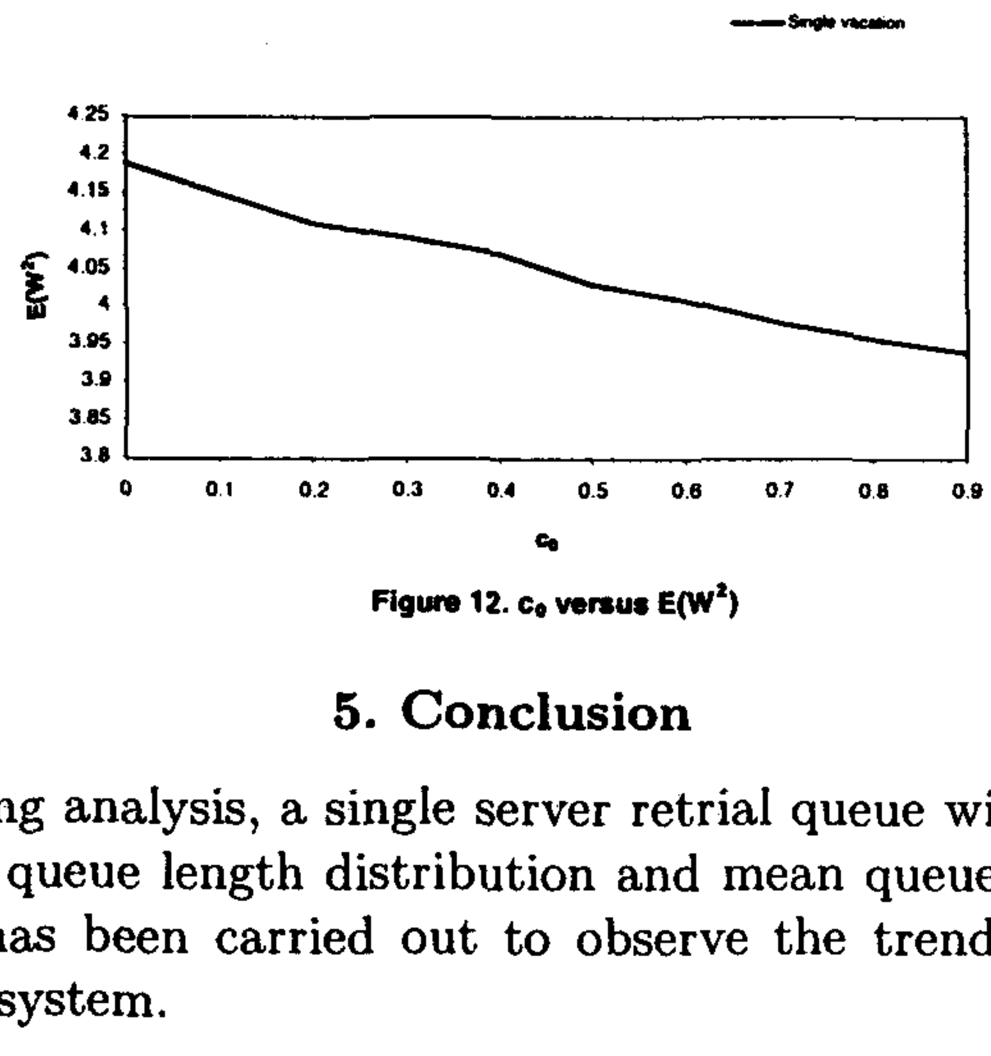


Figure 12. c_0 versus $E(W^2)$

5. Conclusion

In the fore going analysis, a single server retrial queue with vacation is considered to obtain queue length distribution and mean queue length. Extensive numerical work has been carried out to observe the trends of the operating characters of the system.

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