

## A FIFTH ORDER NUMERICAL METHOD FOR SINGULAR PERTURBATION PROBLEMS

P. PRAMOD CHAKRAVARTHY\*, K. PHANEENDRA AND Y.N. REDDY

**ABSTRACT.** In this paper, a fifth order numerical method is presented for solving singularly perturbed two point boundary value problems with a boundary layer at one end point. The two point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a fifth order compact difference scheme is presented for the first order system. An asymptotically equivalent first order equation of the original singularly perturbed two point boundary value problem is obtained from the theory of singular perturbations. It is used in the fifth order compact difference scheme to get a two term recurrence relation and is solved. Several linear and non-linear singular perturbation problems have been solved and the numerical results are presented to support the theory. It is observed that the present method approximates the exact solution very well.

AMS Mathematics Subject Classification : 65L10

*Key words and phrases :* Singular perturbation problems, two-point boundary value problems, ordinary differential equations, boundary layer, fifth order compact difference scheme.

### 1. Introduction

Singular perturbation problems arise very frequently in fluid mechanics, fluid dynamics, elasticity, aero dynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography and other domains of the great world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds numbers, convective heat transport problems with large Peclet numbers, magneto-hydrodynamics duct problems at high Hartman numbers, etc. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. So, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the

---

Received July 7, 2007. \*Corresponding author.  
© 2008 Korean SIGCAM and KSCAM .

solution behaves regularly and varies slowly. If we apply the existing standard numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior. Thus more efficient and simpler computational techniques are required to solve singularly perturbed two-point boundary value problems.

A wide verity of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these, we mention Bender [1], Kevorkian & Cole [5], Nayfeh [6], O' Malley [7], Hemker and Miller [3], Robert [8], Kadalbajoo & Reddy [4]. Higher order Numerical method for two point boundary value problems was presented by Dianyun Peng [2]

In this paper, a fifth order numerical method is presented for solving singularly perturbed two point boundary value problems with a boundary layer at one end point. The two point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a fifth order compact difference scheme is presented for the first order system. An asymptotically equivalent first order equation of the original singularly perturbed two point boundary value problem is obtained from the theory of singular perturbations. It is used in the fifth order compact difference scheme to get a two term recurrence relation and is solved. Several linear and non-linear singular perturbation problems have been solved and the numerical results are presented to support the theory.

## 2. Numerical method

We consider the second order linear differential equation

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \quad (1)$$

with the boundary conditions  $y(0) = \alpha$  and  $y(1) = \beta$

The first order linear system corresponding to the above *BVP* is

$$Y' = A(x)Y + R(x), x \in [a, b] \quad (2)$$

with the boundary conditions  $B_1 Y(0) + B_2 Y(1) = D$ , where  $A, B_1$  and  $B_2$  are  $2 \times 2$  matrices.  $Y, R, D$  are two dimensional vectors.

Now we divide the interval  $[0, 1]$  into  $N$  equal parts with constant mesh length  $H$ . Let  $0 = x_0, x_1, x_2, \dots, x_N = 1$  be the mesh points. Again we divide each subinterval  $[x_i, x_{i+1}]$  into four equal smaller sub intervals. Let  $t_1, t_2, \dots, t_5$  are the grids in the subinterval  $[x_i, x_{i+1}]$  and corresponding values of the variables and its derivatives are  $Y_1, Y_2, Y_3, Y_4, Y_5$  and  $Y'_1, Y'_2, Y'_3, Y'_4, Y'_5$ .

By considering Taylor's expansions of  $Y_1, Y_2, Y_3, Y_4, Y_5$  at the fractional grid  $t_3$  (Ref. Dianyun Peng [2]), we have

$$\frac{h^{n+1}}{(n+1)} Y_3^{(n+1)} = \sum_{j=1}^5 a_j^n Y_j + a_6^n Y'_3 + O(h^6 Y_3^{(6)}), n = 1, 2, 3, 4 \quad (3)$$

where  $h = \frac{x_{i+1} - x_i}{4}$  and the coefficients  $a_j^n$  are given by:

$$\begin{aligned} a_1^1 &= \frac{-1}{24}, a_2^1 = \frac{2}{3}, a_3^1 = \frac{-5}{4}, a_4^1 = \frac{2}{3}, a_5^1 = \frac{-1}{24}, a_6^1 = 0 \\ a_1^2 &= \frac{1}{48}, a_2^2 = \frac{-2}{3}, a_3^2 = 0, a_4^2 = \frac{2}{3}, a_5^2 = \frac{-1}{48}, a_6^2 = \frac{-5}{4} \\ a_1^3 &= \frac{1}{24}, a_2^3 = \frac{-1}{6}, a_3^3 = \frac{1}{4}, a_4^3 = \frac{-1}{6}, a_5^3 = \frac{1}{24}, a_6^3 = 0 \\ a_1^4 &= \frac{-1}{48}, a_2^4 = \frac{1}{6}, a_3^4 = 0, a_4^4 = \frac{-1}{6}, a_5^4 = \frac{1}{48}, a_6^4 = \frac{1}{4} \end{aligned} \quad (4)$$

By taking the Taylor's series expansions of  $Y'_1, Y'_2, Y'_3, Y'_4, Y'_5$  at the grid point  $t_3$  and substituting (3), we get

$$Y'_k = \frac{1}{h} \sum_{j=1}^5 b_j^k Y_j + b_6^k Y'_3 + O(h^5 Y_3^{(6)}) \text{ for } k = 1, 2, 4, 5 \quad (5)$$

$$\text{where } b_j^1 = -4a_j^1 + 12a_j^2 - 32a_j^3 + 80a_j^4 + Sgn(j-6)$$

$$b_j^2 = -2a_j^1 + 3a_j^2 - 4a_j^3 + 5a_j^4 + Sgn(j-6)$$

$$b_j^4 = 2a_j^1 + 3a_j^2 + 4a_j^3 + 5a_j^4 + Sgn(j-6)$$

$$b_j^5 = 4a_j^1 + 12a_j^2 + 32a_j^3 + 80a_j^4 + Sgn(j-6)$$

$$Sgn(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

The variable  $Y$  and its derivative  $Y'$  at grids  $t_1, t_2, \dots, t_5$  subject to equations

$$Y'_j = A_j Y_j + R_j, j = 1, 2, 3, 4, 5 \quad (6)$$

where  $A_j$  and  $R_j$  are values of  $A$  and  $R$  at grids  $t_j$ . Substituting (6) in (5), we get six linear algebraic equations with respect to five unknown variables  $Y_1, Y_2, Y_3, Y_4, Y_5$ .

By eliminating  $Y_2, Y_3, Y_4$  from the above equations a relation between  $Y_1$  and  $Y_5$  can be obtained as follows:

$$\frac{1}{h} S_i Y_i + \frac{1}{h} T_i Y_{i+1} = F_i \text{ for } i = 1, 2, \dots, N-1 \quad (7)$$

where  $S_i$  and  $T_i$  are  $2 \times 2$  matrices and  $F_i$  is a two dimensional vector. By assuming

$$c_1 = b_2^1 b_4^5 - b_2^5 b_4^1$$

$$W_1 = ((b_5^5 b_4^1 - b_5^1 b_4^5)I - h b_4^1 A_5)/c_1,$$

$$W_2 = ((b_3^5 b_4^1 - b_3^1 b_4^5)I + h(b_6^5 b_4^1 - b_6^1 b_4^5)A_3)/c_1,$$

$$W_3 = ((b_1^5 b_4^1 - b_4^5 b_1^1)I + h b_4^5 A_1)/c_1,$$

$$\begin{aligned}
G_1 &= (b_4^5 R_1 - b_4^1 R_5 + (b_6^5 b_4^1 - b_6^1 b_4^5) R_3) / c_1, \\
W_4 &= ((b_1^5 b_2^1 - b_1^1 b_2^5) I + h b_2^5 A_1) / c_2, \\
W_5 &= ((b_2^1 b_3^5 - b_2^5 b_3^1) I + h (b_6^5 b_2^1 - b_6^1 b_2^5) A_3) / c_2, \\
W_6 &= ((b_2^1 b_5^5 - b_2^5 b_5^1) I - h b_2^1 A_5) / c_2, \\
G_2 &= (b_2^5 R_1 - b_2^1 R_5 + (b_6^5 b_2^1 - b_6^1 b_2^5) R_3) / c_2, \\
W_7 &= b_1^2 I + (b_2^2 - h A_2) W_3 + b_4^2 W_4, \\
W_8 &= b_3^2 I + b_4^2 W_5 + h b_6^2 A_3 + (b_2^2 I - h A_2) W_2, \\
W_9 &= b_5^2 I + b_4^2 W_6 + (b_2^2 I - h A_2) W_1, \\
G_3 &= R_2 - b_6^2 R_3 - (b_2^2 I - h A_2) G_1 - b_4^2 G_2, \\
W_{10} &= b_2^4 W_3 + (b_4^4 - h A_4) W_4 + b_1^4 I, \\
W_{11} &= b_3^4 I + b_2^4 W_2 + h b_6^4 A_3 + (b_4^4 - h A_4) W_5, \\
W_{12} &= b_5^4 I + b_2^4 W_1 + (b_4^4 - h A_4) W_6, \\
G_4 &= R_4 - b_6^4 R_3 - b_2^4 G_1 - (b_4^4 - h A_4) G_2
\end{aligned}$$

We get  $S_i = W_{11}W_7 - W_8W_{10}$ ,  $T_i = W_{11}W_9 - W_8W_{12}$ ,  $F_i = G_3W_{11} - W_8G_4$ . Now we consider a linear singularly perturbed two-point boundary value problem of the form:

$$\varepsilon y''(x) + [a(x)y(x)]' + b(x)y(x) = f(x), \quad x \in [0, 1] \quad (8)$$

with

$$y(0) = \alpha, \quad (9a)$$

and

$$y(1) = \beta \quad (9b)$$

where  $\varepsilon$  is a small positive parameter ( $0 < \varepsilon \ll 1$ ) and  $\alpha, \beta$  are known constants. We assume that  $a(x), b(x)$  and  $f(x)$  are sufficiently continuously differentiable functions in  $[0, 1]$ . Further more, we assume that  $a(x) \geq M > 0$  throughout the interval  $[0, 1]$ , where  $M$  is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of  $x = 0$ .

First we obtain the reduced problem by setting  $\varepsilon = 0$  in equation (8) and solve it for the solution with an appropriate boundary condition. Let  $y_0(x)$  be the solution of the reduced problem

$$[a(x)y_0(x)]' + b(x)y_0(x) = f(x) \text{ with } y_0(1) = \beta.$$

We now set up the approximation equation to given equation (8) as follows:

$$\varepsilon y''(x) + [a(x)y(x)]' + b(x)y_0(x) = f(x) \quad (10)$$

where we simply replaced  $y(x)$  by  $y_0(x)$  in the last term of left hand side of the equation. Now we rewrite the equation (10) in the form

$$\varepsilon y''(x) + [a(x)y(x)]' = H(x) \quad (11)$$

where  $H(x) = f(x) - b(x)y_0(x)$ .

By integrating (11), we obtain

$$\varepsilon y'(x) + a(x)y(x) = P(x) + K \quad (12)$$

where  $P(x) = \int H(x)dx$  and  $K$  is a constant to be determined.

In order to determine  $K$ , we introduce the condition that the reduced equation of (12) should satisfy the boundary condition  $y(1) = \beta$ .

$$\begin{aligned} \text{i.e., } y(1) &= \frac{1}{a(1)} [P(1) + K] = \beta \\ \therefore K &= a(1)\beta - P(1) \end{aligned} \quad (13)$$

**Remark:** This choice of  $K$  ensure that the solution of the reduced equation of (8)-(9) satisfies the reduced equation of (12). Hence, the equation (13) is a first order equation which is asymptotically equivalent to the linear singularly perturbed two-point boundary value problem (8).

It is used in the fifth order compact difference scheme (7) to get a two term recurrence relation.

The system (7) i.e.,  $\frac{1}{h}S_i Y_i + \frac{1}{h}T_i Y_{i+1} = F_i$  gives the following equations

$$S_{11}y_i + S_{12}y'_i + T_{11}y_{i+1} + T_{12}y'_{i+1} = hf_1 \quad (14)$$

$$S_{21}y_i + S_{22}y'_i + T_{21}y_{i+1} + T_{22}y'_{i+1} = hf_2 \quad (15)$$

By eliminating  $y'_{i+1}$  from the above equations we have,

$$\begin{aligned} (S_{11}T_{22} - T_{12}S_{21})y_i + (S_{12}T_{22} - T_{12}S_{22})y'_i + \\ (T_{11}T_{22} - T_{12}T_{21})y_{i+1} = h(T_{22}f_1 - T_{12}f_2) \end{aligned} \quad (16)$$

From the asymptotic boundary condition (12) we have

$$y'(x) = \frac{1}{\varepsilon}(P(x) + K - a(x)y(x))$$

Substituting this in (16) and simplifying, we get a two term recurrence relation as follows:

$$y_{i+1} = \frac{1}{\varepsilon(T_{22}T_{11} - T_{12}T_{21})} \begin{pmatrix} h\varepsilon(T_{22}f_1 - T_{12}f_2) - (P_i + K) \\ (T_{22}S_{12} - S_{22}T_{12}) - (\varepsilon(T_{22}S_{11} - S_{21}T_{12})) \\ - a_i(T_{22}S_{12} - T_{12}S_{22}))y_i \end{pmatrix} \quad (17)$$

The condition  $y_0 = \alpha$  is used to solve the above two term recurrence relation in Forward process.

### 3. Numerical examples

To demonstrate the applicability of the method we have applied it to three linear singular perturbation problems with left-end boundary layer. These examples have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The approximate solution is compared with the exact solution.

**Example 3.1.** Consider the following homogeneous singular perturbation problem from Bender and Orszag [[1], page 480; problem 9.17 with  $\alpha = 0$ ]

$$\varepsilon y''(x) + y'(x) - y(x) = 0; x \in [0, 1]$$

with  $y(0) = 1$  and  $y(1) = 1$ .

The exact solution is given by  $y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{(e^{m_2} - e^{m_1})}$   
Where  $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon)$  and  $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/(2\varepsilon)$

The numerical results are given in tables 1(a), 1(b) for  $\varepsilon = 10^{-3}$  and  $10^{-4}$  respectively.

**Example 3.2.** Now consider the following non-homogeneous singular perturbation problem

$$\varepsilon y''(x) + y'(x) = 1 + 2x; x \in [0, 1] \text{ with } y(0) = 0 \text{ and } y(1) = 1.$$

The exact solution is given by  $y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{(1 - e^{-1/\varepsilon})}$

The numerical results are given in tables 2(a), 2(b) for  $\varepsilon = 10^{-3}$  and  $10^{-4}$  respectively.

**Example 3.3.** Finally we consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [[5], page 33; equations 2.3.26 and 2.3.27 with  $\alpha = -1/2$ ]

$$\varepsilon y''(x) + (1 - \frac{x}{2})y'(x) - \frac{1}{2}y(x) = 0; x \in [0, 1]$$

with  $y(0) = 0$  and  $y(1) = 1$ .

We have chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [[6], page 148; equation 4.2.32] as our 'exact' solution: with  $y(0) = 0$  and  $y(1) = 1$ .

$$y(x) = \frac{1}{2-x} - \frac{1}{2}e^{-(x-x^2/4)/\varepsilon}$$

The numerical results are given in tables 3(a), 3(b) for  $\varepsilon = 10^{-3}$  and  $10^{-4}$  respectively.

#### 4. Nonlinear problems

Nonlinear singular perturbation problems were converted as a sequence of linear singular perturbation problems by using Quasilinearization method. The outer solution is taken to be the initial approximation. The approximate solution is compared with the exact solution. To demonstrate the applicability of the method, we have applied it on three nonlinear singular perturbation problems with left-end boundary layer.

**Example 4.1.** Consider the following singular perturbation problem from Bender and Orszag [1], page 463; equations: 9.7.1]

$$\varepsilon y''(x) + 2y'(x) + e^{y(x)} = 0; x \in [0, 1]$$

with  $y(0) = 0$  and  $y(1) = 0$ .

The linear problem concerned to this example is

$$\varepsilon y''(x) + 2y'(x) + \frac{2}{x+1}y(x) = \left(\frac{2}{x+1}\right) \left[\log_e\left(\frac{2}{x+1}\right) - 1\right]$$

We have chosen to use Bender and Orszag's uniformly valid approximation [1], page 463; equation: 9.7.6] for comparison,

$$y(x) = \log_e\left(\frac{2}{x+1}\right) - (\log_e 2)e^{-2x/\varepsilon}$$

For this example, we have boundary layer of thickness  $O(\varepsilon)$  at  $x = 0$ . [cf. Bender and Orszag [1]].

The numerical results are given in table 4(a), 4(b) for  $\varepsilon = 10^{-3}$  and  $10^{-4}$  respectively.

**Example 4.2.** Now consider the following singular perturbation problem from Kevorkian and Cole [5], page 56; equation 2.5.1]

$$\varepsilon y''(x) + y(x)y'(x) - y(x) = 0; x \in [0, 1]$$

with  $y(0) = -1$  and  $y(1) = 3.9995$  The linear problem concerned to this example is

$$\varepsilon y''(x) + (x + 2.9995)y'(x) = x + 2.9995$$

We have chosen to use the Kevorkian and Cole's uniformly valid approximation [5], pages 57 and 58; equations (2.5.5), (2.5.11) and (2.5.14)] for comparison,

$$y(x) = x + c_1 \tanh\left(\left(\frac{c_1}{2}\right)\left(\frac{x}{\varepsilon} + c_2\right)\right)$$

Where  $c_1 = 2.9995$  and  $c_2 = (1/c_1)\log_e[(c_1 - 1)/(c_1 + 1)]$ . For this example also we have a boundary layer of width  $O(\varepsilon)$  at  $x = 0$  [cf. Kevorkian and Cole [5], pages 56-66].

The numerical results are given in table 5(a), 5(b) for  $\varepsilon = 10^{-3}$  and  $10^{-4}$  respectively.

**Example 4.3.** Finally we consider the following singular perturbation problem from O'Malley [7], page 9; equation (1.10) case 2]:

$$\varepsilon y''(x) - y(x)y'(x) = 0; x \in [-1, 1]$$

With  $y(-1) = 0$  and  $y(1) = -1$ .

The linear problem concerned to this example is

$$\varepsilon y''(x) + y'(x) = 0$$

We have chosen to use O'Malley's approximate solution [[7], pages 9 and 10; equations 1.13 and 1.14] for comparison,

$$y(x) = -\frac{(1 - e^{-(x+1)/\epsilon})}{(1 + e^{-(x+1)/\epsilon})}$$

For this example, we have a boundary layer of width  $O(\epsilon)$  at  $x = -1$  [cf. O'Malley [7], pages 9 and 10, eqs. (1.10), (1.13), (1.14), case 2].

The numerical results are given in table 6(a), 6(b) for  $\epsilon = 10^{-3}$  and  $10^{-4}$  respectively.

### 5. Right-end boundary layer problems

Finally, we discuss our method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form:

$$\epsilon y''(x) + [a(x)y(x)]' + b(x)y(x) = f(x), x \in [0, 1] \quad (18)$$

$$\text{with } y(0) = \alpha \quad (19a)$$

$$\text{and } y(1) = \beta \quad (19b)$$

where  $\epsilon$  is a small positive parameter ( $0 < \epsilon < 1$ ) and  $\alpha, \beta$  are known constants. We assume that  $a(x)$ ,  $b(x)$  and  $f(x)$  are sufficiently continuously differentiable functions in  $[0, 1]$ . Further more, we assume that  $a(x) \leq M < 0$  throughout the interval  $[0, 1]$ , where  $M$  is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of  $x = 1$ .

From the asymptotic boundary condition we have

$$\epsilon y'(x) + a(x)y(x) = P(x) + k$$

$$\text{where } P(x) = \int H(x)dx \quad k = a(0)\beta - P(0)$$

$$\text{and } H(x) = f(x) - b(x)y_0(x)$$

Substituting this in (16) and simplifying, we get the two term recurrence relation as follows:

$$y_i = \frac{h\epsilon(T_{22}f_1 - T_{12}f_2) - (P_i + k)(T_{22}S_{12} - S_{22}T_{12}) - \epsilon(T_{22}T_{11} - T_{21}T_{12})y_{i+1}}{\epsilon(S_{11}T_{22} - S_{21}T_{12}) - a_i(S_{12}T_{22} - S_{22}T_{12})} \quad (20)$$

The condition  $y_n = \beta$  is used to solve the above two term recurrence relation in backward process.

### 6. Examples with right-end boundary layer

To illustrate the method for singularly perturbed two point boundary value problems with right-end boundary layer of the underlying interval we considered two examples. The approximate solution is compared with the exact solution.

Table 1(a) Numerical Results of Example 3.1,  $\varepsilon = 10^{-3}$ ,  $h = 10^{-3}$ 

X	Approximate solution	Exact solution
0	1.00000000000000e+000	1.00000000000000e+000
1.00000000000000e-003	6.009594491773668e-001	6.00791797118448e-001
1.00000000000000e-002	3.716055665865143e-001	3.719723959330329e-001
1.00000000000000e-001	4.065696264467130e-001	4.069350064869349e-001
2.00000000000000e-001	4.493289273217870e-001	4.496878534047455e-001
3.00000000000000e-001	4.965852631261653e-001	4.969323412245205e-001
4.00000000000000e-001	5.488115911519812e-001	5.491403645555466e-001
5.00000000000000e-001	6.065306100439920e-001	6.068333955506272e-001
6.00000000000000e-001	6.703199911433010e-001	6.705876925538863e-001
7.00000000000000e-001	7.408181600163022e-001	7.410400559723790e-001
8.00000000000000e-001	8.187306860323287e-001	8.18894188631804e-001
9.00000000000000e-001	9.048373439390537e-001	9.049277257677435e-001
1.00000000000000e+000	9.999999181102541e-001	1.00000000000000e+000

Least square error = 8.948352801616218e-003,  
Maximum error = 3.673097848881368e-004

Table 1(b) Numerical Results of Example 3.1,  $\varepsilon = 10^{-4}$ ,  $h = 10^{-3}$ 

X	Approximate Solution	Exact solution
0	1.00000000000000e+000	1.00000000000000e+000
1.00000000000000e-003	3.975707066749633e-001	3.683129549003972e-001
1.00000000000000e-002	3.719112271849666e-001	3.716134715795036e-001
2.00000000000000e-002	3.756489971586271e-001	3.753478737865929e-001
1.00000000000000e-001	4.069357006070080e-001	4.066062453397311e-001
2.00000000000000e-001	4.497335018376038e-001	4.493649046843792e-001
3.00000000000000e-001	4.970323871152407e-001	4.966200590285670e-001
4.00000000000000e-001	5.493057395814812e-001	5.488445591957479e-001
5.00000000000000e-001	6.070767285174882e-001	6.065609809398519e-001
6.00000000000000e-001	6.709235453980331e-001	6.703468540124427e-001
7.00000000000000e-001	7.414851906261128e-001	7.408404411178888e-001
8.00000000000000e-001	8.194678688637576e-001	8.187471245827021e-001
9.00000000000000e-001	9.056520569656534e-001	9.048464646461863e-001
1.00000000000000e+000	1.000900315253830e+000	1.00000000000000e+000

Least square error = 3.440052272231959e-002  
Maximum error = 2.925775177456608e-002

**Example 6.1.** Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0; x \in [0, 1]$$

with  $y(0) = 1$  and  $y(1) = 0$ .

Clearly, this problem has a boundary layer at  $x = 1$ . i.e.; at the right end of the underlying interval.

The exact solution is given by  $y(x) = \frac{(e^{(x-1)/\varepsilon} - 1)}{(e^{-1/\varepsilon} - 1)}$ . The numerical results are given in tables 7(a), 7(b) for  $\varepsilon = 10^3$  and  $10^{-4}$  respectively.

Table 2(a) Numerical Results of Example 3.2,  $\varepsilon = 10^{-3}, h = 10^{-3}$ 

x	Approximate Solution	Exact solution
0	0	0
1.000000000000000e-003	-6.314863373952746e-001	-6.298573177109006e-001
1.000000000000000e-002	-9.904464958614266e-001	-9.878746908700971e-001
2.000000000000000e-002	-9.802035611420511e-001	-9.776399979429686e-001
1.000000000000000e-001	-8.906966799558036e-001	-8.882000000000000e-001
2.000000000000000e-001	-7.608130758948899e-001	-7.584000000000000e-001
3.000000000000000e-001	-6.109294718339762e-001	-6.086000000000000e-001
4.000000000000000e-001	-4.410458677730626e-001	-4.388000000000000e-001
5.000000000000000e-001	-2.511622637121488e-001	-2.490000000000000e-001
6.000000000000000e-001	-4.127865965123527e-002	-3.920000000000001e-002
7.000000000000000e-001	1.886049444096786e-001	1.906000000000003e-001
8.000000000000000e-001	4.384885484705924e-001	4.404000000000001e-001
9.000000000000000e-001	7.083721525315059e-001	7.101999999999999e-001
1.000000000000000e+000	9.982557565924196e-001	1.000000000000000e+000

Least square error = 6.873942060228279e-002

Maximum error = 2.572721386393861e-003

**Example 6.2.** Now we consider the following singular perturbation problem  $\varepsilon y''(x) - y'(x) - (1 + \varepsilon)y(x) = 0; x \in [0, 1]$  with  $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$ ; and  $y(1) = 1 + 1/e$ .

Clearly this problem has a boundary layer at  $x = 1$ . The exact solution is given by

$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$$

The numerical results are given in table 8(a), 8(b) for  $\varepsilon = 10^{-3}$  and  $10^{-4}$  respectively.

## 7. Discussion and conclusions

A fifth order numerical method is presented for solving singularly perturbed two point boundary value problems with a boundary layer at one end point. A two term recurrence relation is obtained and is solved. The advantage of this two term recurrence relation is it can be solvable by forward or backward process. There is no need of applying any analytical or numerical method to solve the system of equations. Several linear and non-linear singular perturbation problems have been solved and the numerical results are presented to support the theory. Least square error and Maximum error are mentioned. It is observed that the present method approximates the exact solution very well.

Table 2(b) Numerical Results of Example 3.2,  $\varepsilon = 10^{-4}, h = 10^{-3}$

X	Approximate Solution	Exact solution
0	0	0
1.000000000000000e-003	-9.532636737169055e-001	-9.987538091502235e-001
1.000000000000000e-002	-9.899481731124386e-001	-9.897020000000000e-001
1.000000000000000e-001	-8.900568089506290e-001	-8.898200000000001e-001
2.000000000000000e-001	-7.600664043263516e-001	-7.598400000000001e-001
3.000000000000000e-001	-6.100759997020741e-001	-6.098600000000001e-001
4.000000000000000e-001	-4.400855950777966e-001	-4.398800000000001e-001
5.000000000000000e-001	-2.500951904535192e-001	-2.499000000000000e-001
6.000000000000000e-001	-4.010478582924174e-002	-3.99199999999996e-002
7.000000000000000e-001	1.898856187950360e-001	1.900600000000003e-001
8.000000000000000e-001	4.398760234193133e-001	4.400400000000002e-001
9.000000000000000e-001	7.098664280435906e-001	7.10019999999999e-001
1.000000000000000e+000	9.998568326678683e-001	1.000000000000000e+000

Least square error = 4.595203642069600e-002

Maximum error = 4.549013543331804e-002

Table 3(a) Numerical Results of Example 3.3, ,  $\varepsilon = 10^{-3}$ ,  $h = 10^{-3}$

X	Approximate Solution	Exact solution
0	0	0
1.000000000000000e-003	3.934453610441256e-001	3.162644137980682e-001
1.000000000000000e-002	5.025792058328056e-001	5.024892881968422e-001
1.000000000000000e-001	5.263722752192074e-001	5.263157894736842e-001
2.000000000000000e-001	5.555955897330043e-001	5.55555555555556e-001
3.000000000000000e-001	5.882514929144920e-001	5.882352941176471e-001
4.000000000000000e-001	6.249818777031101e-001	6.250000000000000e-001
5.000000000000000e-001	6.665991303676301e-001	6.66666666666666e-001
6.000000000000000e-001	7.141466849976975e-001	7.142857142857143e-001
7.000000000000000e-001	7.689873191649062e-001	7.692307692307694e-001
8.000000000000000e-001	8.329352156202253e-001	8.33333333333334e-001
9.000000000000000e-001	9.084593222235640e-001	9.090909090909091e-001
1.000000000000000e+000	9.990072376340179e-001	1.000000000000000e+000

Least square error = 9.255303301162977e-002

Maximum error = 7.718094724605740e-002

Table 3(b) Numerical Results of Example 3.3,  $\varepsilon = 10^{-4}$ ,  $h = 10^{-3}$

X	Approximate Solution	Exact solution
1.00000000000000e-003	5.009280221751820e-001	5.002273682767412e-001
1.00000000000000e-002	5.025523086031816e-001	5.025125628140703e-001
1.00000000000000e-001	5.263573033387959e-001	5.263157894736842e-001
2.00000000000000e-001	5.555989545224864e-001	5.555555555555556e-001
3.00000000000000e-001	5.882803709391627e-001	5.882352941176471e-001
4.00000000000000e-001	6.250463434466899e-001	6.25000000000000e-001
5.00000000000000e-001	6.667135360225245e-001	6.66666666666666e-001
6.00000000000000e-001	7.143318319297451e-001	7.142857142857143e-001
7.00000000000000e-001	7.692739697859491e-001	7.692307692307694e-001
8.00000000000000e-001	8.333699533928605e-001	8.33333333333334e-001
9.00000000000000e-001	9.091146895961753e-001	9.090909090909091e-001
1.00000000000000e+000	1.000000031197226e+000	1.00000000000000e+000

Least square error = 2.016751660977926e-001

Maximum error = 1.997305935198497e-001

Table 4(a) Numerical Results of Example 4.1,  $\varepsilon = 10^{-3}$ ,  $h = 10^{-3}$

X	Approximate Solution	Exact solution
0	0	0
1.00000000000000e-003	5.983373865663235e-001	5.983404102211222e-001
1.00000000000000e-002	6.831273666719399e-001	6.831968482780944e-001
1.00000000000000e-001	5.977828625956081e-001	5.978370007556204e-001
2.00000000000000e-001	5.107881944738443e-001	5.108256237659907e-001
3.00000000000000e-001	4.307617330414831e-001	4.307829160924542e-001
5.00000000000000e-001	2.876918205906360e-001	2.876820724517809e-001
6.00000000000000e-001	2.231679683515254e-001	2.231435513142098e-001
7.00000000000000e-001	1.625574978161727e-001	1.625189294977747e-001
8.00000000000000e-001	1.054127366806312e-001	1.053605156578264e-001
9.00000000000000e-001	5.135869166296200e-002	5.129329438755048e-002
1.00000000000000e+000	7.812044734680510e-005	0

Least square error = 1.383476602846244e-003

Maximum error = 7.812044734680510e-005

Table 4(b) Numerical Results of Example 4.1,  $\varepsilon = 10^{-4}$ ,  $h = 10^{-3}$

x	Approximate Solution	Exact solution
0	0	0
1.00000000000000e-003	5.563305730154806e-001	6.921476787981791e-001
1.00000000000000e-002	6.830299738197976e-001	6.831968497067772e-001
1.00000000000000e-001	5.977013860298430e-001	5.978370007556204e-001
2.00000000000000e-001	5.107234795896892e-001	5.108256237659907e-001
3.00000000000000e-001	4.307129243796157e-001	4.307829160924542e-001
4.00000000000000e-001	3.566357946191172e-001	3.566749439387324e-001
5.00000000000000e-001	2.876724980629961e-001	2.876820724517809e-001
6.00000000000000e-001	2.231623438655305e-001	2.231435513142098e-001
7.00000000000000e-001	1.625649474989907e-001	1.625189294977747e-001
8.00000000000000e-001	1.054326856796921e-001	1.053605156578264e-001
9.00000000000000e-001	5.139060962386853e-002	5.129329438755048e-002
1.00000000000000e+000	1.215172502346983e-004	0

Least square error = 1.385563137370096e-001

Maximum error = 1.358171057826985e-001

Table 5(a) Numerical Results of Example 4.2,  $\varepsilon = 10^{-3}$ ,  $h = 10^{-3}$ 

X	Approximate Solution	Exact solution
0	-1.00000000000000e+000	-1.00000000000000e+000
1.00000000000000e-003	1.941302542836244e+000	2.456939672318084e+000
1.00000000000000e-002	3.009741750685025e+000	3.009499999998871e+000
1.00000000000000e-001	3.099792505856871e+000	3.099500000000000e+000
2.00000000000000e-001	3.199838732372467e+000	3.199500000000000e+000
3.00000000000000e-001	3.299882153793792e+000	3.299500000000000e+000
4.00000000000000e-001	3.399923018015949e+000	3.399500000000000e+000
5.00000000000000e-001	3.499961544550680e+000	3.499500000000000e+000
6.00000000000000e-001	3.599997928476840e+000	3.599500000000000e+000
7.00000000000000e-001	3.700032343748764e+000	3.699500000000000e+000
8.00000000000000e-001	3.800064945981061e+000	3.799500000000000e+000
9.00000000000000e-001	3.900095874804115e+000	3.899500000000000e+000
1.00000000000000e+000	4.000125255865504e+000	3.999500000000000e+000

Least square error = 5.783440559268844e-001

Maximum error = 5.156371294818398e-001

Table 5(b) Numerical Results of Example 4.2,  $\varepsilon = 10^{-4}$ ,  $h = 10^{-3}$

X	Approximate Solution	Exact solution
0	-1.000000000000000e+000	-1.000000000000000e+000
1.000000000000000e-003	9.191895910718589e-001	3.00049999998871e+000
1.000000000000000e-002	3.004044808343324e+000	3.009500000000000e+000
1.000000000000000e-001	3.099859529904133e+000	3.099500000000000e+000
2.000000000000000e-001	3.199927743025648e+000	3.199500000000000e+000
3.000000000000000e-001	3.299991813333774e+000	3.299500000000000e+000
4.000000000000000e-001	3.400052107315556e+000	3.399500000000000e+000
5.000000000000000e-001	3.500108949447064e+000	3.499500000000000e+000
6.000000000000000e-001	3.600162628048267e+000	3.599500000000000e+000
7.000000000000000e-001	3.700213400184857e+000	3.699500000000000e+000
8.000000000000000e-001	3.800261495793313e+000	3.799500000000000e+000
9.000000000000000e-001	3.900307121169166e+000	3.899500000000000e+000
1.000000000000000e+000	4.000350461930346e+000	3.999500000000000e+000

Least square error = 2.436920365644344e+000

Maximum error = 1.358171057826985e-001

Table 6(a) Numerical Results of Example 4.3,  $\varepsilon = 10^{-3}$ ,  $h = 10^{-3}$

X	Approximate Solution	Exact Solution
0	0	0
1.000000000000000e-003	-8.646003262642733e-001	-7.615941559557649e-001
1.000000000000000e-002	-9.99999979290187e-001	-9.99999958776927e-001
1.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
2.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
3.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
4.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
5.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
6.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
7.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
8.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
9.000000000000000e-001	-9.99999999999999e-001	-1.000000000000000e+000
1.000000000000000e+000	-9.99999999999999e-001	-1.000000000000000e+000

Least square error = 1.045351515956167e-001

Maximum error = 1.030061703085085e-001

Table 6(b) Numerical Results of Example 4.3,  $\varepsilon = 10^{-4}$ ,  $h = 10^{-3}$

X	Approximate Solution	Exact solution
0	0	0
1.00000000000000e-003	-8.042265923099371e-001	-9.999999958776927e-001
1.00000000000000e-002	-9.999999172940025e-001	-1.00000000000000e+000
1.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
2.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
3.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
4.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
6.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
7.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
8.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
9.00000000000000e-001	-1.00000000000000e+000	-1.00000000000000e+000
1.00000000000000e+000	-1.00000000000000e+000	-1.00000000000000e+000

Least square error = 1.996365386642005e-001

Maximum error = 1.957734035677556e-001

Table 7(a) Numerical Results of Example 6.1,  $\varepsilon = 10^{-3}$ ,  $h = 10^{-3}$ 

X	Approximate Solution	Exact solution
0	9.99999999999999e-001	1.00000000000000e+000
1.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
2.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
3.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
4.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
5.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
6.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
7.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
8.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
9.00000000000000e-001	9.99999999999999e-001	1.00000000000000e+000
9.95000000000000e-001	9.932619436414945e-001	9.932620530009145e-001
9.96000000000000e-001	9.816841232958725e-001	9.81684361112659e-001
9.98000000000000e-001	8.646638381505986e-001	8.646647167633875e-001
9.99000000000000e-001	6.321193646719070e-001	6.321205588285580e-001
1.00000000000000e+000	0	0

Least square error = 1.582530823892044e-006

Maximum error = 1.194156651007639e-006

Table 7(b) Numerical Results of Example 6.1,  $\varepsilon = 10^{-4}$ ,  $h = 10^{-3}$

x	Approximate Solution	Exact solution
0	1.00000000000000e+000	1.00000000000000e+000
1.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
2.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
4.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
5.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
6.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
7.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
8.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
9.00000000000000e-001	1.00000000000000e+000	1.00000000000000e+000
9.90000000000000e-001	9.999999999999596e-001	1.00000000000000e+000
9.92000000000000e-001	9.99999999807050e-001	1.00000000000000e+000
9.94000000000000e-001	9.99999907937720e-001	1.00000000000000e+000
9.96000000000000e-001	9.99956074050986e-001	1.00000000000000e+000
9.98000000000000e-001	9.979041481681147e-001	9.99999979388464e-001
9.99000000000000e-001	9.542195256480970e-001	9.999546000702375e-001
1.00000000000000e+000	0	0

Least square error = 4.578317206380173e-002

Maximum error = 4.573507442214053e-002

Table 8(a) Numerical Results of Example 6.2,  $\varepsilon = 10^{-3}$ ,  $h = 10^{-3}$ 

X	Approximate Solution	Exact solution
0	9.990004182316430e-001	1.00000000000000e+000
2.00000000000000e-001	8.177311713096249e-001	8.187307530779819e-001
3.00000000000000e-001	7.398186389133611e-001	7.408182206817180e-001
4.00000000000000e-001	6.693204642672823e-001	6.703200460356393e-001
5.00000000000000e-001	6.055310779442765e-001	6.065306597126334e-001
6.00000000000000e-001	5.478120543256696e-001	5.488116360940265e-001
7.00000000000000e-001	4.955857220230526e-001	4.965853037914095e-001
8.00000000000000e-001	4.483293823488647e-001	4.493289641172216e-001
9.00000000000000e-001	4.055700779722422e-001	4.065696597405991e-001
9.90000000000000e-001	3.706223890184758e-001	3.716216392149593e-001
9.92000000000000e-001	3.701695175406173e-001	3.711670699313101e-001
9.94000000000000e-001	3.715695739775183e-001	3.725572771040055e-001
9.96000000000000e-001	3.866615273517950e-001	3.875964285627507e-001
9.98000000000000e-001	5.029880752417469e-001	5.036808194637676e-001
9.99000000000000e-001	7.353607259514390e-001	7.357592502223562e-001
1.00000000000000e+000	1.367879441171442e+000	1.367879441171442e+000

Least square error = 3.158030483435408e-002

Maximum error = 9.995817683571273e-004

Table 8(b) Numerical Results of Example 6.2,  $\varepsilon = 10^{-4}$ ,  $h = 10^{-3}$

x	Appr0ximate Solution	Exact solution
0	9.999000052023696e-001	1.000000000000000e+000
2.000000000000000e-001	8.186307582803516e-001	8.187307530779819e-001
3.000000000000000e-001	7.407182258840875e-001	7.408182206817180e-001
4.000000000000000e-001	6.702200512380089e-001	6.703200460356393e-001
5.000000000000000e-001	6.064306649150029e-001	6.065306597126334e-001
6.000000000000000e-001	5.487116412963961e-001	5.488116360940265e-001
7.000000000000000e-001	4.964853089937791e-001	4.965853037914095e-001
8.000000000000000e-001	4.492289693195911e-001	4.493289641172216e-001
9.000000000000000e-001	4.064696649429687e-001	4.065696597405991e-001
9.900000000000000e-001	3.714766962244558e-001	3.715766910220457e-001
9.920000000000000e-001	3.707342855199180e-001	3.708342802981957e-001
9.940000000000000e-001	3.699933673415090e-001	3.700933529119613e-001
9.960000000000000e-001	3.692583105049822e-001	3.693539058996320e-001
9.980000000000000e-001	3.706135217540654e-001	3.686159383604543e-001
9.990000000000000e-001	4.139491472910133e-001	3.682928591661881e-001
1.000000000000000e+000	1.367879441171442e+000	1.367879441171442e+000

Least square error = 4.580889717312363e-002,

Maximum error = 4.565628812482525e-002

#### REFERENCES

1. BENDER, C.M., and ORSZAG, S. A., *Advanced Mathematical Methods for Scientists and Engineers*, Mc. Graw-Hill, New York, 1978.
2. DIANYUN PENG , *High order Numerical method for two point boundary value problems*, Vol. 120, pp. 253-259, 1995.
3. HEMKER, P.W. and MILLER, J.J.H.(Editors), *Numerical Analysis of Singular Perturbation Problems*, Academic Press, New York, 1979.
4. KADALBAJOO, M.K., and REDDY, Y.N., *Asymptotic and Numerical Analysis of Singular Perturbations; A Survey*, Applied Mathematics and Computation, Vol.30, No.1, pp. 223-259, 1989.
5. KEVORKIAN, J. and COLE, J. D., *Perturbation Methods in Applied Mathematics*, Springer-Verlag, New York, 1981.
6. NAYFEH, A.H. , *Perturbation Methods*, Wiley, New York, 1973.
7. O'MALLEY, R.E., *Introduction to Singular Perturbations*, Academic Press, New York, 1974.
8. ROBERTS, S.M., *A boundary value technique for singular perturbation problems*, Journal of Mathematical Analysis and Applications, Vol. 87, pp. 489-508, 1982.

**P. Pramod Chakravarthy** received his Ph.D. at National Institute of Technology, Warangal, (India) under the supervision of Professor Y.N. Reddy in 2004. His research interests are Numerical Analysis, Singular Perturbation Problems.

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, Maharashtra-440011, India.

e-mail:pramodpodila@yahoo.co.in

**K. Phaneendra** is persuing Ph.D. in NIT, Warangal.

**Y.N. Reddy** is Professor and Head of the Department of Mathematics at National Institute of Technology, Warangal. His research interests are Numerical Analysis, Boundary Layer theory, Numerical Analysis of Singular Perturbation Problems.

Department of Mathematics, Kakatiya Institute of Technology & Science, Warangal,  
Andhra Pradesh-506015, India.

e-mail:[kollojuphaneendra@yahoo.co.in](mailto:kollojuphaneendra@yahoo.co.in)

Department of Mathematics, National Institute of Technology, Warangal, Andhra Pradesh-  
506004, India.

e-mail:[ynreddy@nitw.ernet.in](mailto:ynreddy@nitw.ernet.in)