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THE APPLICATIONS OF ADDITIVE MAP PRESERVING IDEMPOTENCE IN GENERALIZED INVERSE

HONGMEI YAO* ZHAOBIN FAN AND JIAPEI TANG

ABSTRACT. Suppose $\mathbf{R}$ is an idempotence-diagonalizable ring. Let n and m be two arbitrary positive integers with $n \geq 3$. We denote by $M_n(\mathbf{R})$ the ring of all $n \times n$ matrices over $\mathbf{R}$. Let $\langle \mathfrak{J}_n(\mathbf{R}) \rangle$ be the additive subgroup of $M_n(\mathbf{R})$ generated additively by all idempotent matrices. Let $\mathfrak{V} = \langle \mathfrak{J}_n(\mathbf{R}) \rangle$ or $M_n(\mathbf{R})$. In this paper, by using an additive idempotence-preserving result obtained by Cao (see [4]), I characterize (i) the additive preservers of tripotence from $\mathfrak{V}$ to $M_m(\mathbf{R})$ when 2 and 3 are units of $\mathbf{R}$; (ii) the additive preservers of inverses (respectively, Drazin inverses, group inverses, $\{1\}$-inverses, $\{2\}$-inverses, $\{1,2\}$-inverses) from $M_n(\mathbf{R})$ to $M_m(\mathbf{R})$ when 2 and 3 are units of $\mathbf{R}$.

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1. Introduction

A ring $\mathbf{R}$ is called idempotence-diagonalizable if $\mathbf{R}$ is a connected commutative ring with the multiplicative identity 1 (i.e., $\mathbf{R}$ contains no idempotents except 0 and 1) and every idempotent matrix over $\mathbf{R}$ is similar to a diagonal matrix. Let $\mathbf{R}^*$ denote the subset of $\mathbf{R}$ consisting of all units.

We will hereafter assume that n and m are two arbitrary positive integers with $n \geq 3$. We denote by $M_n(\mathbf{R})$ the ring of all $n \times n$ matrices. A matrix $A \in M_n(\mathbf{R})$ is called idempotent (respectively, tripotent) if $A^2 = A$ (respectively, $A^3 = A$). Let $\mathfrak{J}_n(\mathbf{R})(K_n(\mathbf{R}))$ be the subset of $M_n(\mathbf{R})$ consisting of all idempotent(tripotent) matrices, respectively. The notation $\langle \mathfrak{J}_n(\mathbf{R}) \rangle$ denotes the additive subgroup of $M_n(\mathbf{R})$ generated additively by $\mathfrak{J}_n(\mathbf{R})$. In more detail, $\langle \mathfrak{J}_n(\mathbf{R}) \rangle$ is the subset of $M_n(\mathbf{R})$ consisting of all matrices whose traces are integral multiple of the multiplicative identity 1 of $\mathbf{R}$.

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Let $\mathfrak{V} = \langle \mathfrak{J}_n(\mathbf{R}) \rangle$ or $M_n(\mathbf{R})$. We say that a map $f : \mathfrak{V} \to \mathfrak{M}_m(\mathbf{R})$ is additive if $f(A+B) = f(A)+f(B)$ for any $A, B \in \mathfrak{V}$. An additive map $f : \mathfrak{V} \to \mathfrak{M}_m(\mathbf{R})$ is called a preserver of tripotent if $f(A)$ is tripotent for every tripotent $A \in \mathfrak{V}$.

For a matrix $A \in M_n(R)$, consider the following matrix equations with unknow $X \in M_n(R)$.

$$AX = XA, \tag{1}$$

$$XAX = X, \tag{2}$$

$$A^k XA = A^k \text{ for some positive integer } k. \tag{3}$$

When $k = 1$, (3) turns into

$$AXA = A, \tag{4}$$

We say that X is a $\{1\}$-inverse of A if X satisfies (4), X is a $\{2\}$-inverse of A if X satisfies (2), X is a $\{1,2\}$-inverse of A if X satisfies (2) and (4), X is a Drazin inverse of A if X satisfies (1), (2) and (3), and X is a group inverse of A if X satisfies (1), (2) and (4). An additive map $f : M_n(\mathbf{R}) \to \mathbf{M_m}(\mathbf{R})$ is said to preserve Drazin (respectively, group , $\{1\}$-, $\{2\}$- and $\{1,2\}$-) inverse if $f(B)$ is a Drazin (respectively, group , $\{1\}$-, $\{2\}$- and $\{1,2\}$-) inverse of $f(A)$ whenever B is a Drazin (respectively, group , $\{1\}$-, $\{2\}$- and $\{1,2\}$-) inverse of $A \in M_n(R)$. f is said to preserve inverse if $f(A)$ is nonsingular whenever $A \in M_n(R)$ is nonsingular, and satisfies $f(A)^{-1} = f(A^{-1})$.

Some researchers are interested in the study of Linear/Additive Preserver Problems between different sets of matrices (e.g., [1]- [3],). In Cao [1], he gave the applications of the linear-idempotent preserving result. In [?], they charactered the form of additive-idempotent preserving map, Inspired by these works mentioned above, in this article we characterize:

- **(a):** the additive preservers of tripotence from $\mathfrak{V}$ to $M_m(\mathbf{R})$ when $\mathbf{R}$ is any idempotence-diagonalizable ring with $2, 3 \in \mathbf{R}^*$;
- **(b):** the additive preservers of inverses (respectively, Drazin, group , $\{1\}$-, $\{2\}$- and $\{1,2\}$-) inverses from $M_n(\mathbf{R})$ to $M_m(\mathbf{R})$ when $\mathbf{R}$ is any idempotence-diagonalizable ring with $2, 3 \in \mathbf{R}^*$.

For integers a and b with $a \leq b$, let $[a, b]$ be the set of all integers between a and b. Let I_k be the $k \times k$ identity matrix if $k > 0$ and the 0×0 empty matrix if $k = 0$. Denote by $\otimes$ and $\oplus$ the usual Kronecker product and direct sum of matrices, respectively. For any positive integers i and j, let E_{ij} be the matrix (whose dimensions can be determined by the context) with 1 in the (i, j)-th entry and 0 elsewhere. Let A^T be the transpose of A and trX be the trace of matrix X. For a non-negative integer p and a map $\tau : \mathbf{R} \to \mathbf{M_p}(\mathbf{R})$, we denote by A^τ the block matrix $[\tau(a_{ij})]$ for every matrix $A = [a_{ij}]$ if $p > 0$ and the 0×0 empty matrix if $p = 0$.

2. Main results

In this section we will always assume that $\mathbf{R}$ is an arbitrary idempotence-diagonalizable ring with $2,\ 3 \in \mathbf{R}^*$.

Lemma 1. [5] *If $A \in K_n(\mathbf{R})$, then there exists $P \in GL_n(\mathbf{R})$ such that $P^{-1}AP = I_p \oplus -I_q \oplus 0$, where $p + q = rank A$*

Lemma 2. [4] *Suppose $f : \langle \mathfrak{J}_n(\mathbf{R}) \rangle \to \mathbf{M_m}(\mathbf{R})$ is an additive preserver of idempotence with $f(I_n) = I_m$. Then there are two non-negative integers p_1, p_2 with $(p_1+p_2)n = m$, a nonsingular $m \times m$ matrix P and two ring homomorphisms $\tau_t : \mathbf{R} \to \mathbf{M_{p_t}}(\mathbf{R})$ with $\tau_t(1) = I_{p_t}$, $t = 1, 2$, such that $f(X) = P[X^{\tau_1} \oplus (X^T)^{\tau_2}]P^{-1}$ for any $X \in \langle \mathfrak{J}_n(\mathbf{R}) \rangle$.*

Lemma 3. [4] *A map $f : \langle \mathfrak{J}_n(\mathbf{R}) \rangle \to \mathbf{M_m}(\mathbf{R})$ is an additive preserver of idempotence if and only if there are two non-negative integers p_1, p_2 with $(p_1+p_2)n \le m$, a nonsingular $m \times m$ matrix P, and two ring homomorphisms $\tau_t : \mathbf{R} \to \mathbf{M_{p_t}}(\mathbf{R})$, $t = 1, 2$, such that*

$$f(X) = P[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus 0]P^{-1},\ \forall X \in \langle \mathfrak{J}_n(\mathbf{R}) \rangle. \tag{5}$$

Lemma 4. [4] *A map $f : M_n(\mathbf{R}) \to \mathbf{M_m}(\mathbf{R})$ is an additive preserver of idempotence if and only if there are two non-negative integers p_1, p_2 with $(p_1+p_2)n \le m$, an additive group homomorphism $\sigma : \mathbf{R} \to \mathbf{M_m}(\mathbf{R})$ with $\sigma(1) = 0$, a nonsingular $m \times m$ matrix P and two ring homomorphisms $\tau_t : \mathbf{R} \to \mathbf{M_{p_t}}(\mathbf{R})$, $t = 1, 2$, such that*

$$f(X) = P[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus 0]P^{-1} + \sigma(\mathrm{trX}),\ \forall \mathrm{X} \in \mathrm{M_n}(\mathbf{R}). \tag{6}$$

By an argument similar to [3, Theorem 1], the following theorem can be easily obtained from Lemma 2.

Theorem 1. *A map $f : \langle \mathfrak{J}_n(\mathbf{R}) \rangle \to \mathbf{M_m}(\mathbf{R})$ is an additive preserver of tripotence if and only if there are four non-negative integers $p_i, i \in [1, 4]$, with $(p_1 + p_2 + p_3 + p_4)n \le m$, a nonsingular $m \times m$ matrix P and four ring homomorphisms $\tau_t : \mathbf{R} \to \mathbf{M_{p_t}}(\mathbf{R})$, $t \in [1, 4]$, such that $f(X) = P\Big[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus -X^{\tau_3} \oplus -(X^T)^{\tau_4} \oplus 0\Big]P^{-1}$ for any $X \in \langle \mathfrak{J}_n(\mathbf{R}) \rangle$.*

From which, by an argument similar to Lemma 4, the additive preservers of tripotence from $M_n(\mathbf{R})$ to $M_m(\mathbf{R})$ can be characterized as following:

Theorem 2. *A map $f : M_n(\mathbf{R}) \to \mathbf{M_m}(\mathbf{R})$ is an additive preserver of tripotence if and only if there are four non-negative integers $p_i, i \in [1, 4]$, with $(p_1+p_2+p_3+p_4)n \le m$, an additive group homomorphism $\sigma : \mathbf{R} \to \mathbf{M_m}(\mathbf{R})$ with $\sigma(1) = 0$, a nonsingular $m \times m$ matrix P and four ring homomorphisms $\tau_t : \mathbf{R} \to \mathbf{M_{p_t}}(\mathbf{R})$, $t \in [1, 4]$, such that $f(X) = P\Big[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus -X^{\tau_3} \oplus -(X^T)^{\tau_4} \oplus 0\Big]P^{-1} + \sigma(\mathrm{trX})$ for any $X \in M_n(\mathbf{R})$.*

Theorem 3. *Let* $\mathbf{R}$ *be an arbitrary idempotence-diagonalizable ring with* $2,\ 3 \in \mathbf{R}^*$, *and* n, m *are positive integers with* $n \geq 3$. *Then* $f : M_n(\mathbf{R}) \to \mathbf{M_m}(\mathbf{R})$ *is an additive preserver of inverses if and only if* f *has the form*

$$f(X) = P\Big[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus -X^{\tau_3} \oplus -(X^T)^{\tau_4}\Big]P^{-1} \text{ for any } X \in M_n(\mathbf{R}),$$

where $p_t, t \in [1,4]$ *are non-negative integers with* $(p_1 + p_2 + p_3 + p_4)n = m$, P *is a nonsingular* $m \times m$ *matrix, and* $\tau_t : \mathbf{R} \to \mathbf{M_{p_t}}(\mathbf{R})$, $t \in [1,4]$, *are ring homomorphisms such that* $\tau_t(a)$ *is nonsingular for any* $t \in [1,4]$ *and nonzero* $a \in \mathbf{R}$.

Proof. The "if" part is obvious. Now we prove the "only if" part. By an argument similar to [3, Theorem 2], one can easily derive that

$$T^{-1}f(B)T = f_1(B) \oplus -f_2(B),\ \forall B \in \mathfrak{I}_n(\mathbf{R}), \tag{7}$$

where $f_i(B) \in \mathfrak{I}_{p_i}(\mathbf{R})$ satisfies $f_i(I_n) = I_{p_i}$ for $i = 1,2$. Since any matrix in $\langle \mathfrak{I}_n(\mathbf{R})\rangle$ can be represented as a sum of finitely many matrices in $\mathfrak{I}_n(\mathbf{R})$, we obtain from (7) and the additivity of f that

$$T^{-1}f(A)T = f_1(A) \oplus -f_2(A),\ \forall A \in \langle \mathfrak{I}_n(\mathbf{R})\rangle,$$

where $f_i : \langle \mathfrak{I}_n(\mathbf{R})\rangle \to \mathbf{M_{p_i}}(\mathbf{R})$, $i = 1,2$, are additive preservers of idempotence and satisfy $f_i(I_n) = I_{p_i}$ for $i = 1,2$. By Lemma 2, there are four non-negative integers $p_t, t \in [1,4]$ with $(p_1 + p_2 + p_3 + p_4)n \leq m$, a nonsingular $m \times m$ matrix P and four ring homomorphisms $\tau_t : \mathbf{R} \to \mathbf{M_{p_t}}(\mathbf{R})$, $t \in [1,4]$, such that

$$f(A) = P\Big[A^{\tau_1} \oplus (A^T)^{\tau_2} \oplus -A^{\tau_3} \oplus -(A^T)^{\tau_4}\Big]P^{-1},\ \forall A \in \langle \mathfrak{I}_n(\mathbf{R})\rangle. \tag{8}$$

By an argument similar to that the proof of Lemma 4in [4], one can easily derive

$$f(X) = P\Big[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus -X^{\tau_3} \oplus -(X^T)^{\tau_4}\Big]P^{-1} + \sigma(\mathrm{trX}),\ \forall \mathrm{X} \in \mathrm{M_n}(\mathbf{R}),$$

where σ is an additive group homomorphism from $\mathbf{R}$ to $M_m(\mathbf{R})$ with $\sigma(1) = 0$. Clearly, it remains is to show that $\sigma = 0$.

For any nonsingular $E \in M_2(\mathbf{R})$, let

$$\begin{cases} X_E = E \oplus \pm I_{n-2}, \\ Y_E = E^{\tau_1} \oplus 0 \oplus (E^T)^{\tau_2} \oplus 0 \oplus -E^{\tau_3} \oplus 0 \oplus -(E^T)^{\tau_4} \oplus 0, \\ Z_E = (E^{-1})^{\tau_1} \oplus 0 \oplus ((E^{-1})^T)^{\tau_2} \oplus 0 \oplus -(E^{-1})^{\tau_3} \oplus 0 \oplus -((E^{-1})^T)^{\tau_4} \oplus 0. \end{cases}$$

Then $f(X_E) = Y_E + \sigma(\mathrm{trE}) \pm \mathrm{C}$ and $f(X_E)^{-1} = Z_E + \sigma(\mathrm{trE^{-1}}) \pm \mathrm{C}$, where

$$C = 0 \oplus I_{(n-2)p_1} \oplus 0 \oplus I_{(n-2)p_2} \oplus 0 \oplus -I_{(n-2)p_3} \oplus 0 \oplus -I_{(n-2)p_4}. \tag{9}$$

Thus, $(Y_E + \sigma(\mathrm{trE}) \pm \mathrm{C})(\mathrm{Z_E} + \sigma(\mathrm{trE^{-1}}) \pm \mathrm{C}) = \mathrm{I_m}$, which is equivalent to $\sigma(\mathrm{trE})\mathrm{C} + \mathrm{C}\sigma(\mathrm{trE^{-1}}) = 0$ and

$$Y_E\sigma(\mathrm{trE^{-1}}) + \sigma(\mathrm{trE})\mathrm{Z_E} + \sigma(\mathrm{trE})\sigma(\mathrm{trE^{-1}}) = 0. \tag{10}$$

For every $a \in \mathbf{R}$, if we choose E is $\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} a & 1 \\ 2 & 0 \end{bmatrix}$ in (10), respectively, then $\sigma(a)C = C\sigma(a)$ and $\sigma(a)C = 2^{-1}C\sigma(a)$. Thus,

$$\sigma(a)C = C\sigma(a) = 0. \tag{11}$$

Let $\sigma(a) = (X_{st}), \quad s,t \in [1,8]$. Then, by equation (11), we have

$$\sigma(a) = (X_{st}), s,t \in [1,8] \tag{12}$$

where $X_{2k-1,2k-1} \in M_{2p_k}(\mathbf{R})$, and $X_{2k,2k} = 0(k = 1,\ 2,\ 3,\ 4)$.

Case 1. Suppose $n \geq 4$. Then, for every nonsingular $E \in M_2(\mathbf{R})$, let $X_E = \pm I_{n-2} \oplus E$, by an argument similar to (11), we have

$$\sigma(a)D = D\sigma(a) = 0. \tag{13}$$

where $D = I_{(n-2)p_1} \oplus 0 \oplus I_{(n-2)p_2} \oplus 0 \oplus -I_{(n-2)p_3} \oplus 0 \oplus -I_{(n-2)p_4} \oplus 0$. This, together with (12), implies $\sigma(a) = 0$. Because of the arbitrariness of a, we can obtain that $\sigma = 0$.

Case 2. Suppose $n = 3$. We consider X_E is $\begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$, respectively, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nonsingular, similar to the Case 1, we can derive that $\sigma = 0$, we complete the proof. □

Theorem 4. *Let* $\mathbf{R}$ *be an arbitrary idempotence-diagonalizable ring with* $2,\ 3 \in \mathbf{R}^*$, *and let* n, m *be positive integers with* $n \geq 3$. *Then* $f : M_n(\mathbf{F}) \rightarrow \mathbf{M_m}(\mathbf{F})$ *is an additive preserver of Drazin (respectively, group,* $\{1\}$-, $\{2\}$-, $\{1,2\}$- *) inverses if and only if* f *has the form*

$$f(X) = P\Big[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus -X^{\tau_3} \oplus -(X^T)^{\tau_4} \oplus 0\Big]P^{-1} \textit{for any } X \in M_n(\mathbf{R}),$$

where $p_t, t \in [1,4]$ *are non-negative integers with* $(p_1 + p_2 + p_3 + p_4)n \leq m$, P *is a nonsingular* $m \times m$ *matrix, and* $\tau_t : \mathbf{R} \rightarrow \mathbf{M_{p_t}}(\mathbf{R})$, $t \in [1,4]$, *are ring homomorphisms such that* $\tau_t(a)$ *is nonsingular for any* $t \in [1,4]$ *and nonzero* $a \in \mathbf{R}$.

Proof. The "if" part is obvious. Now we prove the "only if" part. By an argument similar to [3, Theorem 3], one can easily derive that f is an additive map preserving tripotence. By Theorem 2, one can easily derive

$$f(X) = P[X^{\tau_1} \oplus (X^T)^{\tau_2} \oplus -X^{\tau_3} \oplus -(X^T)^{\tau_4} \oplus 0]P^{-1} + \sigma(\mathrm{trX}), \forall \mathrm{X} \in \mathrm{M_n}(\mathbf{R}), \tag{14}$$

where σ is an additive group homomorphism from $\mathbf{R}$ to $M_m(\mathbf{R})$ with $\sigma(1) = 0$. Clearly, it remains is to show that $\sigma = 0$. Now, we only prove $\sigma = 0$ when f preserves $\{1\}$-inverse, other generalized inverses can be proved in the same way.

For any nonsingular $E \in M_2(\mathbf{R})$, it is clear that $E^{-1} \pm I_{n-2}$ is one of $\{1\}$-inverses of $E \in M_2(\mathbf{R})$, and hance $f(E^{-1} \pm I_{n-2})$ is one of $\{1\}$-inverses of

$f(E\pm I_{n-2})$, which implies $f(E\pm I_{n-2})f(E^{-1}\pm I_{n-2})f(E\pm I_{n-2}) = f(E\pm I_{n-2})$. By (14), we have

$$(Y_E+\sigma(\mathrm{trE})\pm \mathrm{C})(\mathrm{Z_E}+\sigma(\mathrm{trE^{-1}})\pm \mathrm{C})(\mathrm{Y_E}+\sigma(\mathrm{trE})\pm \mathrm{C}) = (\mathrm{Y_E}+\sigma(\mathrm{trE})\pm \mathrm{C}), \tag{15}$$

where

$$\begin{cases} Y_E = (E^{\tau_1}\oplus 0\oplus (E^T)^{\tau_2}\oplus 0\oplus -E^{\tau_3}\oplus 0\oplus -(E^T)^{\tau_4}\oplus 0)\oplus 0_\delta, \\ Z_E = ((E^{-1})^{\tau_1}\oplus 0\oplus ((E^{-1})^T)^{\tau_2}\oplus 0\oplus -(E^{-1})^{\tau_3}\oplus 0\oplus -((E^{-1})^T)^{\tau_4}\oplus 0) \\ \qquad \oplus 0_\delta. \end{cases}$$

and

$$C = (0\oplus I_{(n-2)p_1}\oplus 0\oplus I_{(n-2)p_2}\oplus 0\oplus -I_{(n-2)p_3}\oplus 0\oplus -I_{(n-2)p_4})\oplus 0_\delta.$$

Furthermore, noting that $f(E^{-1}\oplus O_{n-2})$ is one of $\{1\}$-inverses of $f(E\oplus O_{n-2})$, we derive

$$(Y_E+\sigma(\mathrm{trE}))(\mathrm{Z_E}+\sigma(\mathrm{trE^{-1}}))(\mathrm{Y_E}+\sigma(\mathrm{trE})) = (\mathrm{Y_E}+\sigma(\mathrm{trE})). \tag{16}$$

The combination of (15) and (16) gives that

$$(Y_E+\sigma(\mathrm{trE}))\mathrm{C}^2+\mathrm{C}^2(\mathrm{Y_E}+\sigma(\mathrm{trE}))+\mathrm{C}(\mathrm{Z_E}+\sigma(\mathrm{trE^{-1}}))\mathrm{C}=0. \tag{17}$$

Replacing E by $2E$ in (17), we have

$$4(Y_E+\sigma(\mathrm{trE}))\mathrm{C}^2+4\mathrm{C}^2(\mathrm{Y_E}+\sigma(\mathrm{trE}))+\mathrm{C}(\mathrm{Z_E}+\sigma(\mathrm{trE^{-1}}))\mathrm{C}=0. \tag{18}$$

Using (17) and (18), we can obtain that

$$\sigma(\mathrm{trE})\mathrm{C}^2+\mathrm{C}^2\sigma(\mathrm{trE})=0, \tag{19}$$

and

$$C\sigma(\mathrm{trE^{-1}})\mathrm{C}=0. \tag{20}$$

Premultiplying C on the both sides of (19), we have $C\mathrm{trE})\mathrm{C}^2+\mathrm{C}\sigma(\mathrm{trE})=0$. This, together with (20) and the arbitrariness of E, implies that $\sigma(\mathrm{trE})\mathrm{C} = \mathrm{C}\sigma(\mathrm{trE})=0$. If we choose $E = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$, then

$$\sigma(a)C = C\sigma(a) = 0. \tag{21}$$

Let $\sigma(a) = (\sum_{st}), s,t\in[1,9]$. Then, by equation (21), we have $\sigma(a) = \left(\sum_{st}\right)$. where $\sum_{2k-1,2k-1}\in M_{2p_k}(\mathbf{R})$, $\sum_{2k,2k}=0(k=1,\ 2,\ 3,\ 4)$,and $\sum_{99}\in M_\delta$.
Again by an argument similar to the Case 1 and Case 2 in the proof of Theorem 3, we obtain that $\sigma(a)=0\oplus\Sigma_{99}$. From the arbitrariness of a, we see that (16) simplifies to $\sigma(\mathrm{trE})=\sigma(\mathrm{trE})\sigma(\mathrm{trE^{-1}})\sigma(\mathrm{trE})$. If we choose E is $\begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$ and

$\begin{bmatrix} a & 1 \\ 2 & 0 \end{bmatrix}$, respectively, then we can derive that $-\sigma(a)^3 = \sigma(a)$ and $-\frac{1}{2}\sigma(a)^3 = \sigma(a)$, which implies $\sigma(a) = 0$, then $\sigma = 0$.
we complete the proof. $\square$

REFERENCES

1. C. G. Cao, *Linear maps preserving idempotence on matrix modules over some rings*, J. Natur. Sci. Heilongjiang Univ. 16(1) (1999), 1–4. (in Chinese)
2. C. G. Cao and X. Zhang, *Additive operators preserving idempotent matrices over fields and applications*, Linear Algebra Appl. 248 (1996), 327–338.
3. C. G. Cao and X. Zhang, *Linear preservers between matrix modules over connected commutative rings*, Linear Algebra Appl. 397 (2005), 355-366.
4. C. G. Cao and X. Zhang, *Additive preservers of idempotence between rings of square matrices*, Int.Math.J.(2007)(to appear) .
5. X. Zhang and C. G. Cao, *Homomorphisms between additive matrix groups which preserve some invariances*, Harbin Press, 2001. (in Chinese)

Hongmei Yao received her master degree at Heilongjiang University in July, 2006. Since 2006, she has been worked at the Harbin Engineering University. In 2007, she was a member of the provincial Natural Science Foundation, which number is 159110120002. Her research interests focus on the preserving problem in Algebra and related graph theory.

College of Science, Harbin Engineering University , Harbin 150001, P. R. China
e-mail: hongmeiyao@@163.com