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POSITIVE SOLUTIONS FOR MULTIPOINT BOUNDARY VALUE PROBLEMS WITH ONE-DIMENSIONAL p-LAPLACIAN OPERATOR

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ABSTRACT. In this paper, we study the existence of positive solutions for the following nonlinear m-point boundary value problem with p-Laplacian:

$$\begin{cases} (\phi_p(u'))' + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \\ u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i), \end{cases}$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s, \quad p > 1, \phi_q = (\phi_p)^{-1}, \; \frac{1}{p} + \frac{1}{q} = 1, 1 \le k \le s \le m-2, \; a_i, \; b_i \in (0,+\infty)$ with $0 < \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i < 1, \; 0 < \sum_{i=1}^{m-2} a_i < 1, \; 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, f \in C([0,1] \times [0,+\infty), [0,+\infty))$. We show that there exists one or two positive solutions by using fixed-point theorem for operator on a cone. The conclusions in this paper essentially extend and improve the known results.

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Key words and phrases : M-point boundary value problem; One-dimensional p-Laplacian operator; positive solutions; fixed-point theorem

1. Introduction

In this paper, we study the existence of positive solutions for the following nonlinear m-point boundary value problem with p-Laplacian

$$\begin{cases} (\phi_p(u'))' + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \\ u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i), \end{cases} \tag{1.1}$$

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where $\phi_p(s)$ is p-Laplacian operator, i.e.,

$$\phi_p(s) = |s|^{p-2}s,\ p > 1, \phi_q = (\phi_p)^{-1},\ \frac{1}{p} + \frac{1}{q} = 1, 1 \le k \le s \le m-2,$$

$$a_i,\ b_i \in (0, +\infty) \quad \text{with} \quad 0 < \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i < 1,\ 0 < \sum_{i=1}^{m-2} a_i < 1,$$

$$0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, f \in C([0,1] \times [0,+\infty), [0,+\infty)).$$

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Movisev [1, 2]. Motivated by the study of [1, 2], Gupta [3] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [4, 5, 6] for some references along this line. Multi-point boundary value problems describe many phenomena in the applied mathematical sciences. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problems (see Moshinsky [7]); many problems in the theory of elastic stability can be handle by the method of multi-point boundary value problems(see Timoshenko [8])

In 2001, Ma [6] studied m-point boundary value problem (BVP)

$$\begin{cases} u''(t) + h(t)f(u) = 0, & 0 \le t \le 1, \\ u(0) = 0, & u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u'(\xi_i) \end{cases}$$

where $\alpha_i > 0$ $(i = 1, 2, \cdots, m-2)$, $\sum\limits_{i=1}^{m-2} \alpha_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $f \in C([0,+\infty), [0,+\infty))$, $h \in C([0,1], [0,+\infty))$. Author established the existence of positive solutions theorems under the condition that f is either superlinear or sublinear.

In [4], Ma and Castaneda studied the following m-point boundary value problem (BVP)

$$\begin{cases} u''(t) + h(t)f(u) = 0, & 0 \le t \le 1, \\ u'(0) = \sum\limits_{i=1}^{m-2} a_i u'(\xi_i), & u(1) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i) \end{cases}$$

where $\alpha_i > 0$, $\beta_i > 0$ $(i = 1, 2, \cdots, m-2)$, $\sum\limits_{i=1}^{m-2} \alpha_i < 1$, $\sum\limits_{i=1}^{m-2} \beta_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $f \in C([0,+\infty), [0,+\infty))$, $h \in C([0,1], [0,+\infty))$. They showed the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

Recently, Ma etal. [5] used the monotone iterative technique in cones to prove the existence of at least one positive solutions for m-point boundary value problem (BVP)

$$\begin{cases} (\phi_p(u'))' + a(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where $0 < \sum_{i=1}^{m-2} b_i < 1$, $0 < \sum_{i=1}^{m-2} a_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a(t) \in L^1[0,1]$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

Motivated by the results mentioned above, in this paper we study the existence of positive solutions of m-point boundary value problem (1.1). We generalize the results in [4, 5, 6].

In the rest of the paper, we make the following assumptions:

(H$_1$) $a_i,\ b_i \in (0,+\infty)$, $0 < \sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i < 1$, $0 < \sum_{i=1}^{m-2} a_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$;

(H$_2$) $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

By a positive solution of BVP (1.1), we understand a function u which is positive on (0, 1) and satisfies the differential equations as well as the boundary conditions in BVP (1.1).

The following well-known result of the fixed point theorems is needed in our arguments.

Lemma 1.1.[10-12] *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_K = D \cap K \neq \emptyset$ and $\overline{D}_K \neq K$. Assume that $A : \overline{D}_K \to K$ is a compact map such that $x \neq Ax$ for $x \in \partial D_K$. Then the following results hold:*

(1) *If $\|Ax\| \leq \|x\|$, $x \in \partial D_K$, then $i(A, D_K, K) = 1$;*

(2) *If there exists $x_0 \in K\backslash\{0\}$ such that $x \neq Ax + \lambda x_0$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i(A, D_K, K) = 0$;*

(3) *Let U be open in X such that $\overline{U} \subset D_K$. If $i(A, D_K, K) = 1$ and $i(A, D_K, K) = 0$, then A has a fixed point in $D_K \backslash \overline{U}_K$. The same result holds if $i(A, D_K, K) = 0$ and $i(A, D_K, K) = 1$.*

2. Preliminaries and Lemmas

In this section , we present some lemmas that are important to our main results.

Lemma 2.1. *Let (H_1) and (H_2) hold. Then for $x \in C^+[0,1]$, the problem*

$$\begin{cases} (\phi_p(u'))' + f(t,x(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum\limits_{i=1}^{m-2} a_i u'(\xi_i), \\ u(1) = \sum\limits_{i=1}^{k} b_i u(\xi_i) - \sum\limits_{i=k+1}^{s} b_i u(\xi_i) - \sum\limits_{i=s+1}^{m-2} b_i u'(\xi_i), \end{cases} \tag{2.1}$$

has a unique solution $u(t) = B_x - \int_t^1 \phi_p^{-1}\left(A_x - \int_0^s f(r,x(r))dr\right)ds$, *where* A_x, B_x *satisfy*

$$\phi_p^{-1}(A_x) = \sum_{i=1}^{m-2} a_i \phi_p^{-1}\left(A_x - \int_0^{\xi_i} f(s,x(s))ds\right), \tag{2.2}$$

$$\begin{aligned} B_x \;&= -\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\sum_{i=1}^{k} b_i \int_{\xi_i}^1 \phi_p^{-1}(A_x - \int_0^s f(r,x(r))dr)ds \\ &\quad - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^1 \phi_p^{-1}(A_x - \int_0^s f(r,x(r))dr)ds \\ &\quad + \sum_{i=s+1}^{m-2} b_i \phi_p^{-1}(A_x - \int_0^{\xi_i} f(s,x(s))ds)\Big). \end{aligned}$$

Define $l = \dfrac{\phi_p\left(\sum_{i=1}^{m-2} a_i\right)}{1-\phi_p\left(\sum_{i=1}^{m-2} a_i\right)}$. *Then there exists a unique*

$A_x \in \left[-l\int_0^1 f(s,x(s))ds, 0\right]$ *satisfying (2.2).*

Proof. The proof is similar to Lemma 2.1[5], we omit the details. □

Lemma 2.2. *Let* (H_1) *and* (H_2) *hold. If* $x \in C^+[0,1]$, *the unique solution of the problem (2.1) satisfies* $u(t) \geq 0$.

Proof. According to Lemma 2.1 we first have$-A_x + \int_0^s f(r,x(r))dr \geq 0$. So

$$\begin{aligned} &u(1) = B_x \\ &= -\frac{1}{1-\sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\sum_{i=1}^{k} b_i \int_{\xi_i}^1 \phi_p^{-1}(A_x - \int_0^s f(r,x(r))dr)ds \\ &\quad - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^1 \phi_p^{-1}(A_x - \int_0^s f(r,(r))dr)ds \end{aligned}$$

$$+\sum_{i=s+1}^{m-2} b_i\phi_p^{-1}\big(A_x-\int_0^{\xi_i} f(s,x(s))ds\big)\Big)$$

$$=\frac{1}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^s b_i}\Big(\sum_{i=1}^k b_i\int_{\xi_i}^1 \phi_p^{-1}\big(-A_x+\int_0^s f(r,x(r))dr\big)ds$$

$$-\sum_{i=k+1}^{s} b_i\int_{\xi_i}^1 \phi_p^{-1}\big(-A_x+\int_0^s f(r,x(r))dr\big)ds$$

$$+\sum_{i=s+1}^{m-2} b_i\phi_p^{-1}\big(-A_x+\int_0^{\xi_i} f(s,x(s))ds\big)\Big)$$

$$\geq\frac{1}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^s b_i}\Big(\sum_{i=1}^k b_i\int_{\xi_k}^1 \phi_p^{-1}\big(-A_x+\int_0^s f(r,x(r))dr\big)ds$$

$$-\sum_{i=k+1}^{s} b_i\int_{\xi_k}^1 \phi_p^{-1}\left(-A_x+\int_0^s f(r,x(r))dr\right)ds\Big)$$

$$=\frac{\left(\sum_{i=1}^k b_i-\sum_{i=k+1}^s b_i\right)\int_{\xi_k}^1 \phi_p^{-1}\left(-A_x+\int_0^s f(r,x(r))dr\right)ds}{1-\sum_{i=1}^k b_i+\sum_{i=k+1}^s b_i}\geq 0.$$

If $t\in[0,1)$, we have

$$\begin{aligned}u(t)&=B_x-\textstyle\int_t^1 \phi_p^{-1}\left(A_x-\int_0^s f(r,x(r))dr\right)ds\\&=u(1)+\int_t^1 \phi_p^{-1}\left(-A_x+\int_0^s f(r,x(r))dr\right)ds\\&\geq u(1)\geq 0.\end{aligned}$$

So $u(t)\geq 0,\ t\in[0,1]$. The proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let* (H_1) *and* (H_2) *hold. If* $x\in C^+[0,1]$, *the unique solution of the problem (2.1) satisfies*

$$\inf_{t\in[0,1]} u(t)\geq\gamma_1\|u\|,$$

where $\gamma_1=\dfrac{(\sum_{i=1}^k b_i-\sum_{i=k+1}^s b_i)(1-\xi_k)}{1-\sum_{i=1}^k b_i\xi_k+\sum_{i=k+1}^s b_i\xi_k}\in(0,1)$.

Proof. Clearly

$$\begin{aligned}u'(t)&=\phi_p^{-1}\left(A_x-\int_0^t f(s,x(s))ds\right)=-\phi_p^{-1}\left(-A_x+\int_0^t f(s,x(s))ds\right)\\&\leq 0.\end{aligned}$$

This implies that

$$\|u\| = u(0), \quad \min_{t\in[0,1]} u(t) = u(1).$$

It is easy to see that $u'(t_2) \le u'(t_1)$ for any $t_1,\ t_2 \in [0,1]$ with $t_1 \le t_2$. Hence $u'(t)$ is a decreasing function on $[0,\ 1]$. This means that the graph of $u(t)$ is concave down on $(0,\ 1)$. So we have

$$u(\xi_k) - u(1)\xi_k \ge (1-\xi_k)u(0).$$

Together with $u(1) = \sum_{i=1}^{k} b_i u(\xi_i) - \sum_{i=k+1}^{s} b_i u(\xi_i) - \sum_{i=s+1}^{m-2} b_i u'(\xi_i)$ and $u'(t) \le 0$ on $[0,\ 1]$, we get

$$\begin{aligned} u(0) &\le \frac{\sum_{i=1}^{k} b_i u(\xi_k) - u(1)\sum_{i=1}^{k} b_i\xi_k - \sum_{i=k+1}^{s} b_i u(\xi_k) + u(1)\sum_{i=k+1}^{s} b_i\xi_k}{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k)} \\ &\le \frac{\sum_{i=1}^{k} b_i u(\xi_i) - u(1)\sum_{i=1}^{k} b_i\xi_k - \sum_{i=k+1}^{s} b_i u(\xi_i) + u(1)\sum_{i=k+1}^{s} b_i\xi_k}{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k)} \\ &\le \frac{u(1)\left(1 - \sum_{i=1}^{k} b_i\xi_k + \sum_{i=k+1}^{s} b_i\xi_k\right)}{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k)} = \frac{u(1)}{\gamma}. \end{aligned}$$

The proof of Lemma 2.3 is completed. □

Now we define $K = \{u \in E|\ \ u \ge 0,\ \ \min_{t\in[0,1]} u(t) \ge \gamma\|u\|\}$, where $\gamma = \gamma_1\gamma_2$, γ_1 is defined in Lemma 2.3 and

$$\gamma_2 = \frac{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1-\xi_k^q)}{\left(1 + \sum_{i=k+1}^{m-2} b_i(1+k) + \sum_{i=s+1}^{m-2} b_i q\right)(1+k)^{q-1}} \in (0,1).$$

Obviously, K is a cone in E. Define an operator $T: K \to C[0,1]$ by setting

$$\begin{aligned} &(Tx)(t) \\ &= -\frac{1}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i}\Big(\sum_{i=1}^{k} b_i \int_{\xi_i}^{1} \phi_p^{-1}(A_x - \int_0^s f(r,x(r))dr)\,ds \\ &\quad - \sum_{i=k+1}^{s} b_i \int_{\xi_i}^{1} \phi_p^{-1}(A_x - \int_0^s f(r,x(r))dr)\,ds \\ &\quad + \sum_{i=s+1}^{m-2} b_i \phi_p^{-1}(A_x - \int_0^{\xi_i} f(s,x(s))ds)\Big) \\ &\quad - \int_t^1 \phi_p^{-1}\left(A_x - \int_0^s f(r,x(r))dr\right) ds. \end{aligned}$$

Lemma 2.4. *$T : K \to K$ is completely continuous.*

Proof. According to Lemma 2.3 we easily obtain

$$Tu \geq 0, \quad \text{and} \quad \inf_{t\in[0,1]} Tu(t) \geq \gamma_1 \|Tu\| \geq \gamma \|Tu\|, \quad \text{for} \quad u \in K,$$

which means that $TK \subset K$. Now we show that T is a completely continuous operator.

(i) We show that T is continuous. First, we prove that A_x is continuous about x.

Suppose $\{x_n\} \subset C^+[0,1]$ with $x_n \to x_0 \in C^+[0,1]$. Let $\{A_n\}(n = 0,1,2\cdots)$ be constants decided by (2.2) corresponding to $x_n(n = 0,1,2\cdots)$. As $x_n \to x_0$ uniformly on $[0,1]$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, we have that for $\varepsilon = 1$, there exists $N > 0$, when $n > N$, for any $r \in [0,1]$,

$$0 \leq f(r, x_n(r)) \leq 1 + f(r, x_0(r)) \leq 1 + \max_{r\in[0,1]} f(r, x_0(r)). \tag{2.3}$$

So,

$$A_n \in \left[-l \int_0^1 f(s, x_n(s))ds, 0\right] \subseteq \left[-l(1 + \max_{r\in[0,1]} f(r, x_0(r))), 0\right],$$

which means $\{A_n\}$ is bounded.

Suppose $\{A_n\}$ does not converge to A_0. Then, there exist two subsequences $\{A_{n_k}^{(1)}\}$ and $\{A_{n_k}^{(2)}\}$ of $\{A_n\}$ with $A_{n_k}^{(1)} \to c_1$ and $A_{n_k}^{(2)} \to c_2$, but $c_1 \neq c_2$. By the construction of $\{A_n\}(n = 0,1,2\cdots)$, we have

$$\phi_p^{-1}(A_{n_k}^{(1)}) = \sum_{i=1}^{m-2} a_i \phi_p^{-1}\left(A_{n_k}^{(1)} - \int_0^{\xi_i} f(s, x_{n_k}^{(1)}(s))ds\right). \tag{2.4}$$

Combining (2.3) and using Lebesgue's dominated convergence theorem in (2.4), we get

$$\begin{aligned}
\phi_p^{-1}(c_1) &= \lim_{n_k\to\infty} \sum_{i=1}^{m-2} a_i \phi_p^{-1}\left(A_{n_k}^{(1)} - \int_0^{\xi_i} f(s, x_{n_k}^{(1)}(s))ds\right) \\
&= \sum_{i=1}^{m-2} a_i \phi_p^{-1}\left(\lim_{n_k\to\infty} A_{n_k}^{(1)} - \lim_{n_k\to\infty} \int_0^{\xi_i} f(s, x_{n_k}^{(1)}(s))ds\right) \\
&= \sum_{i=1}^{m-2} a_i \phi_p^{-1}\left(c_1 - \int_0^{\xi_i} f(s, x_0(s))ds\right).
\end{aligned}$$

Since $\{A_n\}(n = 0,1,2\cdots)$ is unique, we get $c_1 = A_0$. Similarly, $c_2 = A_0$. So, $c_1 = c_2$, which is a contradiction. Therefore, for any $x_n \to x_0$, $A_n \to A_0$, which means $A_x : C^+[0,1] \to R$ is continuous. So the continuity of T is obvious.

(ii) We prove T is compact. Let $\Omega \subset K$ be an bounded set. Then, there exists R, such that $\Omega \subset \{x \in K : ||x|| \leq R\}$. for any $x \in \Omega$, we have

$$0 \leq \int_0^1 f(s, x(s))ds \leq \max_{s\in[0,1],u\in[0,R]} f(s,u) =: M_1.$$

So we get

$$|A_x| \leq lM_1.$$

Therefore,

$$||Tx|| \leq \frac{\left(1 + \sum_{i=k+1}^{s} b_i + \sum_{i=s+1}^{m-2} b_i\right) \phi_p^{-1}\left((l+1)M_1\right)}{1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i},$$

$$||(Tx)'|| \leq \phi_p^{-1}\left((l+1)M_1\right).$$

The Arzela-Ascoli theorem guarantees that $T\Omega$ is relatively compact, which means T is compact. $\square$

We define $K_\rho = \{x(t) \in K : \|x\| < \rho\}$,

$$\begin{aligned} \Omega_\rho &= \left\{x(t) \in K : \min_{0\leq t\leq 1} x(t) < \gamma\rho\right\} \\ &= \left\{x : x \in E,\ x \geq 0, \gamma||x|| \leq \min_{0\leq t\leq 1} x(t) < \gamma\rho\right\}. \end{aligned}$$

Lemma 2.5.[10] *Ω_ρ defined above has the following properties:*

(*a*) $K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$;

(*b*) *Ω_ρ is open relative to K;*

(*c*) *$X \in \partial\Omega_\rho$ if and only if $\min_{0\leq t\leq 1} x(t) = \gamma\rho$;*

(*d*) *If $x \in \partial\Omega_\rho$, then $\gamma\rho \leq x(t) \leq \rho$ for $t \in [0,1]$.*

For the convenience, we introduce the following notations.

$$f_{\gamma\rho}^{\rho} = \min\left\{\min_{0\leq t\leq 1} \frac{f(t,u)}{\phi_p(\rho)} : u \in [\gamma\rho, \rho]\right\},\ f_0^{\rho} = \max\left\{\max_{0\leq t\leq 1} \frac{f(t,u)}{\phi_p(\rho)} : u \in [0,\rho]\right\},$$

$$f^\alpha = \lim_{u\to\alpha} \sup \max_{0\leq t\leq 1} \frac{f(t,u)}{\phi_p(u)},$$

$$f_\alpha = \lim_{u\to\alpha} \inf \max_{0\leq t\leq 1} \frac{f(t,u)}{\phi_p(u)} \quad (\alpha := \infty \text{ or } 0^+),$$

$$m = \frac{\left(1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i\right) q}{\left(1 + \sum_{i=k+1}^{m-2} b_i(1+k) + \sum_{i=s+1}^{m-2} b_i q\right)(1+k)^{q-1}},$$

$$M = \frac{\left(1 - \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{s} b_i\right) q}{\left(\sum_{i=1}^{k} b_i - \sum_{i=k+1}^{s} b_i\right)(1 - \xi_k^q)}.$$

Remark 2.1. *By* (H_1), *it is easy to see that* $0 < m$, $M < \infty$ *and* $M\gamma = M\gamma_1\gamma_2 = \gamma_1 m < m$.

Lemma 2.6. *If* f *satisfies the following condition* $f_0^\rho \le \phi_p(m)$ *and* $x \ne Tx$ *for* $x \in \partial\, K_\rho$, *then* $i(T, K_\rho, K) = 1$.

Proof. For $x \in \partial\, K_\rho$, we have

$$\begin{aligned} -A_x + \int_0^s f(r,x(r))dr &\le l\int_0^1 f(r,x(r))dr + \int_0^s f(r,x(r))dr \\ &\le l\phi_p(m)\phi_p(\rho) + \phi_p(m)\phi_p(\rho)s \\ &= \phi_p(m)\phi_p(\rho)(l+s). \end{aligned}$$

So $\phi_p^{-1}\left(-A_x + \int_0^s f(r,x(r))dr\right) \le m\rho(l+s)^{q-1}$. Therefore,

$$\begin{aligned} (Tx)(t) &= -\frac{1}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i}\Bigg(\sum_{i=1}^k b_i \int_{\xi_i}^1 \phi_p^{-1}(A_x - \int_0^s f(r,x(r))dr)ds \\ &\quad - \sum_{i=k+1}^s b_i \int_{\xi_i}^1 \phi_p^{-1}(A_x - \int_0^s f(r,x(r))dr)ds \\ &\quad + \sum_{i=s+1}^{m-2} b_i\phi_p^{-1}(A_x - \int_0^{\xi_i} f(s,x(s))ds)\Bigg) \\ &\quad - \int_t^1 \phi_p^{-1}\left(A_x - \int_0^s f(r,x(r))dr\right)ds \\ &\le \frac{1}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i}\Bigg(\sum_{i=1}^k b_i \int_0^1 \phi_p^{-1}(-A_x + \int_0^s f(r,x(r))dr)ds \\ &\quad + \sum_{i=s+1}^{m-2} b_i\phi_p^{-1}\left(-A_x + \int_0^s f(r,x(r))dr\right) \\ &\quad + \int_0^1 \phi_p^{-1}\left(-A_x + \int_0^s f(r,x(r))dr\right)ds\Bigg) \\ &\le \frac{m\rho\left(1+\sum_{i=k+1}^{m-2} b_i(1+k) + \sum_{i=s+1}^{m-2} b_i q\right)(1+k)^{q-1}}{\left(1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i\right)q} = \rho, \end{aligned}$$

which implies that $||Tx|| \le ||x||$ for $x \in \partial\, K_\rho$. Hence by Lemma 1.1(1) it follows that $i(T, \Omega_\rho, K) = 1$. □

Lemma 2.7. *If* f *satisfies the following condition* $f_{\gamma\rho}^\rho \ge \phi_p(M\gamma)$ *and* $x \ne Tx$ *for* $x \in \partial\, \Omega_\rho$, *then*

$$i(T, \Omega_\rho, K) = 0.$$

Proof. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $e \in \partial K_1$. We claim that

$$x \neq Tx + \lambda e, \quad x \in \partial\Omega_\rho, \ \lambda > 0.$$

In fact, if not, there exist $x_0 \in \partial\Omega_\rho$ and $\lambda_0 > 0$ such that $x_0 = Tx_0 + \lambda_0 e$. By $f^{\rho}_{\gamma\rho} \geq \phi_p(M\gamma)$, we have

$$-A_x + \int_0^s f(r,x(r))dr \geq \int_0^s f(r,x(r))dr \geq \phi_p(M\gamma)\phi_p(\rho)s.$$

So $\phi_p^{-1}\left(-A_x + \int_0^s f(r,x(r))dr\right) \geq M\gamma\rho s^{q-1}$. Therefore,

$$\begin{aligned}
x_0(t) &= Tx_0(t) + \lambda_0 e \\
&= \frac{1}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i}\Big(\sum_{i=1}^k b_i \int_{\xi_i}^1 \phi_p^{-1}(-A_x + \int_0^s f(r,x_0(r))dr)ds \\
&\quad - \sum_{i=k+1}^s b_i \int_{\xi_i}^1 \phi_p^{-1}(-A_x + \int_0^s f(r,x_0(r))dr)ds \\
&\quad + \sum_{i=s+1}^{m-2} b_i\phi_p^{-1}(-A_x + \int_0^{\xi_i} f(s,x_0(s))ds)\Big) \\
&\quad + \int_t^1 \phi_p^{-1}\left(-A_x + \int_0^s f(r,x_0(r))dr\right)ds + \lambda_0 \\
&\geq \frac{1}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i}\Big(\sum_{i=1}^k b_i \int_{\xi_k}^1 \phi_p^{-1}(-A_x + \int_0^s f(r,x_0(r))dr)ds \\
&\quad - \sum_{i=k+1}^s b_i \int_{\xi_k}^1 \phi_p^{-1}(-A_x + \int_0^s f(r,x_0(r))dr)ds\Big) + \lambda_0 \\
&\geq \frac{\sum_{i=1}^k b_i - \sum_{i=k+1}^s b_i}{1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i} \int_{\xi_k}^1 \phi_p^{-1}(-A_x + \int_0^s f(r,x_0(r))dr)ds\Big) \\
&\geq \frac{(\sum_{i=1}^k b_i - \sum_{i=k+1}^s b_i)M\gamma\rho(1-\xi_k^q)}{(1-\sum_{i=1}^k b_i + \sum_{i=k+1}^s b_i)q} + \lambda_0 = \gamma\rho + \lambda_0.
\end{aligned}$$

This implies that $\gamma\rho \geq \gamma\rho + \lambda_0$ which is a contradiction. Hence by Lemma 1.1(2) it follows that $i(T,\Omega_\rho,K) = 0$. □

3. The main results

We now give our results on the existence of positive solutions of BVP (1.1).

Theorem 3.1. *Suppose conditions (H_1), (H_2) hold, and assume that one of the following conditions hold:*

(H_3) *There exist* $\rho_1,\ \rho_2 \in (0,+\infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that* $f_0^{\rho_1} \le \phi_p(m)$, $f_{\gamma\rho_2}^{\rho_2} \ge \phi_p(M\gamma)$;

(H_4) *There exist* $\rho_1,\ \rho_2 \in (0,+\infty)$ *with* $\rho_1 < \rho_2$ *such that* $f_0^{\rho_2} \le \phi_p(m)$, $f_{\gamma\rho_1}^{\rho_1} \ge \phi_p(M\gamma)$.

Then, the BVP (1.1) has at least one positive solution.

Proof. Assume that (H_3) holds, we show that T has a fixed point u_1 in $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$. By $f_0^{\rho_1} \le \phi_p(m)$ and Lemma 2.6, we have that

$$i(T, K_{\rho_1}, K) = 1.$$

By $f_{\gamma\rho_2}^{\rho_2} \ge \phi_p(M\gamma)$ and Lemma 2.7, we have that

$$i(T, K_{\rho_2}, K) = 0.$$

By Lemma 2.5 (a) and $\rho_1 < \gamma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 1.1 (3) that T has a fixed point u_1 in $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$. When condition ($H_4$) holds, the proof is similar to the above, so we omit it here. □

As a special case of Theorem 3.1, we obtain the following result.

Corollary 3.1. *Suppose conditions* (H_1), (H_2) *hold, and assume that one of the following conditions holds:*

(H_5) $0 \le f^0 < \phi_p(m)$ *and* $\phi_p(M) < f_\infty \le \infty$;

(H_6) $0 \le f^\infty < \phi_p(m)$ *and* $\phi_p(M) < f_0 \le \infty$.

Then, the BVP (1.1) has at least one positive solution.

Theorem 3.2. *Assume conditions* (H_1), (H_2)*hold, and suppose that one of the following conditions holds:*

(H_7) *There exist* $\rho_1,\ \rho_2,\ \rho_3 \in (0,+\infty)$ *with* $\rho_1 < \gamma\rho_2$ *and* $\rho_2 < \rho_3$ *such that*

$$f_0^{\rho_1} \le \phi_p(m),\ f_{\gamma\rho_2}^{\rho_2} \ge \phi_p(M\gamma),\ x \ne Tx,\ \forall\, x \in \partial\Omega_{\rho_2},\ \text{and}\ \ f_0^{\rho_3} \le \phi_p(m);$$

(H_8) *There exist* $\rho_1,\ \rho_2,\ \rho_3 \in (0,+\infty)$ *with* $\rho_1 < \rho_2 < \gamma\rho_3$ *such that*

$$f_0^{\rho_2} \le \phi_p(m),\ f_{\gamma\rho_1}^{\rho_1} \ge \phi_p(M\gamma),\ x \ne Tx,\ \forall\, x \in \partial K_{\rho_2},\ \text{and}\ \ f_{\gamma\rho_3}^{\rho_3} \ge \phi_p(M\gamma).$$

Then, the BVP (1.1) has at least two positive solutions. Moreover, if in (H_7) $f_0^{\rho_1} \le \phi_p(m)$ *is replaced by* $f_0^{\rho_1} < \phi_p(m)$, *then the BVP (1.1) has a third positive solution* $x_3 \in K_{\rho_1}$.

Proof. Assume that condition (H_7) holds, we show that either T has a fixed point x_1 in ∂K_{ρ_1} or $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$. If $x \ne Tx$ for $x \in \partial K_{\rho_1} \cup \partial K_{\rho_3}$. By Lemma 2.6 and Lemma 2.7, we have that $i(T,\ K_{\rho_1},\ K) = 1$, $i(T,\ K_{\rho_3},\ K) = 1$, and $i(T,\ K_{\rho_2},\ K) = 0$. By Lemma 2.5(a) and $\rho_1 < \gamma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 1.1 (3) that T has a fixed point x_1 in $\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$. Similarly, T has a fixed point in $K_{\rho_3}\backslash\overline{\Omega}_{\rho_2}$ when condition (H_8) holds. □

As a special case of Theorem 3.2, we obtain the following result.

Corollary 3.2. *Assume conditions* (H_1), (H_2)*hold. If there exists* $\rho > 0$ *such that one of the following conditions holds:*

(H_9) $0 \le f^0 < \phi_p(m)$, $f^{\rho}_{\gamma\rho} \ge \phi_p(M\gamma)$, $x \ne Tx$, $\forall x \in \partial\Omega_\rho$ *and* $0 \le f^\infty < \phi_p(m)$;

(H_{10}) $\phi_p(m) < f_0 \le \infty$, $f^{\rho}_0 \le \phi_p(m)$, $x \ne Tx$, $\forall x \in \partial\, K_\rho$ *and* $\phi_p(M) < f_\infty \le \infty$, *then the BVP (1.1) has at least two positive solutions.*

4. Examples

Example 4.1. Consider the following five-point boundary value problem with p-Laplacian

$$\begin{cases} (\phi_p(u'))' + u^{\frac{1}{2}}\left[\dfrac{\sqrt[4]{2}}{\sqrt{45}} - \dfrac{1}{50} + \dfrac{3e^{2u}}{142 + 7e^u + e^{2u}}\right] = 0, \quad 0 < t < 1, \\ u'(0) = \dfrac{1}{8}u'(\dfrac{1}{4}) + \dfrac{1}{4}u'(\dfrac{1}{2}) + \dfrac{1}{2}u'(\dfrac{3}{4}), \\ u(1) = u(\dfrac{1}{4}) - \dfrac{1}{2}u(\dfrac{1}{2}) - 4u'(\dfrac{3}{4}), \end{cases} \tag{4.1}$$

where $a_1 = \frac{1}{8}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$, $b_1 = 1$, $b_2 = \frac{1}{2}$, $b_3 = 4$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$, $\xi_3 = \frac{3}{4}$, $p = \frac{3}{2}$, $f(t,u) = u^{\frac{1}{2}}\left[\frac{1}{3} + \frac{64e^{2u}}{120 + 7e^u + e^{2u}}\right]$.

By computing, we can know $q = 3$, $f_\infty = \frac{\sqrt[4]{2}}{\sqrt{45}} - \frac{1}{50} + 3$, $f_0 = \frac{\sqrt[4]{2}}{\sqrt{45}}$, $m = \frac{1}{22\sqrt{2}}$, $M = \frac{56}{13}$. Obviously,

$$f_0 = \frac{\sqrt[4]{2}}{\sqrt{45}} < \frac{\sqrt[4]{2}}{\sqrt{44}} = (m)^{p-1},$$

$$f_\infty = \frac{\sqrt[4]{2}}{\sqrt{45}} - \frac{1}{50} + 3 > 3 > \sqrt{\frac{56}{13}} = (M)^{p-1}.$$

So condition (H_5) hold, by Corollary 3.1, BVP (4.1) has at least one positive solution.

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